



# Generalized permutohedra: Minkowski linear functionals and Ehrhart positivity

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(Polytop)ics, MPI Leipzig, April 7, 2021



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Joint work with...



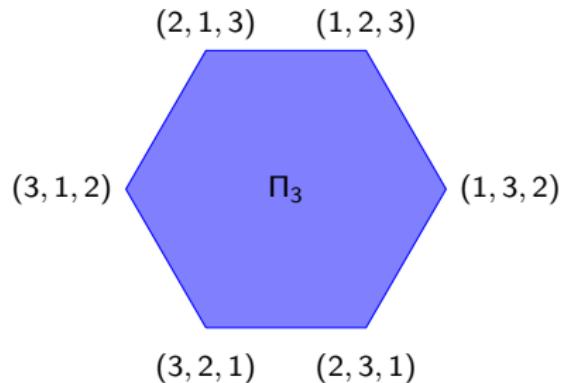
Mohan Ravichandran  
(Bogazici University Istanbul)

# Generalized Permutahedra

# Permutahedra

The **permutahedron** is defined as

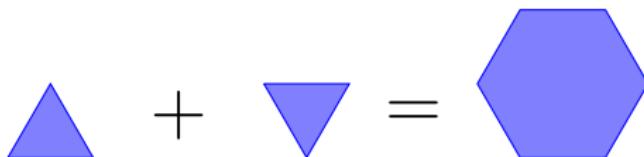
$$\Pi_d = \text{conv}\{(\sigma(1), \dots, \sigma(d)) : \sigma \in S_d\}$$



## Generalized permutohedra

The **Minkowski sum** of two polytopes  $P$  and  $Q$  is defined as

$$P + Q = \{p + q : p \in P, q \in Q\}.$$



A polytope  $P$  is a **generalized permutohedron** if

$$\lambda P + R = \Pi_d$$

for some polytope  $R$  and some  $\lambda > 0$ .

# Generalized permutohedra

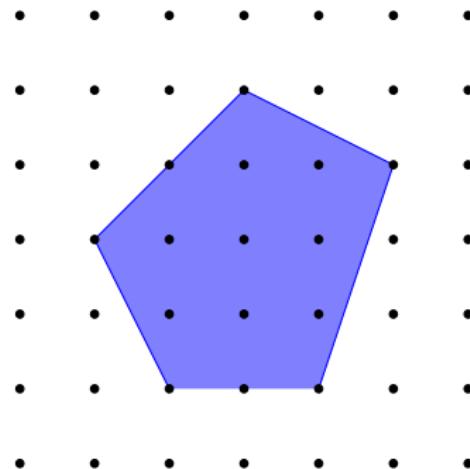
- ▶ Combinatorics
  - ▶ Matroid polytopes
  - ▶ Pitman-Stanley polytopes
  - ▶ associahedra
  - ▶ ...
- ▶ Optimization
  - ▶ Polymatroids
- ▶ Discrete analysis
  - ▶ M-convex sets
- ▶ Game theory
- ▶ Statistics
- ▶ Representation theory
- ▶ ...

# Ehrhart Theory

## Lattice polytopes

A set  $P \subset \mathbb{R}^d$  is a **lattice polytope** if there are  $x_1, \dots, x_m \in \mathbb{Z}^d$  with

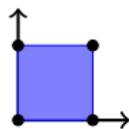
$$P = \text{conv}\{x_1, \dots, x_m\}.$$



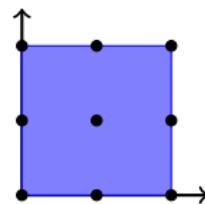
# Discrete volume

The **discrete volume** of  $P$  is

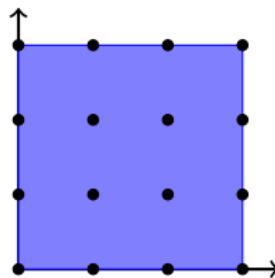
$$E(P) := |P \cap \mathbb{Z}^d|.$$



$n = 1$



$n = 2$



$n = 3$

$$E(nP) = (n+1)^2.$$

# Ehrhart polynomial

Theorem (Ehrhart'62)

For every lattice polytope  $P$  in  $\mathbb{R}^d$

$$E_P(n) := |nP \cap \mathbb{Z}^d|$$

agrees with a polynomial

$$E_P(n) = E_0(P) + E_1(P)n + \cdots + E_{\dim P}(P)n^{\dim P}$$

for  $n \geq 1$ , called the **Ehrhart polynomial** of  $P$ .

Connections to...

- ▶ **enumerative combinatorics**
- ▶ **commutative algebra**
- ▶ **convex geometry**
- ▶ ...

# Coefficients

## Central Questions

- ▶ **Characterization** of Ehrhart polynomials
- ▶ **Interpretation** of coefficients

$E_{\dim P}(P)$  = volume

$E_{\dim P-1}(P)$  = normalized surface area

$E_0(P)$  = 1

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$E_0(P)$  = 1

$\dim P \geq 3$ : **Coefficients can be negative in general!**

## Ehrhart positivity

A polytope  $P$  with  $E_P(n) = \sum E_i(P)n^i$  is **Ehrhart positive** if

$$E_i(P) \geq 0$$

for all  $i$ .

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## Examples

- ▶ zonotopes (Stanley '91)
- ▶ Minkowski sums of standard simplices (Postnikov '09)
- ▶ rank 2 uniform matroids/few elements (De Loera et al '09)
- ▶ hypersimplices (Ferroni '21)
- ▶ generalized permutohedra (Castillo, Liu '18)
  - ▶  $\dim P \leq 6$
  - ▶ 3rd and 4th coefficient

## Ehrhart positivity

Conjecture (De Loera, Haws, Koeppe '09)

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Conjecture (Castillo, Liu '18)

*Generalized permutohedra are Ehrhart positive.*

Theorem (J., Ravichandran 19+; Castillo, Liu 19+)

*If  $P$  is a generalized permutohedron then  $E_1(P) \geq 0$ .*

# Minkowski Linear Functionals

# Minkowski linear functionals

$\Lambda: \mathbb{R}^d$  or  $\mathbb{Z}^d$

$\mathcal{P}(\Lambda)$ : polytopes with vertices in  $\Lambda$ .

A map  $\varphi: \mathcal{P}(\Lambda) \rightarrow \mathbb{R}$  is a **Minkowski linear functional** if for all  $P, Q \in \mathcal{P}(\Lambda)$  and all (integers)  $\lambda, \mu \geq 0$

$$\varphi(\lambda P + \mu Q) = \lambda\varphi(P) + \mu\varphi(Q).$$

## Examples:

- ▶ Volume in  $\mathbb{R}^1$
- ▶  $(d - 1)$ th intrinsic volume in  $\mathbb{R}^d$

## The linear term of the Ehrhart polynomial

Let  $E_1(P)$  be the linear term of the Ehrhart polynomial  $E_P(n)$ .

**Lemma (Böröczky, Ludwig '19)**

For lattice polytopes  $P, Q \in \mathcal{P}(\mathbb{Z}^d)$  and integers  $\lambda, \mu \geq 0$

$$E_1(\lambda P + \mu Q) = \lambda E_1(P) + \mu E_1(Q),$$

that is,  $E_1: \mathcal{P}(\mathbb{Z}^d) \rightarrow \mathbb{R}$  is a Minkowski linear functional.

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*that is,  $E_1: \mathcal{P}(\mathbb{Z}^d) \rightarrow \mathbb{R}$  is a Minkowski linear functional.*

From the Betke-Kneser-Theorem ('85) it follows that

**Corollary**

*$E_1: \mathcal{P}(\mathbb{Z}^d) \rightarrow \mathbb{R}$  is, up to scaling, the unique Minkowski linear functional that is invariant under unimodular transformations.*

## Translation invariance, symmetry and positivity

$\mathcal{P}_d(\Lambda)$ : generalized permutohedra in  $\mathbb{R}^d$  with vertices in  $\Lambda$ .

- ▶ closed under Minkowski sums and (integer) dilations
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- ▶ **translation invariant** if for all  $P \in \mathcal{P}_d(\Lambda)$  and all  $t \in \Lambda$

$$\varphi(P + t) = \varphi(P).$$

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- ▶ **positive** if for all  $P \in \mathcal{P}_d(\Lambda)$

$$\varphi(P) \geq 0.$$

# Minkowski linear functionals on generalized permutohedra

The linear term  $E_1|_{\mathcal{P}_d(\mathbb{Z}^d)}$  of the Ehrhart polynomial is

- ▶ Minkowski linear
- ▶ translation-invariant
- ▶ symmetric

**Q:** Is  $E_1|_{\mathcal{P}_d(\mathbb{Z}^d)}$  positive?

**Q:** Can we characterize all positive, translation invariant, symmetric Minkowski linear functionals on generalized permutohedra?

## Minkowski sums of standard simplices

For  $\emptyset \neq I \subseteq [d]$  let

$$\Delta_I = \text{conv}\{e_i : i \in I\}.$$

For all  $\{y_I\}_{\emptyset \neq I \subseteq [d]}$ ,  $y_I \geq 0$ ,

$$P(\{y_I\}) = \sum_I y_I \Delta_I$$

is a generalized permutohedron.

## Signed Minkowski sums

A **signed Minkowski sum** is a formal sum  $\sum_I y_I \Delta_I$ ,  $y_I \in \mathbb{R}$  that defines a Minkowski difference:

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Theorem (Ardila, Benedetti, Dokter '10)

*Every generalized permutohedron is uniquely given as a signed Minkowski sum  $P = \sum_I y_I \Delta_I$ .*

## Minkowski linear functionals on generalized permutohedra

For all Minkowski linear functionals  $\varphi: \mathcal{P}_d \rightarrow \mathbb{R}$ ,

$$\varphi\left(\sum y_I \Delta_I\right) = \sum y_I \varphi(\Delta_I)$$

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## Observation

- ▶  $\varphi$  uniquely determined by values  $\{\varphi(\Delta_I)\}$  for all  $\emptyset \neq I \subseteq [d]$ .
- ▶ symmetric Minkowski linear functionals are uniquely determined by values on  $\Delta_{i+1} := \Delta_{[i+1]}$  for all  $0 \leq i \leq d - 1$ .

## Signed Minkowski sums

Theorem (J., Ravichandran 19+)

*The sum  $\sum_I y_I \Delta_I$  defines a generalized permutohedron if and only if*

$$\sum_{E \subseteq I \subseteq T} y_I \geq 0$$

*for all 2-element subsets  $E$  and all  $E \subseteq T \subseteq [d]$ .*

## Proof idea

For all  $u \in \mathbb{R}^d \setminus \{0\}$  let

$$P^u = \{x \in P : u^T x = \max_{y \in P} u^T y\}$$

### Theorem (Shephard's Theorem)

A polytope  $P$  is a Minkowski summand of a polytope  $Q$  iff

- (i)  $Q^u$  vertex  $\Rightarrow$   $P^u$  vertex; and
- (ii)  $Q^u = [p, q]$  edge  $\Rightarrow$   $P^u = \lambda[p, q]$  with  $0 \leq \lambda \leq 1$  (up to translation)  
for all  $u \neq 0$ .

## Proof idea

- ▶  $\sum_I y_I \Delta_I$  generalized permutohedron iff

$$\sum_{I: y_I < 0} (-y_I) \Delta_I \quad \text{Minkowski summand of} \quad \sum_{I: y_I \geq 0} y_I \Delta_I$$

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- ▶ possible edge directions:  $e_i - e_j$ ,  $i \neq j$

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- ▶ possible edge directions:  $e_i - e_j$ ,  $i \neq j$
- ▶  $u \in \mathbb{R}^d \setminus \{0\}$  such  $u_i = u_j$
- ▶ determine

$$\left( \sum_{I: y_I < 0} (-y_I) \Delta_I \right)^u \quad \text{and} \quad \left( \sum_{I: y_I \geq 0} y_I \Delta_I \right)^u$$

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- ▶ determine length of corresponding edges

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- ▶ determine length of corresponding edges
- ▶ obtain inequalities using Shephard's theorem

## Minkowski linear functionals

For every 2-element subset  $E \subseteq [d]$  and any  $E \subseteq T \subseteq [d]$  let

$$v_E^T : \mathcal{P}_d \rightarrow \mathbb{R}$$
$$v_E^T(\Delta_I) = \begin{cases} 1 & \text{if } E \subseteq I \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

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**Theorem (J., Ravichandran 19+)**

$\varphi: \mathcal{P}_d \rightarrow \mathbb{R}$  is a positive, translation invariant, Minkowski linear functional if and only if

$$\varphi = \sum c_E^T v_E^T$$

with  $c_E^T \geq 0$ .

# Symmetric Minkowski linear functionals

Theorem (J. Ravichandran 19+)

$\varphi: \mathcal{P}_d \rightarrow \mathbb{R}$  is a positive, symmetric, translation invariant, Minkowski linear functional if and only if

$$\varphi = \sum_{k=1}^{d-1} c_k f_k$$

for  $c_1, \dots, c_{d-1} \geq 0$  where  $f_k$  is the unique symmetric, translation invariant Minkowski linear functional such that

$$(f_k)(\Delta_{i+1}) = \binom{i+1}{2} \binom{d-i-1}{k-i}$$

for all  $1 \leq i \leq d-1$ .

# Ehrhart positivity

## Theorem (J., Ravichandran +19)

The Minkowski linear functional  $E_1|_{\mathcal{P}_d(\mathbb{Z}^d)}$  is positive.

### Proof idea

- ▶  $E_{\Delta_{i+1}}(n) = \binom{n+i}{i} = \frac{(n+i)(n+i-1)\cdots(n+1)}{i!}$   
 $\Rightarrow E_1(\Delta_{i+1}) = 1 + 1/2 + 1/3 + \cdots + 1/i =: h_i$
- ▶ Solve  $h = \sum_{k=1}^{d-1} c_k f_k$ , i.e.,  
$$h_i = \sum_{k=1}^{d-1} c_k \binom{i+1}{2} \binom{d-i-1}{k-i} \quad \text{for all } 1 \leq i \leq d-1.$$
- ▶ Prove  $c_1, \dots, c_{d-1} \geq 0$

# Conclusion

## Results

- ▶ Characterization of generalized permutohedra as signed Minkowski sums of standard simplices
- ▶ Characterization of positive, (symmetric), translation invariant Minkowski linear functionals
- ▶ Positivity of  $E_1$  for generalized permutohedra
- ▶ In contrast: linear term of solid-angle polynomial can be negative

## Open ends

- ▶ Characterization of  $i$ -homogeneous valuations on generalized permutohedra?
- ▶ Nonnegativity of  $E_2, E_3, E_4 \dots$ ?
- ▶ Other classes of polytopes?

# The End

$$\sum_{j=1}^{d-1} \frac{b_{ij} t_{ij} \binom{d-i-j}{j-i}}{\binom{d-i}{j-i}} \geq \sum_{j=1}^{d-1} b_{ij} t_{ij} \frac{\binom{d-i-j}{j-i}}{\binom{d-i}{j-i}}$$
$$b_{ij}(t) = \sum_{j=1}^i b_{ij} t_{ij}$$
$$\text{Claim: } b_{ij}(t) \geq b_{ij}(1)$$
$$b_{ij}(1) = \frac{\binom{d-i}{j-i} \binom{d-i-j}{j-i}}{\binom{d-i}{j-i}}$$
$$b_i(t) = \sum_{j=1}^i b_{ij}(t)$$
$$b_i(t) \geq \sum_{j=1}^i b_{ij}(1)$$
$$b_i(t) = \sum_{j=1}^i b_{ij}(1) + \sum_{k=1}^{n-i} c_k \binom{n-i-k}{n-i-k}$$
$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ c_1 & c_2 & \dots & c_n \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ c_1 & c_2 & \dots & c_n \end{pmatrix}$$
$$b_i(t) = \sum_{j=1}^i b_{ij}(1) + \sum_{k=1}^{n-i} c_k \binom{n-i-k}{n-i-k}$$

Generalized permutohedra: Minkowski linear functionals and Ehrhart positivity (K. Jochemko, M. Ravichandran)  
<https://arxiv.org/abs/1909.08448>