Convexity and Optimization

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Optimization

The discipline concerned with identifying the best element according to a criterion from a collection of alternatives

Historically, scientific laws formulated as solutions of variational principles

- Principle of least action
- Principle of maximum entropy

Significant engineering applications from the 20th century onwards

- Computation has played a key role

Optimization as a Solution Concept

Question: What does it mean to solve a problem? How to obtain answer given input data?

- Provide a closed-form expression
- Provide an algorithmic procedure

Question: What kinds of algorithmic procedures?

- Solution of linear system
- Solution of eigenvalue problem
- Solution of optimization problem

In many domains, a problem is viewed as 'solved' if it is formulated as a tractable optimization problem

Examples

Portfolio selection in finance

- Given some assets, design investment strategy to maximize return while constraining risk to a user-specified level

Model selection in data science

- Given data and a class of statistical models, identify model that best fits the data

Circuit design in electrical engineering

Supply chain and inventory management in logistics

Truss structure design in mechanical engineering

Questions about an Optimization Problem

A mathematical optimization problem in $\mathbb{R}^n$ may be formulated as:

$$\text{maximize } f(x) \text{ subject to } x \in \mathcal{S}$$

- $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function
- $\mathcal{S} \subset \mathbb{R}^n$ is the constraint set
- $x \in \mathbb{R}^n$ is the decision variable

Questions

- How do we certify that a claimed maximizer is indeed a maximizer?
- What is the geometry associated to the set of maximizers?

Outline

Convex optimization problems as a canonical class of problems

Convex sets

Geometry of the solution set of a convex program

Duality of convex sets

Face structure of convex sets

Canonical Form for an Optimization Problem

The optimization problem

$$\text{maximize } f(x) \text{ subject to } x \in \mathcal{S}$$

may be reformulated as follows

$$\text{maximize } t \text{ subject to } x \in \mathcal{S},\ \ f(x) \geq t$$

Here $(x, t) \in \mathbb{R}^{n+1}$ is the decision variable

Without loss of generality, we may consider problems with linear objective functions

Canonical Form for an Optimization Problem

Definition: Consider a set $\mathcal{S} \subset \mathbb{R}^n$. The *convex hull* of $\mathcal{S}$ is defined as

$$\mathsf{conv}(\mathcal{S}) \triangleq \left\{ \sum_{i=1}^{k} \lambda_i \; x^{(i)} \mid x^{(i)} \in \mathcal{S}, \lambda \in \mathbb{R}^k_+, 1'\lambda = 1 \right\}$$

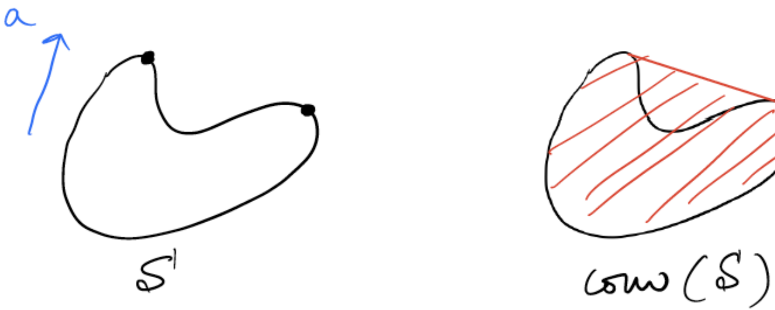
This is the collection of all *convex combinations* of elements of $\mathcal{S}$.

Proposition: Fix a linear functional $a \in (\mathbb{R}^n)^\star$ and a set $\mathcal{S} \subset \mathbb{R}^n$. We have that

$$\sup\{\langle a, x\rangle \mid x \in \mathcal{S}\} = \sup\{\langle a, x\rangle \mid x \in \mathsf{conv}(\mathcal{S})\}$$

(Proof as exercise)

Canonical Form of an Optimization Problem



Canonical Form of an Optimization Problem

Definition: A set that is equal to its convex hull is called a *convex set*. A compact convex set with non-empty interior is called a *convex body*.

Without loss of generality, we may consider optimization problems with linear objective functions and convex constraint sets

- Called convex optimization
- Caveat: need to compute convex hulls efficiently

Our focus is on the geometry of convex optimization

Convex Sets

Convex Optimization

Key object of interest is convex constraint set

We will focus on convex bodies to ensure that maximizers exist

Recall earlier questions

- How do we certify that a claimed maximizer is indeed a maximizer?
- What is the geometry associated to the set of maximizers?

We will address these in the context of convex optimization

Certifying Optimality

Consider optimization problem

$$\text{maximize } \langle a, x\rangle \text{ subject to } x \in \mathcal{C}$$

with $\mathcal{C} \subset \mathbb{R}^n$ a convex body and $a \in (\mathbb{R}^n)^\star$

A point $\hat{x} \in \mathcal{C}$ is a maximizer if

$$\langle a, x\rangle \leq \langle a, \hat{x}\rangle \quad \forall x \in \mathcal{C}$$

Definition: Consider a convex set $\mathcal{C} \subset \mathbb{R}^n$ and let $y \in \mathcal{C}$. The *normal cone* at y with respect to $\mathcal{C}$ is defined as

$$\mathcal{N}_{\mathcal{C}}(y) \triangleq \{w \in (\mathbb{R}^n)^\star \mid \langle w, z\rangle \leq \langle w, y\rangle,\ \forall z \in \mathcal{C}\}$$

Certifying Optimality

Going back to our optimization problem

$$\text{maximize } \langle a, x \rangle \text{ subject to } x \in \mathcal{C}$$

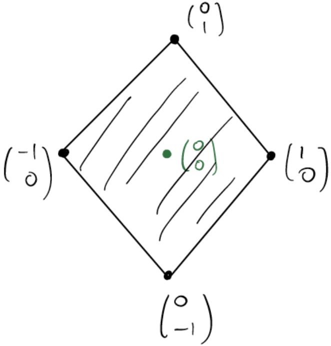
with $\mathcal{C} \subset \mathbb{R}^n$ a convex body and $a \in (\mathbb{R}^n)^\star$

A point $\hat{x} \in \mathcal{C}$ is a maximizer if

$$a \in \mathcal{N}_{\mathcal{C}}(\hat{x})$$

Follows from definitions of normal cone and of a maximizer

Certifying Optimality



Linear functionals that attain maximum at $\begin{pmatrix}1\\0\end{pmatrix}$ are $\{a \mid a_1 \geq a_2, a_1 \geq -a_2\}$

What do Optimal Solution Sets Look Like?

Consider again the optimization problem

$$\text{maximize } \langle a, x\rangle \text{ subject to } x \in \mathcal{C}$$

with $\mathcal{C} \subset \mathbb{R}^n$ a convex body and $a \in (\mathbb{R}^n)^\star$

If $\hat{x}, \tilde{x} \in \mathcal{C}$ are both optimal solutions, then so is any point on the line segment connecting $\hat{x}, \tilde{x}$

More generally, solution set is of the form

$$\{x \mid \langle a, x\rangle = v\} \cap \mathcal{C}$$

Question: Can we say more?

Faces of a Convex Set

Definition: Let $\mathcal{C} \subset \mathbb{R}^n$ be a convex set. A subset $\mathcal{F} \subset \mathcal{C}$ is a *face* of $\mathcal{C}$ if no point in $\mathcal{F}$ can be expressed as a convex combination of points in $\mathcal{C} \backslash \mathcal{F}$.

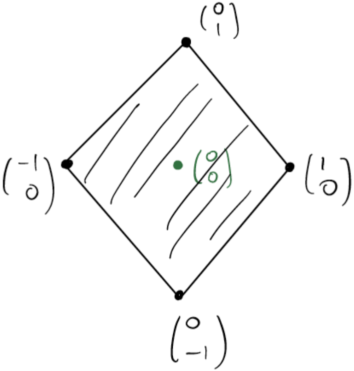
Some related definitions

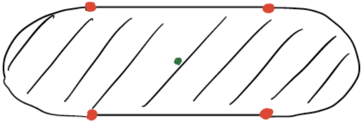
- An element x of a convex set $\mathcal{C}$ is an *extreme point* if $\{x\}$ is a face of $\mathcal{C}$
- A face $\mathcal{F}$ of a convex set $\mathcal{C}$ is an *exposed face* of $\mathcal{C}$ if $\mathcal{F}$ can be expressed as

$$\mathcal{F} = \mathcal{C} \cap \{x \mid \langle u, x \rangle = c\}$$

  for $u \in (\mathbb{R}^n)^\star, \ c \in \mathbb{R}$

Faces of a Convex Set





Solutions Sets of Convex Programs

Consider optimization problem

$$\text{maximize } \langle a, x \rangle \text{ subject to } x \in \mathcal{C}$$

with $\mathcal{C} \subset \mathbb{R}^n$ a convex body and $a \in (\mathbb{R}^n)^\star$

Proposition: The collection of maximizers of the above optimization problem is an exposed face of $\mathcal{C}$.
(Proof as exercise)

Question: How do we connect this proposition with the previous one based on normal cones?

Duality and Convexity

Dual of a Convex Body

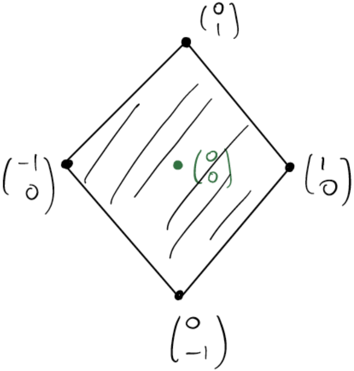
Definition: Consider a convex body $\mathcal{C} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{C})$. The *dual* of $\mathcal{C}$ is denoted $\mathcal{C}^\circ$ and it is defined as:

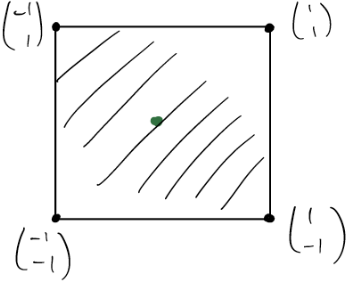
$$\mathcal{C}^\circ \triangleq \{y \mid \langle y, x \rangle \leq 1, \ \forall x \in \mathcal{C}\}$$

This is the collection of linear functionals with a supremum of at most 1 over $\mathcal{C}$

One can check that $\mathcal{C}^\circ$ is a convex body with $0 \in \text{int}(\mathcal{C}^\circ)$

Dual of a Convex Body





Dual of a Convex Body

Dual of a Convex Body

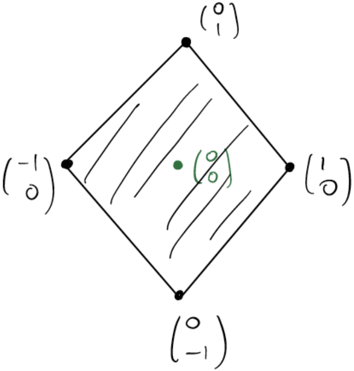
Proposition: Consider a convex body $\mathcal{C} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{C})$. The exposed faces of $\mathcal{C}$ and of $\mathcal{C}^\circ$ are in one-to-one correspondence – for any exposed face $\mathcal{F}$ on the boundary of $\mathcal{C}$, the collection of linear functionals on the boundary of $\mathcal{C}^\circ$ that attain their optimum at $\mathcal{F}$ constitute an exposed face of $\mathcal{C}^\circ$.

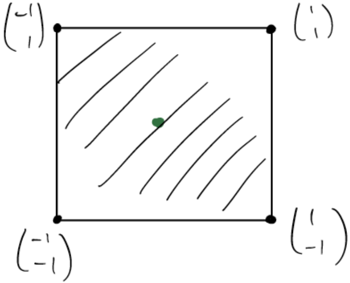
Such pairs of exposed faces are sometimes called *conjugate faces*

Normal cones of $\mathcal{C}$ are exposed faces of $\mathcal{C}^\circ$ (restricted to the boundary of $\mathcal{C}^\circ$)

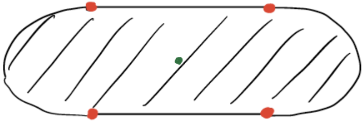
Optimal solution sets and certificates of optimality arise as pairs of conjugate faces

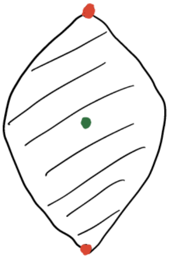
Dual of a Convex Body





Dual of a Convex Body





Facial and Boundary Structure of Convex Bodies

Combinatorial Properties

Proposition: Consider a convex body $\mathcal{C}$. The faces (resp. exposed faces) constitute a lattice with the empty set being the global minimum and $\mathcal{C}$ being the global maximum

- Partial order given by containment
- Join of $\mathcal{F}_1, \mathcal{F}_2$ given by smallest (exposed) face containing $\mathcal{F}_1, \mathcal{F}_2$
- Meet of $\mathcal{F}_1, \mathcal{F}_2$ given by $\mathcal{F}_1 \cap \mathcal{F}_2$

Rich duality between face lattice of $\mathcal{C}$ and of $\mathcal{C}^\circ$

To say more, need to consider specific families of convex bodies

Polyhedra

A *polyhedron* is a finite intersection of halfspaces

$$\mathcal{P} = \{x \mid Ax \leq b\}$$

Here $A : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map and $b \in \mathbb{R}^k$

The problem of optimizing a linear functional over a polyhedron is a *linear program*

Central both to the study of convex geometry and to optimization

Polyhedra

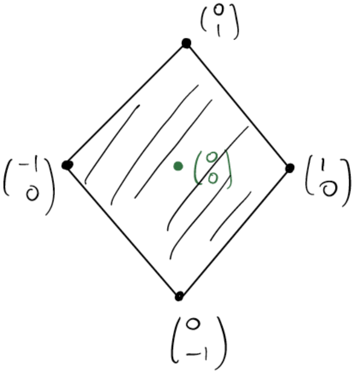
Number of special properties

- Finite face lattice
- All faces are exposed faces
- Projections of polyhedra are polyhedra
- Duals of polyhedra are polyhedra

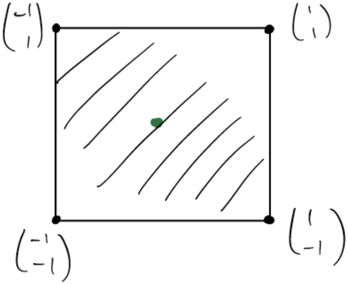
A *polytope* is the convex hull of a finite set

Weyl-Minkowski Theorem: A convex set if a polytope if and only if it is a bounded polyhedron

Polyhedra



$\begin{pmatrix}0\\1\end{pmatrix}$
$\begin{pmatrix}-1\\0\end{pmatrix}$
$\begin{pmatrix}0\\0\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix}$
$\begin{pmatrix}0\\-1\end{pmatrix}$



$\begin{pmatrix}-1\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\end{pmatrix}$
$\begin{pmatrix}-1\\-1\end{pmatrix}$
$\begin{pmatrix}1\\-1\end{pmatrix}$

Spectrahedra

A *spectrahedron* is the convex set defined by a linear matrix inequality

$$\mathcal{S} = \{x \mid \mathcal{A}(x) \preceq B\}$$

for a linear map $\mathcal{A} : \mathbb{R}^n \to \mathbb{S}^k$ and $B \in \mathbb{S}^k$

The problem of optimizing a linear functional over a spectrahedron is a *semidefinite program*

If the image of $\mathcal{A}$ is a subset of diagonal matrices and B is diagonal, then reduce to polyhedra and linear programming

Spectrahedra

Spetrahedra are convex, closed, basic semialgebraic sets

- Cut out by intersection of finitely many polynomial inequalities

Much less known about boundary structure of spectrahedra

- Duals of spectrahedra are not necessarily spectrahedra
- Projected spectrahedra are not spectrahedra, but are still convex and semialgebraic (union of basic semialgebraic sets)
- Faces of spectrahedra are exposed but those of projected spectrahedra are not in general

Spectrahedra

Summary

Goal of this course is to discuss properties of the faces and boundary of convex bodies

Progress on particular families of suitably structured convex bodies

Many open research questions

- Historically, most progress in the polyhedral case
- Much remains to be understood more generally

Subsequent lectures will focus on combinatorial, algebraic, geometric aspects of face structure of convex sets