Non-polyhedral Convex Sets

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MPI MIS Leipzig

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MAX-PLANCK-GESELLSCHAFT

Overview

1. Spectrahedra
2. Projected Spectrahedra
3. Many more convex objects
4. Terracini Convexity

Semialgebraic Sets

A basic closed semialgebraic set is

$$\{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$$

for polynomials $g_i \in \mathbb{R}[x_1, \ldots, x_n]$.

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Tarski-Seidenberg Theorem

The projection of a semialgebraic set is semialgebraic.

Spectrahedra

A set $S \subset \mathbb{R}^n$ is a spectrahedron if

$$S = \{x \in \mathbb{R}^n : A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0\}$$

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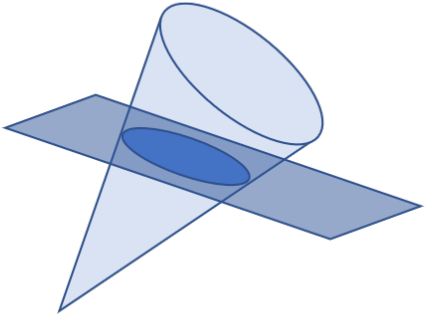

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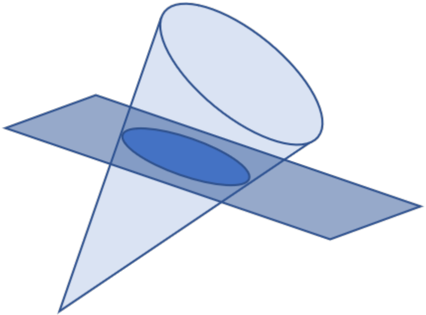

PSD cone $\cap$ an affine subspace of symmetric matrices.

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$$\begin{aligned} x+2 &\geq 0 \\ 3(x+2) - y^2 &\geq 0 \\ 15 - (-x+1)^2 &\geq 0 \\ \det(A(x)) &\geq 0 \end{aligned}$$

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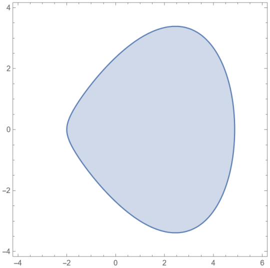

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- All faces are exposed (Exercise)
- Rigidly Convex

Definition

A polynomial p is a real zero polynomial at $u \in \mathbb{R}^n$ if $p(u) > 0$ and for every nonzero $w \in \mathbb{R}^n$ the complex zeros of the univariate polynomial $p(u + tw) \in \mathbb{R}[t]$ are all real.

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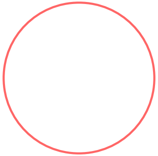
Example

Let $p = 1 - x_1^2 - x_2^2$ and $u = (0, 0)$.



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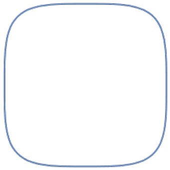

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where $W = \sum A_i w_i$.

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$$p(tw) = \det(I + tW)$$

where $W = \sum A_iw_i$. Since W is symmetric, the zeros of this polynomial are all real, hence p is a real zero polynomial at the origin.

Definition

A subset $C \subseteq \mathbb{R}^n$ is called rigidly convex if there is a point $u \in \mathbb{R}^n$ and a polynomial p which is a real zero polynomial at u such that C equals the closure of the connected component of $\{x \in \mathbb{R}^n : p(x) > 0\}$ at u.

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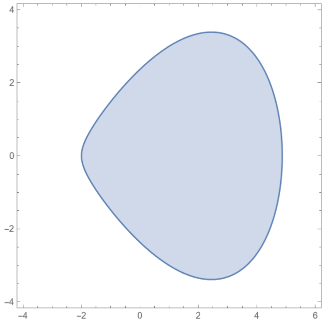

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Facts

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An algebraic interior is rigidly convex if and only if its minimal polynomial is a real zero polynomial at any of its interior points.

Rigid Convexity

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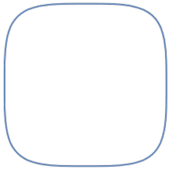

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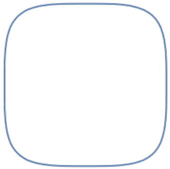
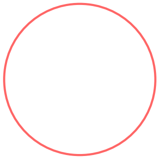


versus the disk $f = 1 - x_1^2 - x_2^2$



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Proof outline of converse

If $S \subset \mathbb{R}^2$ is rigidly convex, then it is an algebraic interior. Let $p(x_1, x_2)$ be its minimal polynomial of degree d.

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where A_1, A_2 are $d \times d$ symmetric matrices. Therefore

$$S = \{(x_1, x_2) : I + A_1 x_1 + A_2 x_2 \succeq 0\}$$

is a spectrahedron.

Symmetric Determinantal Representations

A polynomial $p \in \mathbb{R}[x]$ has a symmetric determinantal representation if $p(x) = \det A(x)$ for some

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Counterexample

$p = (1 + x_1^2)^2 - x_2^2 - \cdots - x_n^2$ for $n \geq 4$

Summary

S is a spectrahedron

S is rigidly convex

S is an algebraic interior

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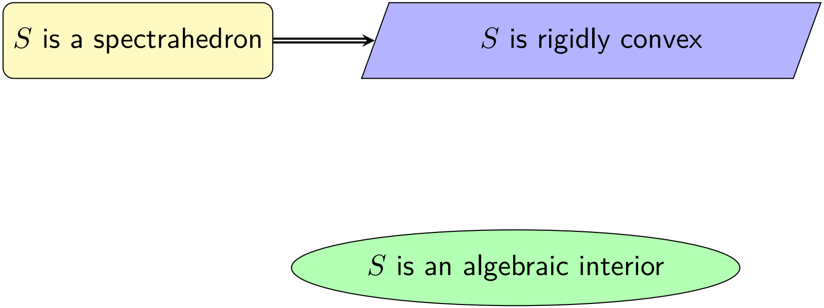
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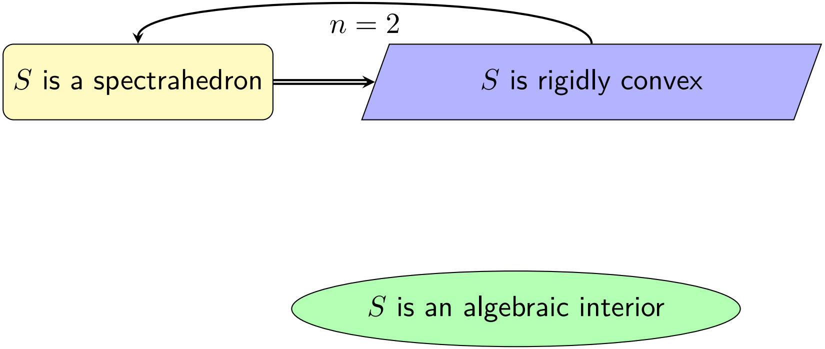

$n = 2$
S is a spectrahedron
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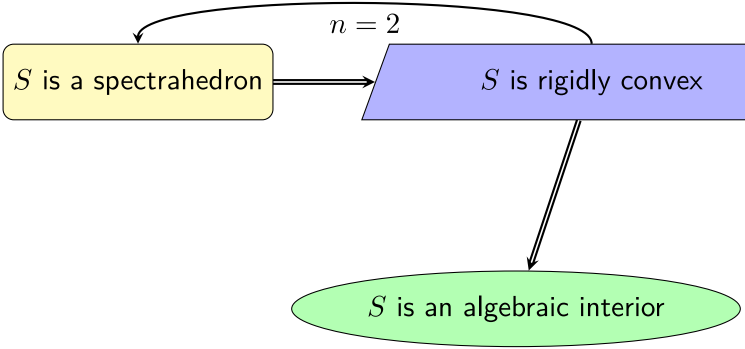

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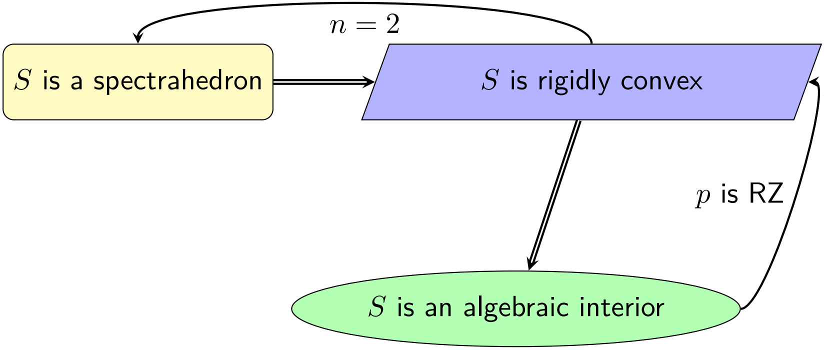

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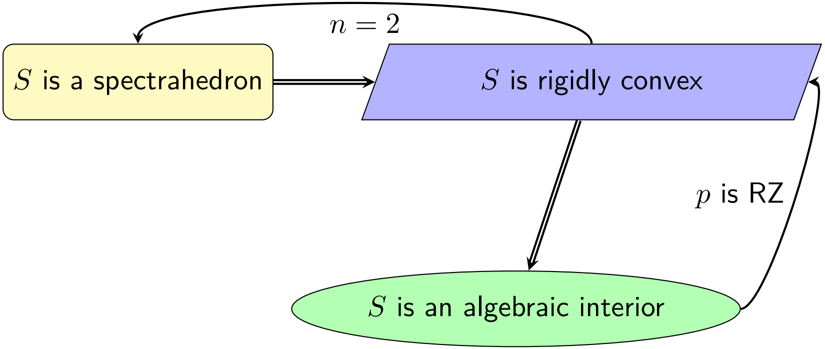

Open Question

Is every rigidly convex algebraic interior of $\mathbb{R}^n$ a spectrahedron?

Spectraplex

The set of PSD matrices with trace one:

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Elliptope

The set of PSD matrices with ones on the diagonal:

$$\{X \in \mathcal{S}^n : X \succeq 0, X_{ii} = 1, i = 1, \ldots, n\}$$

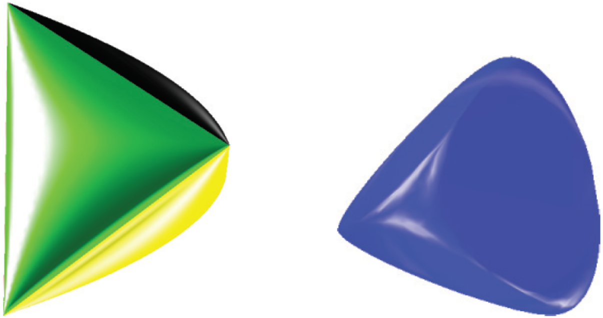

Figure 5.8. *The elliptope* $P = \mathcal{E}_3$ *and its dual convex body* P°.

Spectrahedra

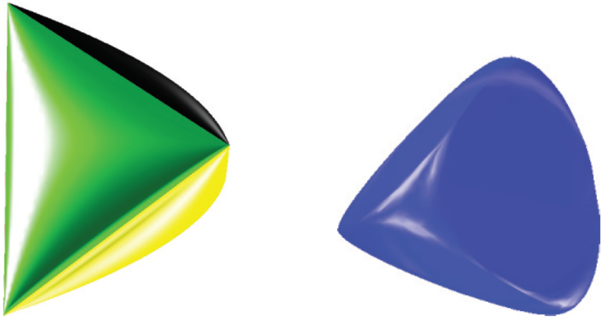

Figure 5.8. *The elliptope $P = \mathcal{E}_3$ and its dual convex body P°.*

The dual body of the elliptope is a *projected spectrahedron*.

Projected Spectrahedra

Consider the spectrahedron

$$\begin{pmatrix} 1+x & y & 0 & 0 \\ y & 1-x & 0 & 0 \\ 0 & 0 & 1+z & 0 \\ 0 & 0 & 0 & 1-z \end{pmatrix} \succeq 0$$

Projected Spectrahedra

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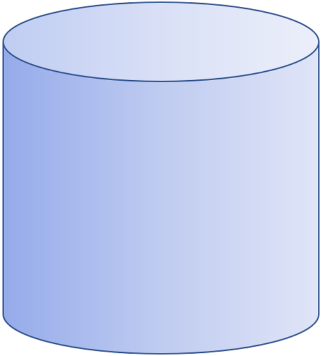

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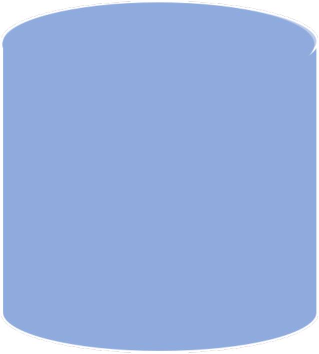

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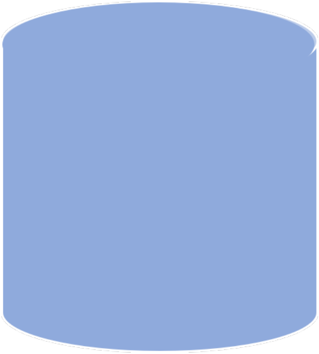

Not a spectrahedron!

Definition

A set $P \subset \mathbb{R}^n$ is a projected spectrahedron if there exists a spectrahedron $S \in \mathbb{R}^{n+k}$ such that

$$P = \{x \in \mathbb{R}^n : (x, y) \in S \text{ for some } y \in \mathbb{R}^k\}.$$

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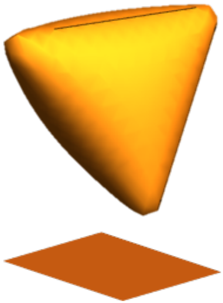

Projected Spectrahedra

Some examples

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- TV Screen: $\{(x_1, x_2) : x_1^4 + x_2^4 \leq 1\}$
- For $X \in \mathcal{S}^n$, let $s_k(X)$ be the sum of the k largest eigenvalues. Then

$$\{(X,t) \in \mathcal{S}^n \times \mathbb{R} : s_k(X) \leq t\}.$$

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Projected spectrahedra are closed under projection, duality, and convex hull of finite unions.

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Examples

- Permutahedron, S_n-orbitope
- Sphere, $SO(n)$-orbitope

Orbitopes are convex semialgebraic sets and some are spectrahedra or projected spectrahedra,

Orbitopes

An orbitope is the convex hull of the orbit of an element v in a real representation V of a compact group G,

$$\text{conv}(G \cdot v) = \{g \cdot v : g \in G\} \subset V$$

Examples

- Permutahedron, S_n-orbitope
- Sphere, $SO(n)$-orbitope

Orbitopes are convex semialgebraic sets and some are spectrahedra or projected spectrahedra, but there are orbitopes which are not projected spectrahedra!

Want to learn more about orbitopes?

Raman Sanyal, Frank Sottile, and Bernd Sturmfels,
Orbitopes,
Mathematatika. 2011, pp. 275-314.

Tim Kobert,
Spectrahedral and semidefinite representability of orbitopes,
PhD thesis (Universität Konstanz, 2019)

Definition

Let $K, L \subset \mathbb{R}^n$ be compact sets. The Hausdorff distance of $K, L \subset \mathbb{R}^n$ is

$$\delta(K, L) := \max\{\sup_{x \in K} \inf_{y \in L} ||x - y||, \sup_{x \in L} \inf_{y \in K} ||x - y||\}.$$

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Hausdorff metric

- The metric space (C^n, δ) is complete and every bounded sequence has a convergent subsequence.

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Hausdorff metric

- The metric space (C^n, δ) is complete and every bounded sequence has a convergent subsequence.
- Every bounded sequence of convex bodies has a subsequence that converges to a convex body.

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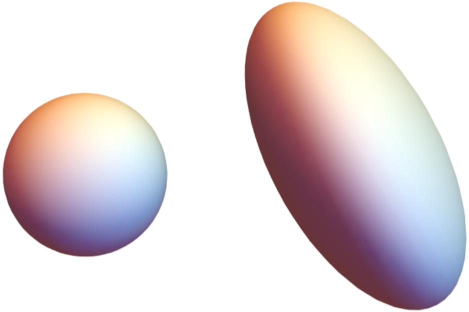

Let K be a convex body in $\mathbb{R}^n$. The projection body ΠK of K is the centered convex body defined by the support function,

$$h_{\Pi K}(u) = \mathsf{Vol}_{n-1}(K|u^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| dS(K, u)$$

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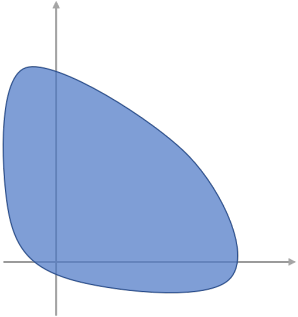

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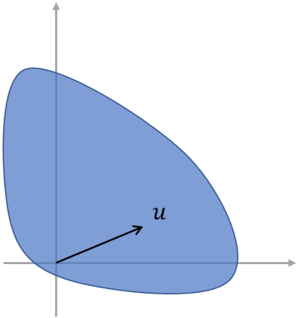

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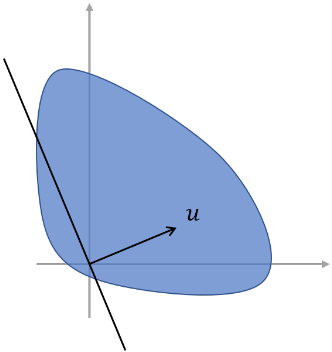

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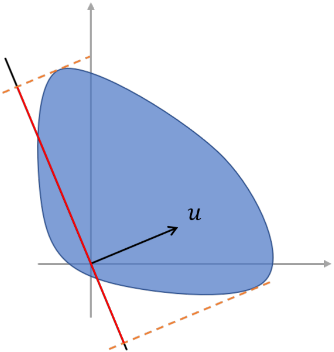

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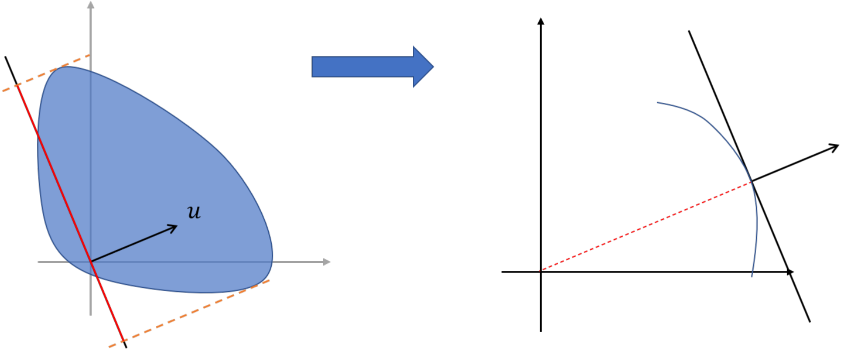

Theorem

A projection body is a centered zonoid. Conversely, every centered full dimensional zonoid in $\mathbb{R}^n$ is the projection body of a unique centered convex body.

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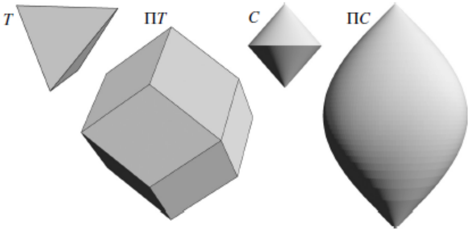

Figure 4.1. Projection bodies.

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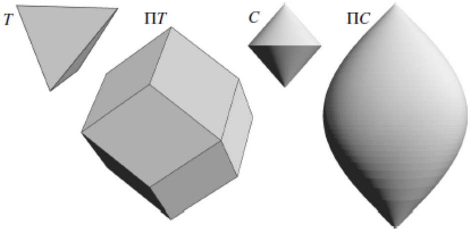

Figure 4.1. Projection bodies.

Theorem

A convex body in $\mathbb{R}^n$ is a zonoid if and only if it is a Hausdorff limit of finite Minkowski sums of n-dimensional ellipsoids.

Want to learn more about projection bodies and zonoids?

Richard Harding Gardner,
Geometric Tomography,
Cambridge University Press, New York. 2006

Rolf Schneider,
Convex bodies: the Brunn-Minkowski theory,
Cambridge University Press, Cambridge. 2014

Many more convex objects

Intersection Bodies

Ask Marie and Katalin

Many more convex objects

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Minkowski Sums of Disks

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Many more convex objects

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Minkowski Sums of Disks

Ask Chiara

Convex Hulls in Buildings

Ask Mima and Marvin

Combinatorics of convex sets

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What other combinatorial properties of polytopes can be generalized to other convex sets and how?

$\rightarrow$ One answer for neighborliness of convex cones is via Terracini convexity.

Definition

A subset $C \subset R^n$ is a cone if for all $\lambda \geq 0$ and $x \in C$, $\lambda x \in C$.

Terracini Convexity

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- A set $\mathcal{F} \subseteq C$ is a *face* of C if for $x, y \in C$, $x + y \in \mathcal{F}$ implies that $x, y \in \mathcal{F}$.
- We say that a cone is pointed if it contains no lines, $C \cap -C = \{0\}$.

Definition

A polytope is called k-neighborly if every set of k or fewer vertices forms a face.

Definition

A pointed polyhedral cone is called k*-neighborly if every set of* k *or fewer extreme rays forms a face.*

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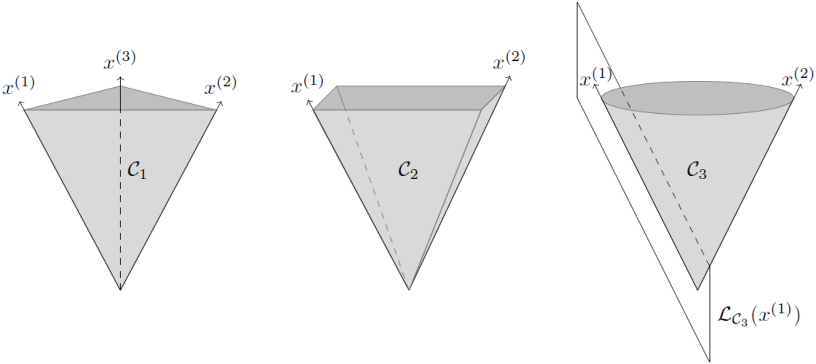

Let $C \subset \mathbb{R}^n$ be a closed, pointed, convex cone.
Define $\mathcal{K}_C(x) = \text{cone}\{z - x : z \in C\}$ to be the cone of feasible directions into C at $x \in C$.

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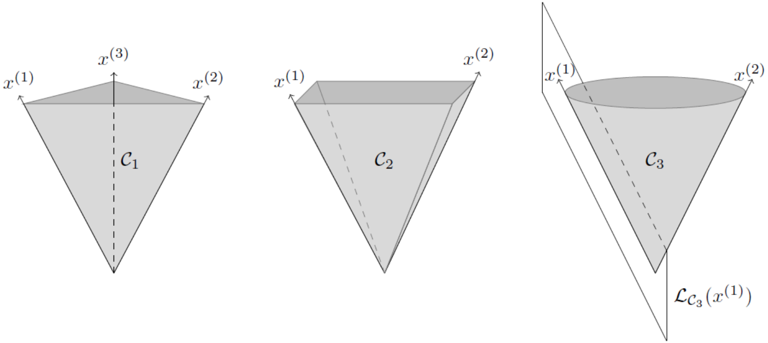
Definition

A closed, pointed, convex cone C is k-Terracini convex if for any collection $x^{(1)}, \ldots, x^{(k)}$ of generators of extreme rays of C,

$$\mathcal{L}_C(\sum_{i=1}^{k} x^{(i)}) = \sum_{i=1}^{k} \mathcal{L}_C(x^{(i)})$$

If C is k-Terracini convex for all k, then C is Terracini convex.

Terracini Convexity



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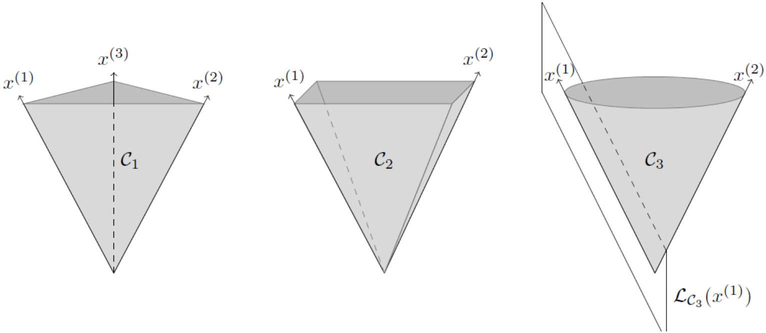

Examples

- A pointed k-neighborly polyhedral cone is k-Terracini convex.
- The nonnegative orthant is Terracini convex
- Any smooth pointed convex cone is Terracini convex.
- PSD cone is Terracini convex

Terracini Convexity

The normal cone to a convex cone C and x is

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Proposition

A closed, pointed, convex cone $C \subset \mathbb{R}^n$ is k-Terracini convex if and only if for any collection $x^{(1)}, \dots, x^{(k)}$ of generators of extreme rays of C,

$$\mathit{span}\left(\bigcap_{i=1}^{k} \mathcal{N}_C(x^{(i)})\right) = \bigcap_{i=1}^{k} \mathit{span}(\mathcal{N}_C(x^{(i)})).$$

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$\longrightarrow$ PSD cone is Terracini convex (Exercise)

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then this collection is a chain of faces. The height of the poset of faces of C, denoted $\mathcal{H}(C)$, is the length of the longest chain.

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$$\dim(C)\text{-Terracini convex} \implies \text{Terracini convex.}$$

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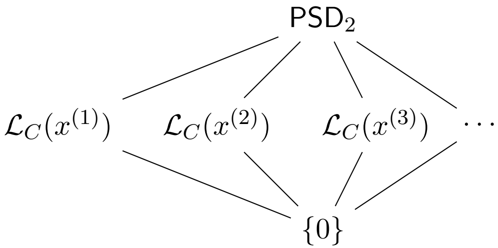

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Definition

- *A linear map* $A : \mathbb{R}^d \to \mathbb{R}^n$ *satisfies the exact recovery property if, for any* $x^* \in \mathbb{R}^d_+$ *with* $|\textit{support}(x^*)| \leq k$*, the unique optimal solution of the linear programming problem* $\min\{x_1 + \cdots + x_n : Ax = Ax^*, x \geq 0\}$ *is* x^**.*

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- *Consider a linear map $B : \mathbb{R}^d \to \mathbb{R}^N$. The cone $B(\mathbb{R}^d_+)$ satisfies the Terracini convexity property if it is pointed, it has d extreme rays, and it is k-Terracini convex.*

Theorem

Consider a linear map $A : \mathbb{R}^d \to \mathbb{R}^n$ *that is surjective and define the linear map* $B : \mathbb{R}^d \to \mathbb{R}^{n+1}$ *as*

$$Bx = \begin{pmatrix} Ax \\ x_1 + \cdots + x_d \end{pmatrix}.$$

Suppose that null$(A) \cap \mathbb{R}^n_{++} \neq \emptyset$*. Fix a positive integer* $k < d$*. The map* A *satisfies the exact recovery property if and only if the cone* $B(\mathbb{R}^d_+)$ *satisfies the Terracini convexity property.*

Proof.

Exact recovery property $\iff$ unique preimage property $\iff$ Terracini convexity property. □

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Terracini Convexity

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- *A linear map* $\mathcal{A} : \mathcal{S}^d \to \mathbb{R}^n$ *satisfies the exact recovery property if for any* $X^* \in \mathcal{S}^d_+$ *with rank*$(X^*) \leq k$*, the unique optimal solution of the semidefinite programming problem* $\min\{\text{trace}(X) : \mathcal{A}(X) = \mathcal{A}(X^*), X \succeq 0\}$ *is* X^**.*

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- *Consider a linear map $\mathcal{B} : \mathcal{S}^d \to \mathbb{R}^N$. The cone $\mathcal{B}(\mathcal{S}^d_+)$ satisfies the Terracini convexity property if it is closed and pointed, its extreme rays are in one-to-one correspondence with those of $\mathcal{S}^d_+$, and it is k-Terracini convex.*

Theorem

Consider a linear map $\mathcal{A} : \mathcal{S}^d \to \mathbb{R}^n$ and fix a positive integer $k < d$.

1. *Suppose $\mathcal{A}$ is surjective and null$(\mathcal{A}) \cap \mathcal{S}^n_{++} \neq \emptyset$. Define $\mathcal{B} : \mathcal{S}^d \to \mathbb{R}^{n+1}$ as $\mathcal{B}(X) = \begin{pmatrix} \mathcal{A}(X) \\ \text{trace}(X) \end{pmatrix}$. If $\mathcal{A}$ satisfies the exact recovery property, then $\mathcal{B}(\mathcal{S}^d_+)$ satisfies the Terracini convexity property.*
2. *Let $n > \binom{d+1}{2} - \binom{d-k+1}{2}$. Suppose there exists an open set $\mathfrak{S}$ in the space of linear maps from $\mathcal{S}^d$ to $\mathbb{R}^n$ with the following properties:*
   - *$\mathcal{A} \in \mathfrak{S}$.*
   - *For each $\tilde{\mathcal{A}} \in \mathfrak{S}$, $\tilde{\mathcal{A}}$ is surjective and $null(\tilde{\mathcal{A}}) \cap \mathcal{S}^d_{++} \neq \emptyset$.*
   - *For each $\tilde{\mathcal{A}} \in \mathfrak{S}$ with associated $\tilde{\mathcal{B}}$, the cone $\tilde{\mathcal{B}}(\mathcal{S}^d_+)$ satisfies the Terracini convexity property.*

   Then the map $\mathcal{A}$ satisfies the exact recovery property.

Terracini Convexity

Venkat Chandrasekaran and James Saunderson,
Terracini Convexity,
Preprint: arxiv.org/abs/2010.00805. 2020

Neighborliness of convex cones $\rightarrow$ Terracini Convexity.

Neighborliness of convex cones $\to$ Terracini Convexity.

Exercise

Pick up Convex Polytopes by Grünbaum or Lectures on Polytopes by Ziegler. For each chapter/section/subsection/theorem/property, what is a generalization to non-polyhedral convex sets?

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Thank you!