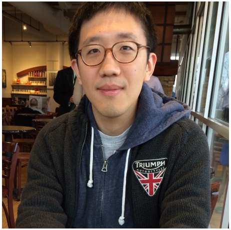
A tale of two polytopes: the bipermutahedron and harmonic polytope

Federico Ardila

San Francisco State University
Universidad de Los Andes

(Polytop)ics
Max Planck Institute, April 9, 2021

Part of Part 1 is with Graham Denham + June Huh (15-20).
Part 2 is joint work with Laura Escobar (20).

Lagrangian geometry of matroids. [ADH20]
`https://arxiv.org/abs/2004.13116`

The harmonic polytope. [AE20]
`https://arxiv.org/abs/2006.03078`

The bipermutahedron. [A20]
`https://arxiv.org/abs/2008.02295`

The plan

0. Why study them? A very short origin story.

1. What is the bipermutahedral fan?
2. What is the bipermutahedron? What do we know about it?

3. What is the harmonic fan?
4. What is the harmonic polytope? What do we know about it?

5. Why study them? A more detailed origin story. (If time.)

A very brief origin story

Given a matroid M of rank r,

f**-vector** = |coeffs| of $\chi_M(q)$ h**-vector** = |coeffs| of $\chi_M(q+1)$

Theorem.

1. [Adiprasito-Huh-Katz '15] $f_0, f_1, \ldots, f_r$ **is log-concave.**
Conjectured by Rota 71, Welsh 71, 76, Heron 72, Mason 72.

2. [Ardila-Denham-Huh '20] $h_0, h_1, \ldots, h_r$ **is log-concave.**
Conjectured by Brylawski 82, Dawson 83, Hibi 89.

A very brief origin story

Given a matroid M of rank r,

f**-vector** = |coeffs| of $\chi_M(q)$ h**-vector** = |coeffs| of $\chi_M(q+1)$

Theorem.

1. [Adiprasito-Huh-Katz '15] $f_0, f_1, \ldots, f_r$ **is log-concave.**
Conjectured by Rota 71, Welsh 71, 76, Heron 72, Mason 72.

2. [Ardila-Denham-Huh '20] $h_0, h_1, \ldots, h_r$ **is log-concave.**
Conjectured by Brylawski 82, Dawson 83, Hibi 89.

[AHK 15]: tropical geom., alg. comb., combin. Hodge th.
ambient space: Bergman fan Σ_M in permutahedral fan Σ_n

[ADH 20]: Lagrangian geom., alg. comb., combin. Hodge th.
ambient: conormal fan $\Sigma_{M,M^\perp}$ in the **bipermutahedral fan** $\Sigma_{n.n}$
(or any simplicial subdivision of the **harmonic fan** $K_{n,n}$)

The permutahedral fan as a moduli space

Permutahedral fan Σ_n in $N_n = \mathbb{R}^n/\mathbb{R}$:

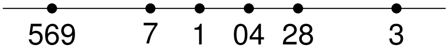
Hyperplane arrangement $x_i = x_j$ for $i \neq j$ in N_n.

The permutahedral fan as a moduli space

Permutahedral fan Σ_n in $N_n = \mathbb{R}^n/\mathbb{R}$:

Hyperplane arrangement $x_i = x_j$ for $i \neq j$ in N_n.

Moduli space: n-tuples of points in $\mathbb{R}$ (mod. common translation)

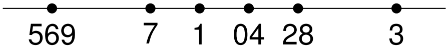


The permutahedral fan as a moduli space

Permutahedral fan Σ_n in $N_n = \mathbb{R}^n/\mathbb{R}$:

Hyperplane arrangement $x_i = x_j$ for $i \neq j$ in N_n.

Moduli space: n-tuples of points in $\mathbb{R}$ (mod. common translation)



Stratification: relative order

Strata: ordered set partitions $3|28|04|1|7|569$

The bipermutahedral fan as a moduli space

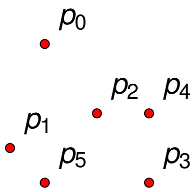
[FA-Denham-Huh 20] **Bipermutahedral fan** $\Sigma_{n,n}$ in $N_n \times N_n$:

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)

The bipermutahedral fan as a moduli space

[FA-Denham-Huh 20] **Bipermutahedral fan** $\Sigma_{n,n}$ in $N_n \times N_n$:

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)



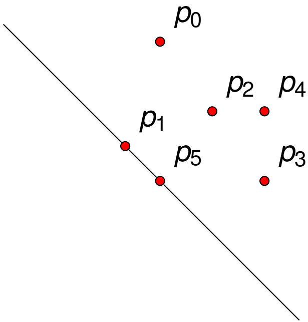
Stratification: • draw lowest supporting -45° diagonal ℓ
• record relative order of x and y projections onto ℓ

Strata: **bisequences** 34|2|035|1|24|0

The bipermutahedral fan as a moduli space

[FA-Denham-Huh 20] **Bipermutahedral fan** $\Sigma_{n,n}$ in $N_n \times N_n$:

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)



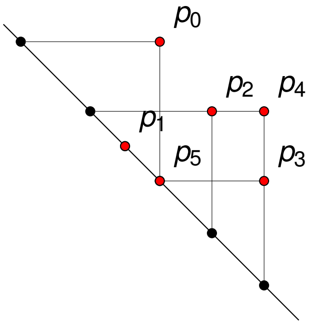
Stratification: • draw lowest supporting -45° diagonal ℓ
• record relative order of x and y projections onto ℓ

Strata: **bisequences** 34|2|035|1|24|0

The bipermutahedral fan as a moduli space

[FA-Denham-Huh 20] **Bipermutahedral fan** $\Sigma_{n,n}$ in $N_n \times N_n$:

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)



Stratification: • draw lowest supporting -45° diagonal ℓ

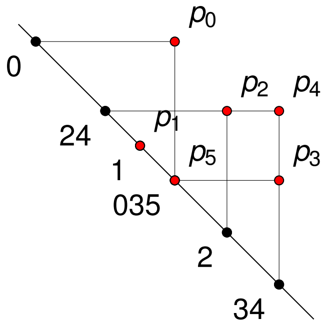
• record relative order of x and y projections onto ℓ

Strata: **bisequences** 34|2|035|1|24|0

The bipermutahedral fan as a moduli space

[FA-Denham-Huh 20] **Bipermutahedral fan** $\Sigma_{n,n}$ in $N_n \times N_n$:

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)

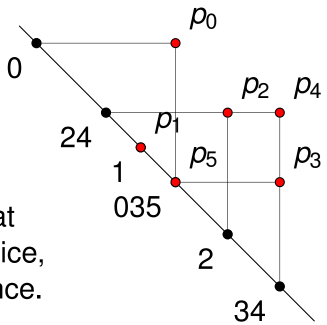


Stratification: • draw lowest supporting -45° diagonal ℓ
• record relative order of x and y projections onto ℓ

Strata: **bisequences** 34|2|035|1|24|0

The bipermutahedral fan as a moduli space

Bipermutahedral fan $\Sigma_{n,n}$ in $N_n \times N_n$:
Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)



Strata: **bisequences** on $[n]$

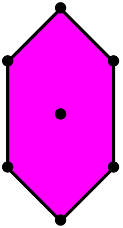
Sequences $\mathcal{B} = B_1|\cdots|B_m$ such that
- each number appears once or twice,
- some number appears exactly once.

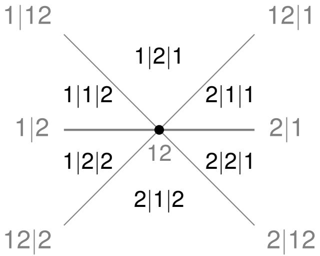
Ex: 34|2|035|1|24|0

The bipermutahedron

Permutahedral fan Σ_n: Normal fan of permutahedron Π_n.

Bipermutahedral fan $\Sigma_{n,n}$: Normal fan of **bipermutahedron** $\Pi_{n,n}$.



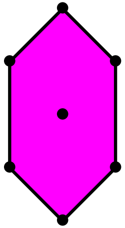


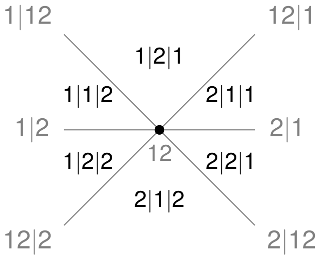
Prop. [FA-Denham-Huh 20, FA 20] The bipermutahedron is

$$\sum_{e\in[n]} x_e = \sum_{e\in[n]} y_e = 0,$$

$$\sum_{s\in S} x_s + \sum_{t\in T} y_t \geq -(|S| + |S - T|)(|T| + |T - S|) \qquad \text{for each } S|T.$$

Combinatorial structure of the bipermutahedron





- faces: **bisequences** 12|45|4|235

- vertices: **bipermutations** 1|5|4|1|3|4|2|5|3. $(2n)!/2^n$
  (one number appears once, others twice)

- facets: **bisubsets** 1245|235 $3^n - 3$
  ($S, T \neq \emptyset$, not both $[n]$, with $S \cup T = [n]$)

The f-vector of the bipermutahedron

Prop. [FA 20] If $f_d(\Sigma_{n,n}) = \#$ of d-dim. faces of $\Sigma_{n,n}$,

$$\sum_{d,n} f_{d-2}(\Sigma_{n,n}) \frac{x^d}{d!} \frac{y^n}{n!} = \frac{F(x, e^y)}{e^x}$$

where

$$F(\alpha, \beta) = \sum_{n \geq 0} \frac{\alpha^n \beta^{\binom{n}{2}}}{n!}$$

is the two variable Rogers-Ramanujan function.

($F(\alpha, \beta)$ also arises in the generating functions for the (arithmetic) Tutte polynomials of classical root systems! (Mphako-Banda 00, FA 02, De Concini-Procesi 08, FA-Castillo-Henley 15) Connection?)

The h-vector of the bipermutahedron

The bipermutahedron is simple; consider its h-polynomial:

$$h_n(x) = h_0(\Pi_{n,n}) + h_1(\Pi_{n,n})x + \cdots + h_{2n-2}(\Pi_{n,n})x^{2n-2}$$

We call it the **biEulerian polynomial**, because

The h-vector of the bipermutahedron

The bipermutahedron is simple; consider its h-polynomial:

$$h_n(x) = h_0(\Pi_{n,n}) + h_1(\Pi_{n,n})x + \cdots + h_{2n-2}(\Pi_{n,n})x^{2n-2}$$

We call it the **biEulerian polynomial**, because

Prop. [FA 20] The h-vector of the bipermutahedron $\Pi_{n,n}$ is

$$h_i(\Pi_{n,n}) = \text{ \# of bipermutations of } [n] \text{ with } i \textbf{ descents.}$$

The *h*-vector of the bipermutahedron

The bipermutahedron is simple; consider its *h*-polynomial:

$$h_n(x) = h_0(\Pi_{n,n}) + h_1(\Pi_{n,n})x + \cdots + h_{2n-2}(\Pi_{n,n})x^{2n-2}$$

We call it the **biEulerian polynomial**, because

Prop. [FA 20] The *h*-vector of the bipermutahedron $\Pi_{n,n}$ is

$$h_i(\Pi_{n,n}) = \text{ \# of bipermutations of } [n] \text{ with } i \textbf{ descents.}$$

Observation: this sequence is log-concave. How to prove it?

The h-vector of the bipermutahedron

Let Δ = standard triangle in $\mathbb{R}^3$.

Prop. [FA 20] There's a unimodular triangulation of $\Delta \times \cdots \times \Delta$ that is combinatorially isomorphic to (the triple cone over) $\Sigma_{n,n}$.

The *h*-vector of the bipermutahedron

Let Δ = standard triangle in $\mathbb{R}^3$.

Prop. [FA 20] There's a unimodular triangulation of $\Delta \times \cdots \times \Delta$ that is combinatorially isomorphic to (the triple cone over) $\Sigma_{n,n}$.

Prop. [FA 20] By Ehrhart theory, the biEulerian polynomial is

$$\frac{h_n(x)}{(1-x)^{2n+1}} = \sum_{k \geq 0} \binom{k+2}{2}^n x^k$$

The *h*-vector of the bipermutahedron

Let Δ = standard triangle in $\mathbb{R}^3$.

Prop. [FA 20] There's a unimodular triangulation of $\Delta \times \cdots \times \Delta$ that is combinatorially isomorphic to (the triple cone over) $\Sigma_{n,n}$.

Prop. [FA 20] By Ehrhart theory, the biEulerian polynomial is

$$\frac{h_n(x)}{(1-x)^{2n+1}} = \sum_{k\geq 0} \binom{k+2}{2}^n x^k$$

Prop. [FA 20] (thanks to Katharina Jochemko!)

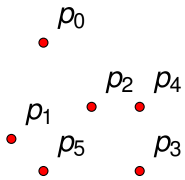
- All roots of the biEulerian polynomial are real and negative.
- The *h*-vector of the bipermutahedron is log-concave.

The harmonic fan

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)

The harmonic fan

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)

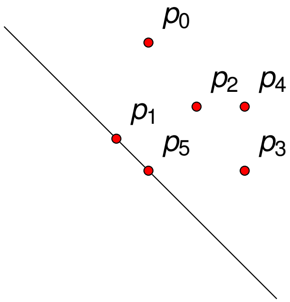


Stratification: • record points on supporting -45° diagonal ℓ
• record relative order of x projections onto ℓ
• record relative order of y projections onto ℓ

Strata: **harmonic triples** $(15; 35|1|24|0, 34|2|05|1)$

The harmonic fan

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)

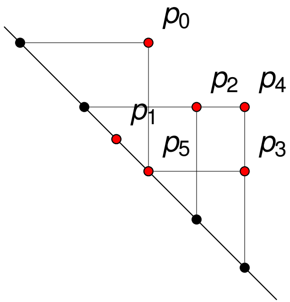


Stratification: • record points on supporting -45° diagonal ℓ
• record relative order of x projections onto ℓ
• record relative order of y projections onto ℓ

Strata: **harmonic triples** $(15; 35|1|24|0, 34|2|05|1)$

The harmonic fan

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)

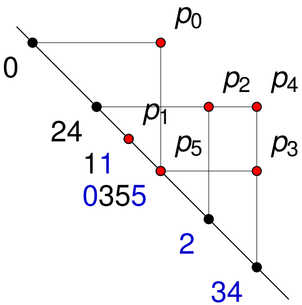


Stratification: • record points on supporting -45° diagonal ℓ
• record relative order of x projections onto ℓ
• record relative order of y projections onto ℓ

Strata: **harmonic triples** $(15; 35|1|24|0, 34|2|05|1)$

The harmonic fan

Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)



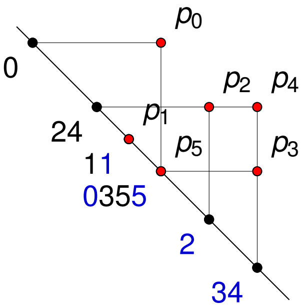
Stratification: • record points on supporting $-45°$ diagonal ℓ

• record relative order of x projections onto ℓ

• record relative order of y projections onto ℓ

The harmonic fan

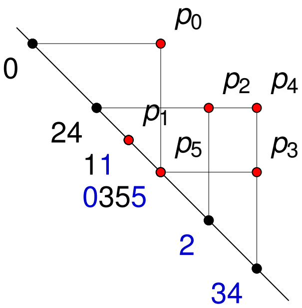
Moduli space: n-tuples of points in $\mathbb{R}^2$ (mod common translation)



Stratification: • record points on supporting -45° diagonal ℓ
• record relative order of x projections onto ℓ
• record relative order of y projections onto ℓ

Strata: **harmonic triples** $(15; 35|1|24|0, 34|2|05|1)$

The bipermutahedral fan refines the harmonic fan.

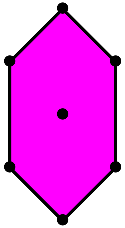


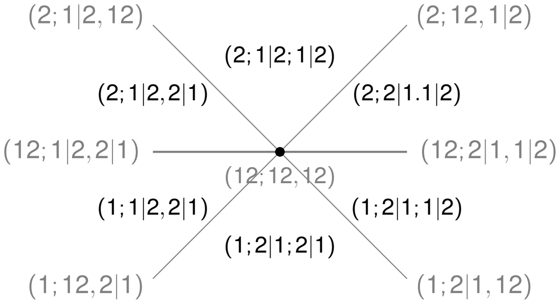
Harmonic fan: **harmonic triple** $(15; 35|1|24|0, 34|2|05|1)$

Bipermut. fan: **bipermutation** $34|2|035|1|24|0$

(The bipermutation determines the harmonic triple.)

The harmonic polytope



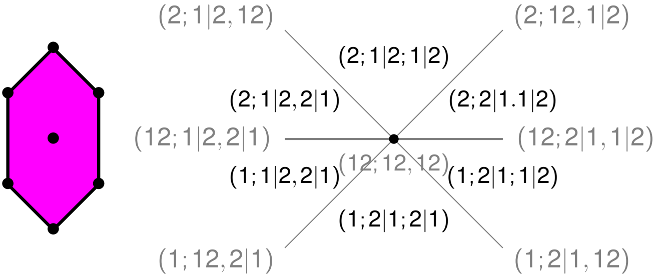


Def./Prop. [FA - Escobar 20] The harmonic polytope is

$$\sum_{e\in[n]} x_e = \sum_{e\in[n]} y_e = \frac{n(n+1)}{2} + 1,$$

$$\sum_{s\in S} x_s + \sum_{t\in T} y_t \geq \frac{|S|(|S|+1) + |T|(|T|+1)}{2} + 1 \qquad \text{for each } S|T.$$

Combinatorial structure of the harmonic polytope



Prop. [FA-Escobar 20] Faces of polytope $\longleftrightarrow$ harmonic triples

- f-vector: we have a formula
- # of facets $= 3^n - 3$
- # of vertices $= (n!)^2 \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$!

Minkowski quotients

Biperm. fan refines harm. fan $\Longleftrightarrow \lambda H_{n,n}$ is a summand of $\Pi_{n,n}$.

Minkowski quotient $P/Q := max\{\lambda \,:\, P = \lambda Q + R \text{ for some } R\}$

Minkowski quotients

Biperm. fan refines harm. fan $\Longleftrightarrow$ $\lambda H_{n,n}$ is a summand of $\Pi_{n,n}$.

Minkowski quotient $P/Q := max\{\lambda \, : \, P = \lambda Q + R$ for some $R\}$

Prop. [FA 20] $\quad \Pi_{n,n}/H_{n,n} = 2$

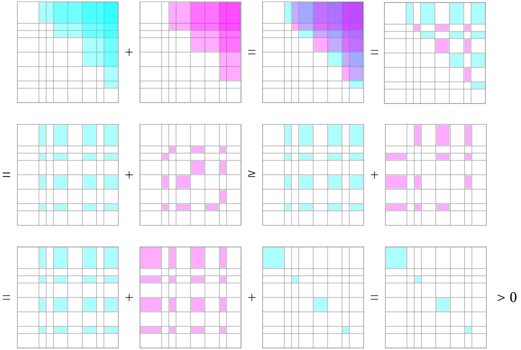
Minkowski quotients

Biperm. fan refines harm. fan $\iff$ $\lambda H_{n,n}$ is a summand of $\Pi_{n,n}$.

Minkowski quotient $P/Q := max\{\lambda \, : \, P = \lambda Q + R \text{ for some } R\}$

Prop. [FA 20] $\quad \Pi_{n,n}/H_{n,n} = 2$

Proof:



Volume

$$\begin{aligned} H_{n,n} &= (\Pi_n \times 0) + (0 \times \Pi_n) + conv(\mathrm{e}_i + \mathrm{f}_i \,:\, 1 \leq i \leq n) \\ &= \sum_{i<j} [\mathrm{e}_i, \mathrm{e}_j] + \sum_{i<j} [\mathrm{f}_i, \mathrm{f}_j] + conv(\mathrm{e}_i + \mathrm{f}_i \,:\, 1 \leq i \leq n) \end{aligned}$$

A sum of (twisted) simplices – almost a gen. permutahedron.

Volume

$$\begin{aligned} H_{n,n} &= (\Pi_n \times 0) + (0 \times \Pi_n) + conv(\mathrm{e}_i + \mathrm{f}_i \, : \, 1 \leq i \leq n) \\ &= \sum_{i<j} [\mathrm{e}_i, \mathrm{e}_j] + \sum_{i<j} [\mathrm{f}_i, \mathrm{f}_j] + conv(\mathrm{e}_i + \mathrm{f}_i \, : \, 1 \leq i \leq n) \end{aligned}$$

A sum of (twisted) simplices – almost a gen. permutahedron.

Theorem. (FA - Escobar 20)

$$Vol(H_{n,n}) = \sum_{\Gamma} \frac{\deg(X_\Gamma)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \deg(v)^{\deg(v)-2}$$

Γ = connected bipartite multigraphs on edges $[n]$
X_Γ = (embedded) toric variety given by toric ideal of Γ

Volume

Theorem. [AE 20] Summing over conn. bip. graphs on edges $[n]$

$$Vol(H_{n,n}) = \sum_{\Gamma} \frac{\deg(X_\Gamma)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \deg(v)^{\deg(v)-2}$$

$\deg(X_\Gamma)$ = deg. of toric variety given by toric ideal of Γ
$= i(P_\Gamma^-) = \#$ of lattice points of trimmed gen. perm. P_Γ^- (Postnikov 05)

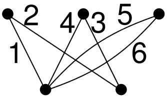
Volume

Theorem. [AE 20] Summing over conn. bip. graphs on edges $[n]$

$$Vol(H_{n,n}) = \sum_{\Gamma} \frac{\deg(X_\Gamma)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \deg(v)^{\deg(v)-2}$$

$\deg(X_\Gamma) =$ deg. of toric variety given by toric ideal of Γ
$= i(P_\Gamma^-) = \#$ of lattice points of trimmed gen. perm. P_Γ^- (Postnikov 05)

$\Gamma =$



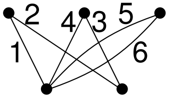
Toric ideal $\langle z_1 z_3 - z_2 z_4, z_5 - z_6 \rangle$ has degree 2.
Polytope $P_\Gamma^- = (\Delta_{abc} + \Delta_{ab}) - \Delta_{abc} = \Delta_{ab}$ has 2 lattice points

Volume

Theorem. [AE 20] Summing over conn. bip. graphs on edges $[n]$

$$Vol(H_{n,n}) = \sum_{\Gamma} \frac{\deg(X_\Gamma)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \deg(v)^{\deg(v)-2}$$

$\deg(X_\Gamma) =$ deg. of toric variety given by toric ideal of Γ
$= i(P_\Gamma^-) = \#$ of lattice points of trimmed gen. perm. P_Γ^- (Postnikov 05)

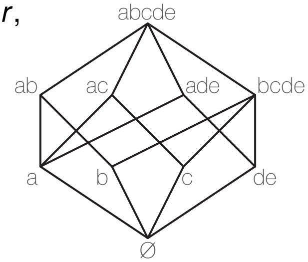
$\Gamma =$ 

Toric ideal $\langle z_1 z_3 - z_2 z_4, z_5 - z_6 \rangle$ has degree 2.
Polytope $P_\Gamma^- = (\Delta_{abc} + \Delta_{ab}) - \Delta_{abc} = \Delta_{ab}$ has 2 lattice points

(This is $\mathrm{MVol}(\mathrm{e}_{12}, \mathrm{e}_{34}, \mathrm{e}_{56}, \mathrm{f}_{14}, \mathrm{f}_{23}, \mathrm{f}_{45}, \mathrm{f}_{56}, D_{123456}, D_{123456}, D_{123456}) = 2$.)

Origin story: Lagrangian geometry of matroids

Given a matroid M on n elements, rank r,



f-vector = |coeffs| of $\chi_M(q)$
h-vector = |coeffs| of $\chi_M(q+1)$

Ex: $n=5$ $r=3$ $f=(1,4,5,2)$ $h=(1,1,0,0)$

Theorem.

1. [Adiprasito-Huh-Katz '15] $f_0, f_1, \ldots, f_r$ **is log-concave.**
Conjectured by Rota 71, Welsh 71, 76, Heron 72, Mason 72.

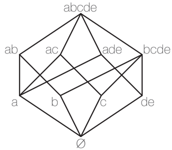
2. [Ardila-Denham-Huh '20] $h_0, h_1, \ldots, h_r$ **is log-concave.**
Conjectured by Brylawski 82, Dawson 83, Hibi 89.

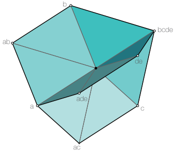
Log-concavity of f-vector: geometry of matroids

[Adiprasito–Huh–Katz 15]

(tropical geometry, alg combinatorics, combin. Hodge theory)

1. Use the **Bergman fan** Σ_M as a geometric model for M.
   $(r-1)$-dim fan in N_n, $\mathrm{Supp}(\Sigma_M) = \mathit{Trop}(M)$ [FA-Klivans 06]





2. Find classes α, β in the Chow ring $A^\bullet(\Sigma_M)$ with

$$deg(\alpha^{r-i}\beta^i) = f_i \qquad (1 \leq i \leq r)$$

3. Prove the Hodge-Riemann relations for the fan Σ_M.
   They imply $(deg(\alpha^{r-i}\beta^i) : 0 \leq i \leq r)$ is log-concave.

Log-conc of h-vector: Lagrangian geom of matroids

[Ardila–Denham–Huh 20]
(Lagrangian geometry, alg combin., combin. Hodge theory)

1. Use the **conormal fan** $\Sigma_{M,M^\perp}$ as a geometric model for M.
   $(n-2)$-dim fan in $N_n \times N_n$

2. Find classes γ, δ in the Chow ring $A^\bullet(\Sigma_{M,M^\perp})$ with

$$deg(\gamma^i \delta^{n-2-i}) = h_{r-i} \qquad (1 \leq i \leq r)$$

3. Prove the Hodge-Riemann relations for the fan $\Sigma_{M,M^\perp}$.
   They imply $(deg(\gamma^i \delta^{n-2-i}) : 0 \leq i \leq r)$ is log-concave.

How to construct the conormal fan $\Sigma_{M,M^\perp}$?

Varchenko's **critical set varieties** offer hints/requirements:

1. Support($\Sigma_{M,M^\perp}$) "should be" $Trop(M) \times Trop(M^\perp)$.
   Tropical analog of conormal bundle.
2. $\Sigma_{M,M^\perp}$ "should be" simplicial, so the Chow ring is tractable.
   Try: $\Sigma_{M,M^\perp} = \Sigma_M \times \Sigma_{M^\perp}$?

How to construct the conormal fan $\Sigma_{M,M^\perp}$?

Varchenko's **critical set varieties** offer hints/requirements:

1. Support($\Sigma_{M,M^\perp}$) “should be" $\mathit{Trop}(M) \times \mathit{Trop}(M^\perp)$.
   Tropical analog of conormal bundle.
2. $\Sigma_{M,M^\perp}$ “should be" simplicial, so the Chow ring is tractable.
   Try: $\Sigma_{M,M^\perp} = \Sigma_M \times \Sigma_{M^\perp}$?
3. There “should be" classes γ and δ with $\gamma^i \delta^{n-2-i} = h_{r-i}$ $(*)$
   - γ “should be" the pullback of α along
     $$\pi : \Sigma_M \times \Sigma_{M^\perp} \to \Sigma_M, \quad \pi(x,y) = x$$
   - δ "should be" the pullback of α along
     $$\sigma : \Sigma_M \times \Sigma_{M^\perp} \to \Delta_n, \quad \sigma(x,y) = x+y$$
     where Δ_n is the normal fan of the standard simplex.
   - Geometry predicts $(*)$, prove it algebro-combinatorially.

How to construct the conormal fan $\Sigma_{M,M^\perp}$?

Varchenko's **critical set varieties** offer hints/requirements:

1. Support($\Sigma_{M,M^\perp}$) "should be" $Trop(M) \times Trop(M^\perp)$.
   Tropical analog of conormal bundle.
2. $\Sigma_{M,M^\perp}$ "should be" simplicial, so the Chow ring is tractable.
   Try: $\Sigma_{M,M^\perp} = \Sigma_M \times \Sigma_{M^\perp}$?
3. There "should be" classes γ and δ with $\gamma^i \delta^{n-2-i} = h_{r-i}$ $(*)$
   - γ "should be" the pullback of α along
     $$\pi : \Sigma_M \times \Sigma_{M^\perp} \to \Sigma_M, \quad \pi(x,y) = x$$
   - δ "should be" the pullback of α along
     $$\sigma : \Sigma_M \times \Sigma_{M^\perp} \to \Delta_n, \quad \sigma(x,y) = x + y$$
     where Δ_n is the normal fan of the standard simplex.
   - Geometry predicts $(*)$, prove it algebro-combinatorially.

Problem: σ is not a map of fans!

How to construct the conormal fan $\Sigma_{M,M^\perp}$?

Problem: $\sigma : \Sigma_M \times \Sigma_{M^\perp} \to \Delta_E$, $\sigma(x,y) = x + y$ not a map of fans!

Solution: Subdivide $\Sigma_M \times \Sigma_{M^\perp}$ so σ is a map of fans. Pero how?

How to construct the conormal fan $\Sigma_{M,M^\perp}$?

Problem: $\sigma : \Sigma_M \times \Sigma_{M^\perp} \to \Delta_E$, $\sigma(x, y) = x + y$ not a map of fans!

Solution: Subdivide $\Sigma_M \times \Sigma_{M^\perp}$ so σ is a map of fans. Pero how?

Idea: Do it simultaneously for all matroids on E.

[FA – Klivans 06]

Permutahedral fan Σ_E resolved this issue for all Bergman fans:

$$\Sigma_M := \mathrm{Trop}(M) \cap \Sigma_E$$

[FA – Denham – Huh 20]

Bipermutahedral fan $\Sigma_{E,E}$ resolves this for all conormal fans:

$$\Sigma_{M,M^\perp} := (\mathrm{Trop}(M) \times \mathrm{Trop}(M^\perp)) \cap \Sigma_{E,E}$$

How to construct the bipermutahedral fan?

As usual, it is a science (harmonic) and an art (bipermutahedral).

How to construct the bipermutahedral fan?

As usual, it is a science (harmonic) and an art (bipermutahedral).

What do we want?

A **nice** complete fan Σ in $N_n \times N_n$ such that:

a. $\pi_1 : \Sigma \to \Sigma_n, \quad \pi(x,y) = x$ is a map of fans

b. $\pi_2 : \Sigma \to \Sigma_n, \quad \pi(x,y) = y$ is a map of fans

c. $\sigma \;: \Sigma \to \Delta_n, \quad \sigma(x,y) = x+y$ is a map of fans

where Σ_n = braid fan and Δ_n = fan of $\mathbb{P}^{n-1}$.

d. It is the normal fan of a polytope.

How to construct the bipermutahedral fan?

As usual, it is a science (harmonic) and an art (bipermutahedral).

What do we want?

A **nice** complete fan Σ in $N_n \times N_n$ such that:

a. $\pi_1 : \Sigma \to \Sigma_n, \quad \pi(x, y) = x$ is a map of fans
b. $\pi_2 : \Sigma \to \Sigma_n, \quad \pi(x, y) = y$ is a map of fans
c. $\sigma \;\; : \Sigma \to \Delta_n, \quad \sigma(x, y) = x + y$ is a map of fans

where Σ_n = braid fan and Δ_n = fan of $\mathbb{P}^{n-1}$.

d. It is the normal fan of a polytope.

Try 1: Σ = coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

How to construct the bipermutahedral fan?

As usual, it is a science (harmonic) and an art (bipermutahedral).

What do we want?
A **nice** complete fan Σ in $N_n \times N_n$ such that:

a. $\pi_1 : \Sigma \to \Sigma_n, \quad \pi(x,y) = x$ is a map of fans
b. $\pi_2 : \Sigma \to \Sigma_n, \quad \pi(x,y) = y$ is a map of fans
c. $\sigma \ : \Sigma \to \Delta_n, \quad \sigma(x,y) = x + y$ is a map of fans

where Σ_n = braid fan and Δ_n = fan of $\mathbb{P}^{n-1}$.

d. It is the normal fan of a polytope.

Try 1: Σ = coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

This is the **harmonic fan/polytope** $H_{n,n}$.

Good news: It has all these properties + beautiful combinatorics.
Bad news: It is not simplicial. How to compute in its Chow ring?

How to construct the bipermutahedral fan?

As usual, it is a science (harmonic) and an art (bipermutahedral).

What do we want?

A **nice** complete fan Σ in $N_n \times N_n$ such that:

a. $\pi_1 : \Sigma \to \Sigma_n, \quad \pi(x,y) = x$ is a map of fans
b. $\pi_2 : \Sigma \to \Sigma_n, \quad \pi(x,y) = y$ is a map of fans
c. $\sigma \; : \Sigma \to \Delta_n, \quad \sigma(x,y) = x + y$ is a map of fans

where Σ_n = braid fan and Δ_n = fan of $\mathbb{P}^{n-1}$.

d. It is the normal fan of a polytope.

Try 1: Σ = coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

This is the **harmonic fan/polytope** $H_{n,n}$.

Good news: It has all these properties + beautiful combinatorics.
Bad news: It is not simplicial. How to compute in its Chow ring?
Note: The harmonic fan is canonical. Any solution must refine it!

How to define the bipermutahedral fan?

We want a **nice, polytopal, simplicial** fan with these properties.

Try 1: $H_{n,n}$ = coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

Try 2: $\Sigma_{n,n}$ = **nice polytopal simplicial** refinement of $H_{n,n}$.

How to define the bipermutahedral fan?

We want a **nice, polytopal, simplicial** fan with these properties.

Try 1: $H_{n,n}$ = coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

Try 2: $\Sigma_{n,n}$ = **nice polytopal simplicial** refinement of $H_{n,n}$.

The **bipermutohedral fan** $\Sigma_{n,n}$ is the nicest one we could find.

How to define the bipermutahedral fan?

We want a **nice, polytopal, simplicial** fan with these properties.

Try 1: $H_{n,n}$ = coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

Try 2: $\Sigma_{n,n}$ = **nice polytopal simplicial** refinement of $H_{n,n}$.

The **bipermutohedral fan** $\Sigma_{n,n}$ is the nicest one we could find.
The **bipermutohedron** $\Pi_{n,n}$ is crucial in [ADH20]!
($\Rightarrow$ nef cone of $\Sigma_{M,M^\perp}$ is non-empty $\Rightarrow$ log-concavity)

To conclude, there is much more (fun!) work to be done:
– Chern-Schwartz-MacPherson classes of matroids
– Lagrangian combinatorics of matroids

muchas gracias

(part 1 of) [ADH20]: `https://arxiv.org/abs/2004.13116`

[AE20]: `https://arxiv.org/abs/2006.03078`

[A20]: `https://arxiv.org/abs/2008.02295`