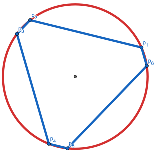
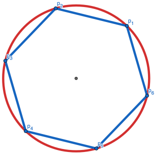
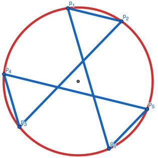
Normally inscribable polytopes

Raman Sanyal
Goethe-Universität Frankfurt





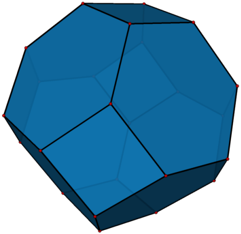


joint work with Sebastian Manecke

arXiv 2012.07724

Polytopes and inscribability

Polytope $P \subset \mathbb{R}^d$ is the convex hull of finitely many points.



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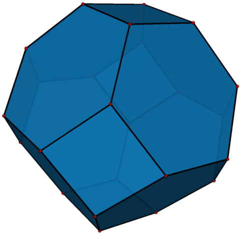

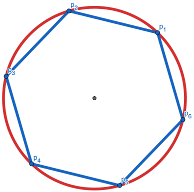

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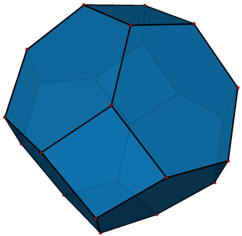

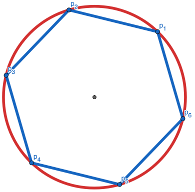

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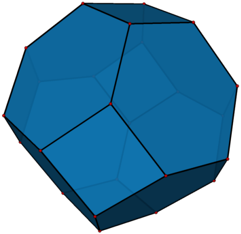

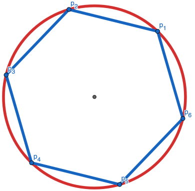

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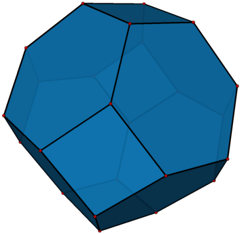

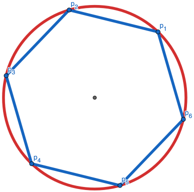

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Obviously true for convex polygons (2-dimensional polytopes).

Why care about inscribed polytopes?

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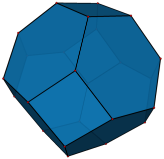
Orbit polytopes / discrete orbitopes

$P = conv(G \cdot p)$, $G \subset O(d)$ finite group

E.g.: Permutahedra



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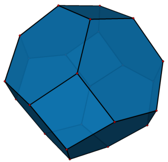


Ideal hyperbolic polyhedra

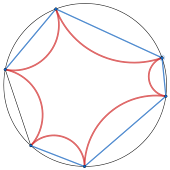
hyperbolic space $B_d^\circ = \{x \in \mathbb{R}^d : \|x\|_2 < 1\}$

hyperplanes: $H \cap B_d^\circ$, $H \subset \mathbb{R}^d$ is *usual* hyperplane

hyperbolic polytopes: *usual* polytopes $P \subseteq B_d$.

$S^{d-1} = \partial B_d$ are points at infinity

ideal hyperbolic polyhedra have all vertices in infinity



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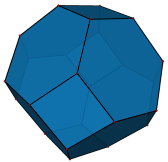

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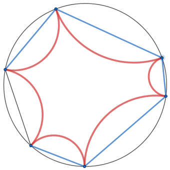
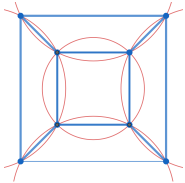


Delaunay subdivisions

subdivisions with empty circumsphere condition

inscribed polytopes under stereographic projection



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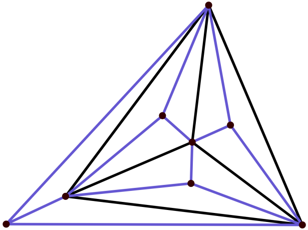

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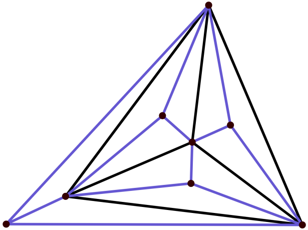

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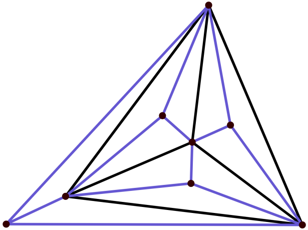

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Theorem (Rivin 1992)

Checking if a planar 3-connected graph can be realized as an inscribed polytope can be done in polynomial time.

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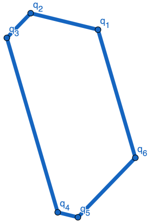

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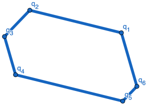

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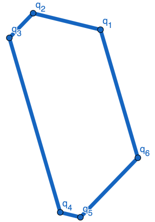

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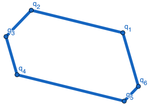

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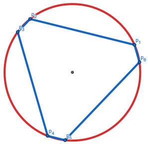

Question

When is P normally equivalent to an inscribed polytope P'?
$\longrightarrow$ call such P normally inscribable

Inscribed cones

Inscribed cone of $P \subset \mathbb{R}^d$

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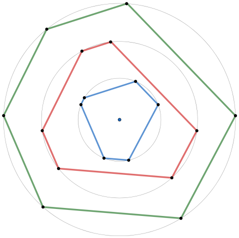

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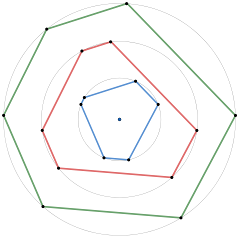

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Corollary

If P is normally inscribable with symmetry group G, then there is $P' \in \mathcal{I}_+(P)$ with symmetry group G.

First implications

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For generic linear function $\ell(x)$, let $p \in P$, $q \in Q$ be maximizers.

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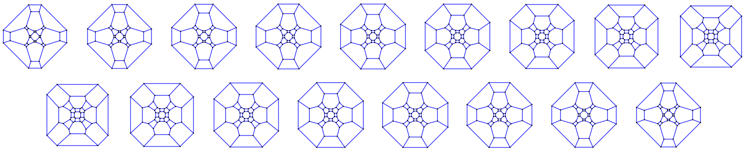

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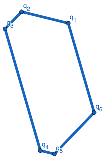

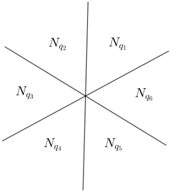

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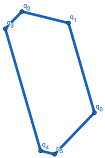

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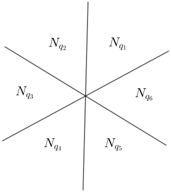
$$P \simeq P' \quad \iff \quad \mathcal{N}(P) = \mathcal{N}(P')\,.$$



Equivalence classes represented by complete, projective fans.

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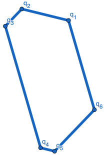

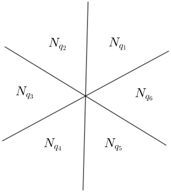

Type cone (or nef cone) of a fan $\mathcal{N}$

$$\mathcal{T}_+(\mathcal{N}) \ := \ \{P \subset \mathbb{R}^d : \mathcal{N}(P) = \mathcal{N}\} \ / \text{ translation}$$

Inscribed cone of $\mathcal{N}$

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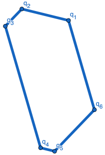

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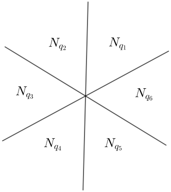

Equivalence classes represented by complete, projective fans.

Type cone (or nef cone) of a fan $\mathcal{N}$

$$\mathcal{T}_+(\mathcal{N}) \ := \ \{P \subset \mathbb{R}^d : \mathcal{N}(P) = \mathcal{N}\} \ / \text{ translation}$$

Inscribed cone of $\mathcal{N}$

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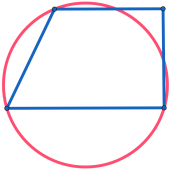

is a subcone of $\mathcal{T}_+(\mathcal{N})$.

$\mathcal{N}$ is inscribable if $\mathcal{I}_+(\mathcal{N}) \neq \varnothing$.

Example: Braid Arrangement $\mathcal{A}_{d-1}$ and Permutahedra

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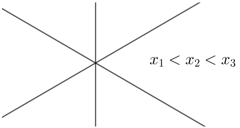

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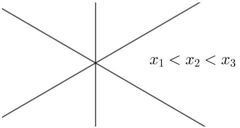

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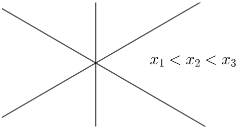

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$\mathcal{I}_+(\mathcal{A}_{d-1}) \cong \{x_1 < x_2 < \cdots < x_d\}$ (linear order cone)

Permutahedron

$$p \;\mapsto\; \Pi(p) := conv\{(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(d)}) : \sigma\}$$

$\Rightarrow$ every inscribed permutahedron is $\mathfrak{S}_d$-symmetric.

Local reflections

Lemma

Let $Q \in \mathcal{I}_+(\mathcal{N})$*. Then* Q *is uniquely determined by a single vertex.*

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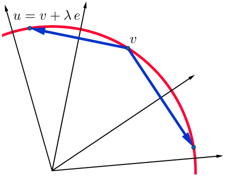
Normal cone $N_v = \{c : \langle e_i, c\rangle \leq 0 \text{ for } i = 1, \dots, k\}$.

Neighboring vertices are on the rays $\{v + \lambda_i e_i : \lambda_i \geq 0\}$.

If $z \in \mathbb{R}^d$ is the center of the inscribing sphere, then

$$\|v + \lambda_i e_i - z\| \;=\; \|v - z\|\,.$$

At most one solution $\lambda_i > 0$.



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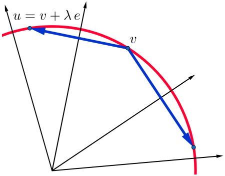

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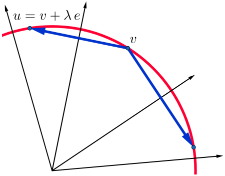

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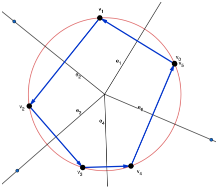

Let's see this!

Virtually inscribable fans

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Dual graph $G(\mathcal{N}) = (\mathcal{N}, E)$
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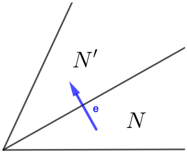
Associate $s_{NN'} : \mathbb{R}^d \to \mathbb{R}^d$ reflection in $e^{\perp}$.



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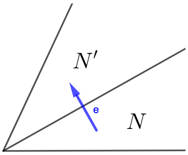

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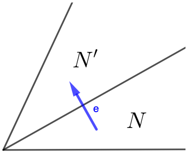

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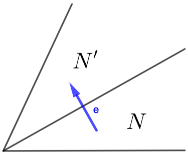

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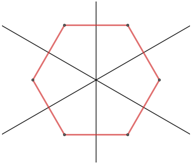
Inscribed zonotopes

Zonotope

$$Z \; = \; [-z_1, z_1] + [-z_2, z_2] + \cdots + [-z_k, z_k]$$

for some $z_1, \ldots, z_k \in \mathbb{R}^d \setminus \{0\}$.



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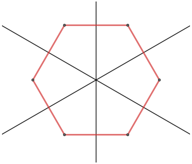

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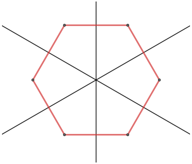

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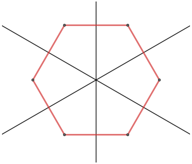

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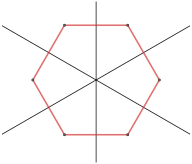

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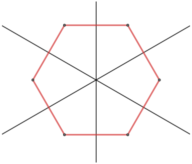
Exercise

Which zonotopes are inscribable?

Zonotopes and hyperplane arrangements

$Z = [-z_1, z_1] + [-z_2, z_2] + \cdots + [-z_k, z_k]$ gives rise to hyperplane arrangement $\mathcal{H} = \{H_i = z_i^\perp : i = 1, \ldots, k\}$



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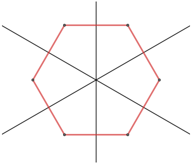

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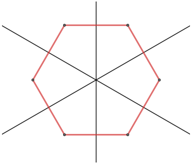

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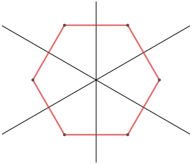

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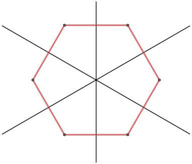

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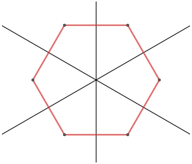

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Zonotopes essentially the only polytopes with that property.

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($\mathcal{N}(Z \times Z') = \mathcal{N}(Z) \oplus \mathcal{N}(Z')$ is reducible.)

Arrangement $\mathcal{H}$ is simplicial if all regions linearly isomorphic to $\mathbb{R}^d_{\geq 0}$.

Theorem

If P is inscribed and $\mathcal{N}(P) = \mathcal{H}$ for some arrangement, then $\mathcal{H}$ is simplicial.

$\longrightarrow$ simplicial arrangements are fascinating but rare!

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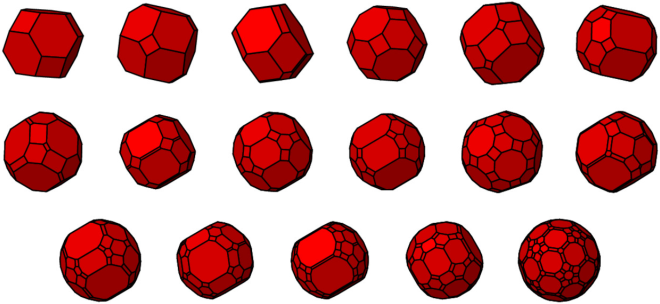

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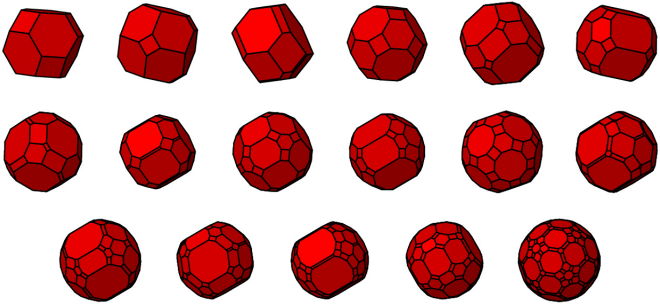
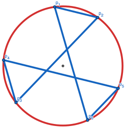


Conjecture: Every inscribed 3-zonotope comes from a restriction.

Inscribed virtual polytopes

How to interpret this?



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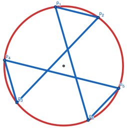

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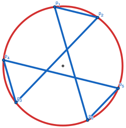

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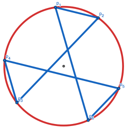

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Polytopes with fixed fan $\mathcal{N}$ form monoid $\mathcal{T}_+$ under Minkowski addition

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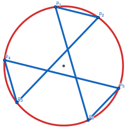

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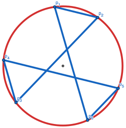

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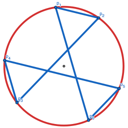

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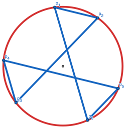

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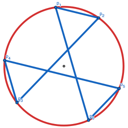

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$\mathcal{T} = \mathcal{T}_+ - \mathcal{T}_+$ but in general $\mathcal{I} \neq \mathcal{I}_+ - \mathcal{I}_+$.

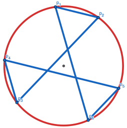
Corollary

If $\mathcal{N}$ is a 2-dim odd normal fan, then $\mathcal{I}(\mathcal{N}) \neq 0$.

Routed particle trajectories

How to interpret this?

Think of closed piecewise-linear trajectory of particle in a ball bouncing off the boundary in a random direction.



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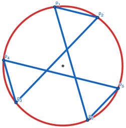

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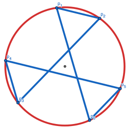

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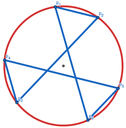

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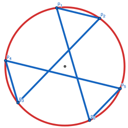

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Every full-dimensional fan $\mathcal{N}$ determines a routing scheme (G, α).

Theorem

Trajectories for (G, α) are precisely the inscribed virtual polytopes $P' \simeq P$.

new connection: inscribed polytopes, PL functions, and particle trajectories!

Routing schemes and reflection groupoids

Groupoid: Small category, every morphism invertible.

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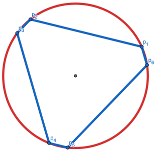
Proposition

If $\mathcal{N}$ is virtually inscribed, then $\hom_R(v_0)$ is generated by finitely many reflections.

Paper on the arXiv: 2012.07724

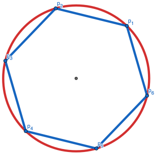
Inscribed normally equivalent polytopes

- rich structure ($\mathcal{I}_+(\mathcal{N})$ open cone)
- effectively computable



Strongly inscribable arrangements

- subclass of simplicial arrangements
- restrictions/localizations of reflection arrangements
- Conjecture: Not more!



Inscribable virtually polytopes

- natural notion, group structure

Routed trajectories and reflection groupoids

- interesting structures — deserve further study!