Algebra and Convexity

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Minicourse on Convex Geometry
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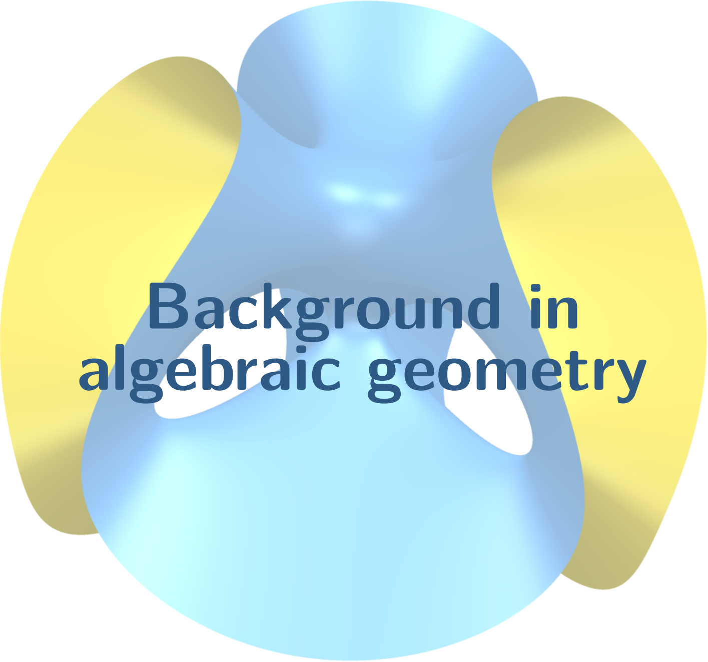
MAX-PLANCK-GESELLSCHAFT



Background in algebraic geometry

Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal.

Definition

The variety associated to I is the set

$$\mathcal{V}(I) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid f(x_1, \dots, x_n) = 0 \, \forall f \in I\}.$$

If $I = \langle f_1, \dots, f_k \rangle$ then we will also write $\mathcal{V}(I) = \mathcal{V}(f_1, \dots, f_k)$.

Definition

A variety $\mathcal{V}(I)$ is called irreducible if it cannot be written as a union of two proper subvarieties in $\mathbb{C}^n$. Namely,

$$\mathcal{V}(I) = \mathcal{V}(J_1) \cup \mathcal{V}(J_2) \quad \Longrightarrow \quad \mathcal{V}(I) = \mathcal{V}(J_1) \text{ or } \mathcal{V}(I) = \mathcal{V}(J_2).$$

When $I = \langle f \rangle$, the associated variety is called a hypersurface.

Remarks:

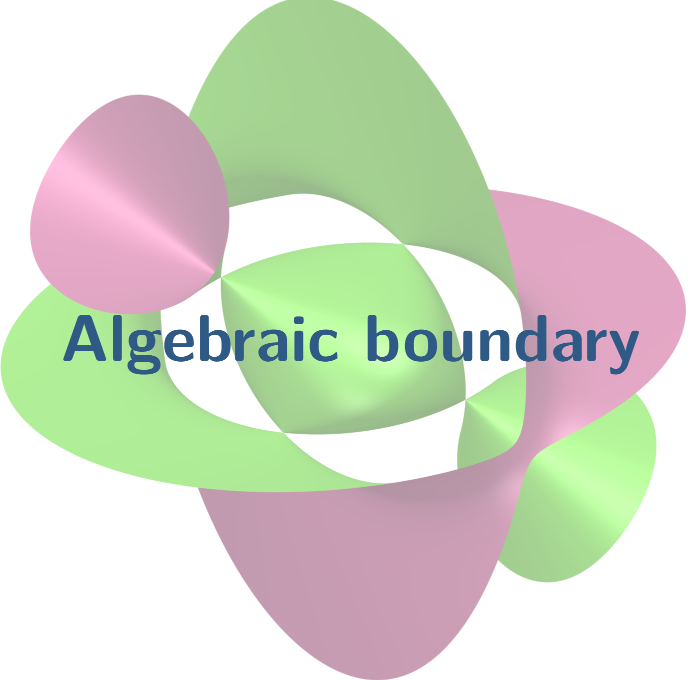
- irreducible hypersurface $\longleftrightarrow$ irreducible polynomial;
- degree of the hypersurface $\longleftrightarrow$ degree of the (reduced) polynomial.

Today we are going to use the following topology on $\mathbb{C}^n$.

Definition

Declare the sets $\mathcal{V}(I)$, for every ideal $I \subset \mathbb{R}[x_1, \dots, x_n]$, to be closed. They form a basis of a topology. Such a topology is called the $\mathbb{R}$–Zariski topology of $\mathbb{C}^n$.

Algebraic boundary



Let $S \subset \mathbb{R}^n$ be a semialgebraic set and denote by ∂S its Euclidean boundary.

Definition

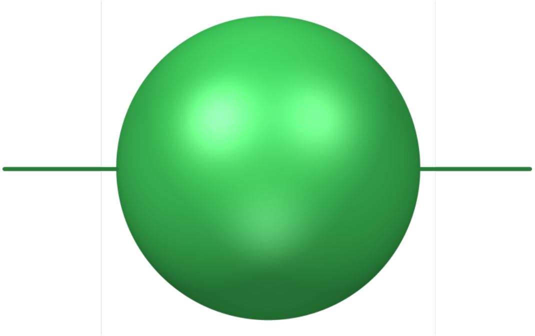
The algebraic boundary of S, denoted $\partial_a S$, is the closure in $\mathbb{C}^n$, with respect to the Zariski topology, of ∂S.

If $K \subset \mathbb{R}^n$ is a semialgebraic convex body with non–empty interior, then $\partial_a K$ is a hypersurface.

Proposition

A convex body with non–empty interior is semialgebraic if and only if its algebraic boundary is a hypersurface.

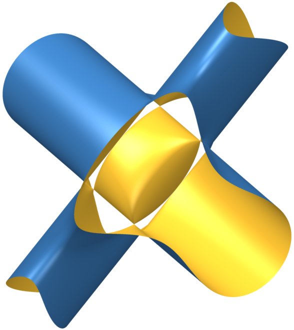
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid y^2(x^2 + y^2 + z^2 - 1) \leq 0,\ z^2(x^2 + y^2 + z^2 - 1) \leq 0\}$$



$$\partial_a S = \mathcal{V}(x^2 + y^2 + z^2 - 1) \cup \mathcal{V}(y, z)$$

Consider the spectrahedron associated to the matrix

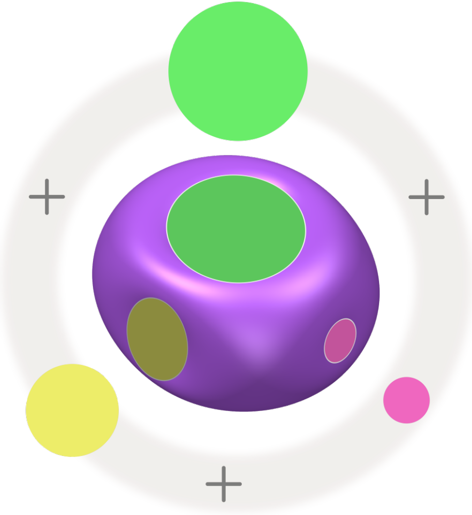
$$M = \begin{pmatrix} 1 & x & y & 0 & 0 \\ x & 1 & 0 & 0 & z \\ y & 0 & 1 & x+y & 0 \\ 0 & 0 & x+y & 1 & 0 \\ 0 & z & 0 & 0 & 1 \end{pmatrix}$$



Its algebraic boundary is the vanishing locus $\mathcal{V}(\det M)$, namely

$$x^4+2x^3y+x^2y^2+x^2z^2+2xyz^2+2y^2z^2-2x^2-2xy-2y^2-z^2+1=0$$

$$K = \{x=0, y^2+z^2 \leq 1\} + \{y=0, x^2+z^2 \leq 4\} + \{z=0, x^2+y^2 \leq 9\}$$



$$x^{24} + 4x^{22}y^2 + \ldots\ldots$$
$$\ldots\ldots + 110075314176 = 0$$

together with 6 hyperplanes

A subset $C \subset \mathbb{R}^n$ is called (convex) cone if (it is convex and) for all $\lambda \geq 0$ and all $x \in C$, $\lambda x \in C$.

Some definitions:

- The *conic hull* of a set $D \subset \mathbb{R}^n$ is

$$\text{cone}(D) = \{\lambda_1 x_1 + \ldots + \lambda_k x_k \mid k \in \mathbb{N}, \lambda_i \geq 0, x_i \in D\}\,;$$

- A *ray* is a set of the form $\mathbb{R}_{\geq 0} x$;
- We say that a cone is *pointed* if it contains no lines;
- A *basis* of a cone C is its intersection $C \cap H$ with an hyperplane non containing the origin, such that

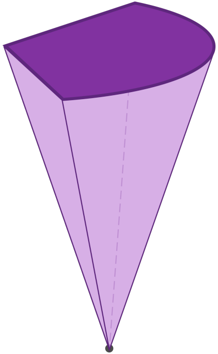
$$\text{cone}(C \cap H) = C.$$

C has a compact basis if and only if C is pointed and closed.

Convex body $\rightsquigarrow$ Convex cone

Let $K \subset \mathbb{R}^n$ be a convex body and consider the map $\phi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ such that $x \mapsto (1, x)$. Then $C_K = \operatorname{cone}(\phi(K))$ is a closed pointed cone, with non–empty interior.

This procedure gives a bijection between the faces of K and C_K.



Algebraic boundary

$$\partial_a K \quad \rightsquigarrow \quad \partial_a C_K$$

$$\partial_a(\text{compact basis of } C) \quad \leftsquigarrow \quad \partial_a(C, \text{ pointed and closed})$$

A point $(1, x)$ belongs to the boundary of C_K if and only if $x \in \partial K$. Therefore

$$\partial_a(C_K) = \widehat{X}$$

where X is the closure of $\partial_a K$ in $\mathbb{P}^n_{\mathbb{C}}$ and $\widehat{X}$ is the affine cone over X.

More precisely: $\partial_a K \subset \mathbb{C}^n \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, with the usual embedding $(x_1, \ldots, x_n) \mapsto [1, x_1, \ldots, x_n]$, and

$$\widehat{X} = \left\{ x \in \mathbb{C}^{n+1} \mid \text{ the line through } x \text{ and the origin is in } X \right\}.$$

Algebraic boundary

For all "nice" cones:

Corollary

If $C \subset \mathbb{R}^{n+1}$ is a semialgebraic pointed closed convex cone, its algebraic boundary is a hypersurface and an algebraic cone. In particular it is the affine cone over the projectivization of $\partial_a C$ in $\mathbb{P}^n_{\mathbb{C}}$.

Meaning: $\partial_a C$ is defined by homogeneous equations in $\mathbb{C}^{n+1}$, so it makes sense to think of it inside $\mathbb{P}^n_{\mathbb{C}}$. Then take the affine cone over this projective variety: you get $\partial_a C$ back!

Duality

For a convex set $K \subset \mathbb{R}^n$ we define the polar/dual convex set as

$$K^\circ = \{\ell \in (\mathbb{R}^n)^* \mid \ell(x) \leq 1 \, \forall x \in K\}.$$

In the case of a convex cone $C \subset \mathbb{R}^{n+1}$, the definition above is equivalent to

$$\left\{\ell \in \left(\mathbb{R}^{n+1}\right)^* \mid \ell(x) \leq 0 \, \forall x \in C\right\}$$

and we will denote the dual convex cone by $C^\vee$ in order to emphasize that it is a cone.

- $(K^\circ)^\circ = \mathrm{cl}\,(\mathrm{conv}(K \cup 0))$. In particular if K is a convex body containing the origin, $(K^\circ)^\circ = K$. If C is a closed convex cone, then $(C^\vee)^\vee = C$.

- $K_1 \subset K_2 \implies K_2^\circ \subset K_1^\circ$;

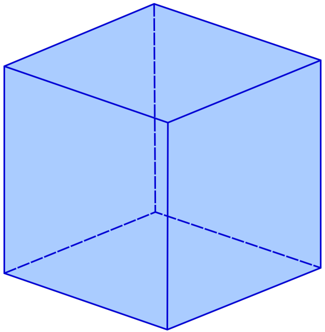
- $(K_1 \cap K_2)^\circ = \mathrm{conv}(K_1^\circ \cup K_2^\circ)$;

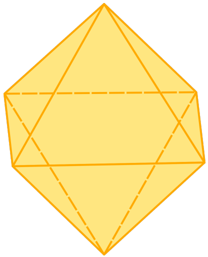
- for all $g \in GL_n(\mathbb{R})$, $(g \cdot K)^\circ = g^{-T} \cdot K^\circ$;

- let K be the unit ball of the L^p–norm, then K° is the unit ball of the L^q–norm, with $\frac{1}{p} + \frac{1}{q} = 1$;

- let P be a polytope with the origin in its interior, then P° is a polytope as well.

$$P = \operatorname{conv}\{(\pm 1, \pm 1, \pm 1)\} \quad \rightsquigarrow \quad P^\circ = \operatorname{conv}\{\pm e_1, \pm e_2, \pm e_3\}$$





$$\partial_a P = \mathcal{V}\Big((x^2-1)(y^2-1)(z^2-1)\Big)$$

$$\partial_a P^\circ = \mathcal{V}\Big(\prod(1 - x \pm y \pm z)\prod(1 + x \pm y \pm z)\Big)$$

Let F be a face of a convex set $K \subset \mathbb{R}^n$, we define the dual face F° as the set of linear functionals $\ell \in (\mathbb{R}^n)^*$ that attain the maximum over K on F.

Remarks:

- F° is an exposed face of K°;
- if K is a convex body containing the origin, then

$$F^\circ = \{\ell \in K^\circ \mid \ell(x) = 1 \, \forall x \in F\};$$

- analogously, for a convex cone C we have that

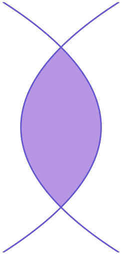
$$F^\circ = \{\ell \in C^\vee \mid \ell(x) = 0 \, \forall x \in F\};$$

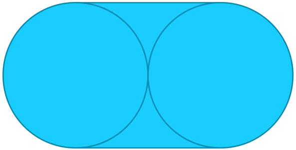
- "biduality": if F is an exposed face of K, the dual of $F^\circ \subset K^\circ$ is exactly F.

exposed extreme point of K	$\rightsquigarrow$	inclusion maximal face of K°
inclusion maximal face of K	$\not\rightsquigarrow$	exposed extreme point of K°

Example

Consider the convex body $K = \{y^2 - 2x - 1 \leq 0, y^2 + 2x - 1 \leq 0\}$.
Its dual convex body is $K^\circ = \text{conv}\{(x-1)^2 + y^2 = 1, (x+1)^2 + y^2 = 1\}$.





"homogenization commutes with duality"

Recall that $C_K = \operatorname{cone}(\phi(K))$, where

$$\begin{aligned} \phi : \mathbb{R}^n &\to \mathbb{R}^{n+1} \\ x &\mapsto (1, x) \end{aligned}$$

then $(C_K)^\vee = C_{K^\circ}$.

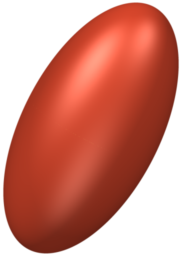
$K:$

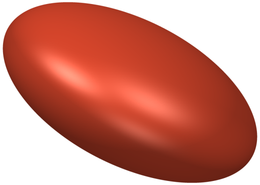
$$x^2 + y^2 + \frac{1}{4}z^2 \leq 1$$

$\rightsquigarrow$

$K^\circ:$

$$x^2 + y^2 + 4z^2 \leq 1$$





Dual variety!

Let $I \subset \mathbb{R}[x_0, \ldots, x_n]$ be a homogeneous ideal and consider $X = \mathcal{V}(I) \subset \mathbb{P}^n_{\mathbb{C}}$. Then, if $c = \operatorname{codim} X$, the singular locus $\mathrm{Sing}(X)$ is a subvariety of X defined by the vanishing of the $c \times c$ minors of the Jacobian matrix $J(X)$.

If $I = \langle p_1, \ldots, p_k \rangle$, then $J(X) = \left(\frac{\partial p_i}{\partial x_j}\right)_{i,j}$ is a $k \times (n+1)$ matrix.

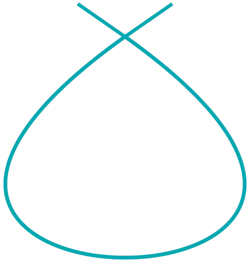
Definition

The *regular points* of X are $X_{\mathrm{reg}} = X \setminus \mathrm{Sing}(X)$.

Consider the plane cubic curve $X = \mathcal{V}(-y^3 + x^2 + y^2 + y - 1)$. We are in the case $n = k = c = 1$. The Jacobian matrix is just the gradient of the defining polynomial:

$$(2x, -3y^2 + 2y + 1)$$

and by definition $\mathrm{Sing}(X) = \mathcal{V}(2x, -3y^2 + 2y + 1) \cap X = \{(0, 1)\}$.



Hence $X_{\mathrm{reg}} = X \setminus \{(0, 1)\}$.

$(\mathbb{P}^n_{\mathbb{C}})^* =$ hyperplanes of $\mathbb{P}^n_{\mathbb{C}}$.

The point $v = [v_0, \ldots, v_n] \in (\mathbb{P}^n_{\mathbb{C}})^*$ represents the hyperplane $v_0 x_0 + \ldots + v_n x_n = 0$ in $\mathbb{P}^n_{\mathbb{C}}$. So v is said to be tangent to X at $x \in X_{\mathrm{reg}}$ if x belongs to the hyperplane associated to v and if the vector $(v_0, \ldots, v_n)$ lies in the row span of $J(X)$ at x.

Definition

The *conormal variety* $\mathrm{CN}(X)$ of X is the closure of the set

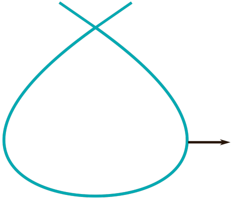
$$\left\{(x, v) \in \mathbb{P}^n_{\mathbb{C}} \times (\mathbb{P}^n_{\mathbb{C}})^* \mid x \in X_{\mathrm{reg}}, v \text{ is tangent to } X \text{ at } x\right\}.$$

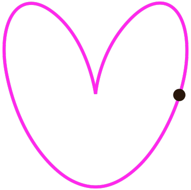
Consider the projection $\pi : \mathbb{P}^n_{\mathbb{C}} \times (\mathbb{P}^n_{\mathbb{C}})^* \to (\mathbb{P}^n_{\mathbb{C}})^*$ onto the second factor. The dual variety of X is $X^* = \pi\,(\mathrm{CN}(X))$. More precisely, it is the closure of the set

$$\{v \in (\mathbb{P}^n_{\mathbb{C}})^* \mid v \text{ is tangent to } X \text{ at some regular point}\}\,.$$

Example

Let $X = \{-y^3 + x^2 + y^2 + y - 1 = 0\}$.
Its dual curve is $X^* = \{32x^4 + 13x^2y^2 + 4y^4 - 18x^2y + 4y^3 - 27x^2 = 0\}$.





- $\dim X^* \leq n-1$;
- If X is a smooth (i.e. $X = X_{\text{reg}}$) hypersurface of degree d, then $\deg X^* = d(d-1)^{n-1}$;
- If X is an irreducible projective variety, then $(X^*)^* = X$.

Theorem

Let K be a semialgebraic convex body and define $S = \partial K^\circ \cap (\partial_a K^\circ)_{\text{reg}}$. Then every element $\ell \in S$ supports a point of ∂K. S is open, semialgebraic and dense in ∂K°.

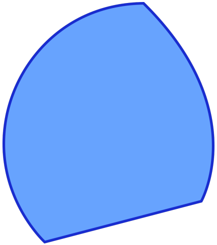
Theorem

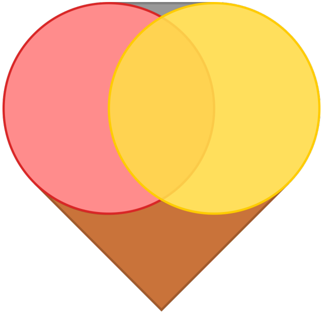
Let C be a semialgebraic pointed and closed convex cone with non–empty interior; define $S = \partial C^\vee \cap (\partial_a C^\vee)_{\text{reg}}$. Then every element $\ell \in S$ supports a ray. S is open, semialgebraic and dense in $\partial C^\vee$.

The previous result is NOT an "if and only if"!

Example: lemon and strawberry ice cream

$K = \{x^2 + y^2 - 1 \leq 0, y^2 + 2x - 1 \leq 0, x - (1 + 2\sqrt{2})y - 2 \leq 0\}$,
$K^\circ = \operatorname{conv}\left\{x^2 + y^2 - 1 \leq 0, (x-1)^2 + y^2 - 1 \leq 0, \{(\tfrac{1}{2}, -\tfrac{1}{2} - \sqrt{2})\}\right\}$.





Irreducible components and extreme points

We say that a point $x \in \partial K$ is an extreme point of the convex set K if $x = ty + (1-t)z$ implies $x = y = z$.

Theorem (Krein–Milman)

Let $K \subset \mathbb{R}^n$ be a compact convex set, then K is the convex hull of its extreme points.

Analogously for closed pointed convex cones, with extreme rays.

Notation:

$\mathrm{Ex}_a(K)$: is the Zariski closure in $\mathbb{C}^n$ of the union of the extreme points of the convex semialgebraic set $K \subset \mathbb{R}^n$.
$\mathrm{Exr}_a(C)$: is the Zariski closure in $\mathbb{C}^{n+1}$ of the union of the extreme rays of the convex semialgebraic cone $C \subset \mathbb{R}^{n+1}$

"It's enough to look at extreme points"

Let $C \subset \mathbb{R}^{n+1}$ be a pointed, closed, semialgebraic cone with non–empty interior.

Result 1.

The dual variety to the algebraic boundary of C is contained in the Zariski closure of the extreme rays of the dual convex cone:

$$(\mathbb{P}\partial_a C)^* \subset \mathbb{P}\mathsf{Exr}_a(C^\vee).$$

Result 2.

The dual variety to the Zariski closure of the extreme rays of C is contained in the algebraic boundary of the dual convex cone:

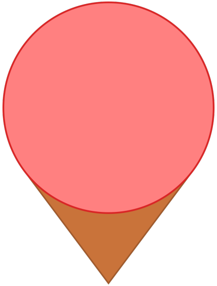
$$(\mathbb{P}\mathsf{Exr}_a(C))^* \subset \mathbb{P}\partial_a C^\vee.$$

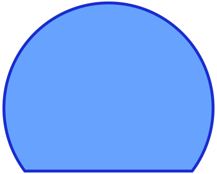
Corollary

$$(\mathbb{P}\partial_a C)^* = \mathbb{P}\mathsf{Exr}_a(C^\vee) \qquad \text{BUT} \qquad (\mathbb{P}\mathsf{Exr}_a(C^\vee))^* \neq \mathbb{P}\partial_a C$$

Example: strawberry ice cream

Consider the convex body $K = \operatorname{conv}\left\{x^2 + y^2 - 1 \leq 0, \{(0, -\frac{5}{3})\}\right\}$.
Its dual body is $K^\circ = \{x^2 + y^2 - 1 \leq 0, y \geq -\frac{3}{5}\}$.



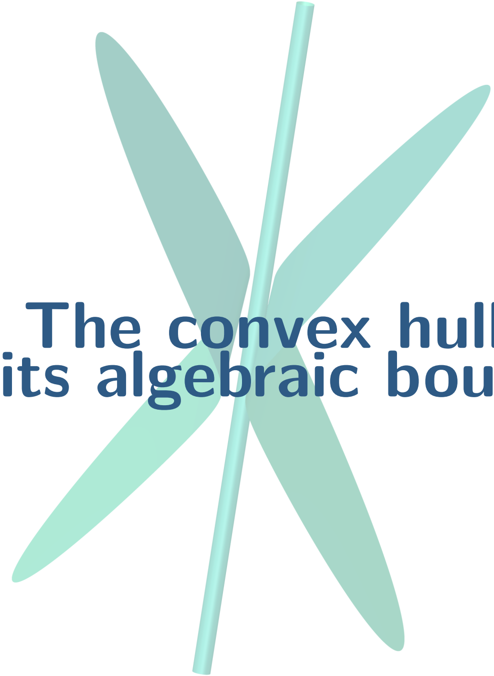


Do you want to know more about the irreducible components?
Take a look at

Rainer Sinn,
Algebraic Boundaries of Convex Semi-algebraic Sets,
Research in the Mathematical Sciences, 2, No. 1 (2015)

Do you want to know more about duality?
Take a look at

Philipp Rostalski and Bernd Sturmfels,
Dualities in convex algebraic geometry ,
Rendiconti di Mathematica, Serie VII, 30, 285-327 (2010)



The convex hull and its algebraic boundary

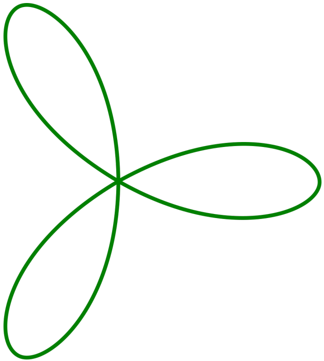
Let X be a compact variety in $\mathbb{R}^n$. In this section we assume

$$K = \mathrm{conv}(X).$$

How can we describe $\partial_a K$?

Example: the trefoil

Consider the plane curve $C = \{(x^2 + y^2)^2 = x(x^2 - 3y^2)\}$.



Example: the trefoil

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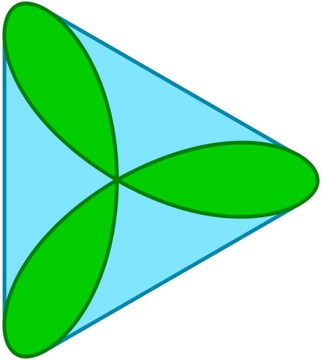


Its algebraic boundary is given by the curve itself, together with three lines.

A line is said to be bitangent to $\overline{C} \subset \mathbb{P}^2_{\mathbb{C}}$ if it is tangent to $\overline{C}$ at two distinct points.

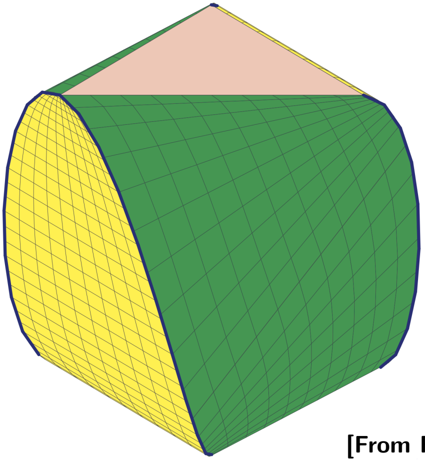
Plücker formula

Let $\overline{C}$ be a generic smooth plane curve of degree $d \geq 2$. Then the number of bitangents of $\overline{C}$ is

$$\frac{(d-3)(d-2)d(d+3)}{2}.$$

Therefore we can give a bound to the degree of the algebraic boundary of $K = \mathrm{conv}(C)$, namely

$$\deg \partial_a K \leq d + \frac{(d-3)(d-2)d(d+3)}{2}.$$



[From Ranestad, Sturmfels (2012)]

edge surface $+$ tritangent planes

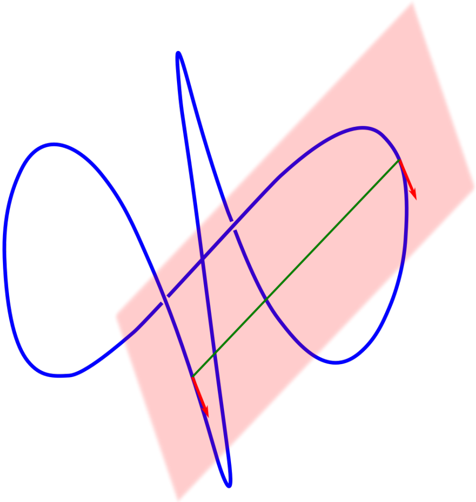
Consider a curve $C \subset \mathbb{R}^3$. Any two distinct points $p_1, p_2 \in C$ span a so called *bisecant line*. Such line is called a stationary bisecant line if the tangent lines to C at p_1 and p_2 lie in a common plane.

Definition

The union of all stationary bisecant lines is called the *edge surface* of C.

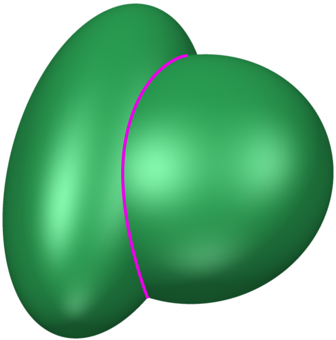
Convex hull 〉 back to previous example

$$C = \{x^2 - y^2 - xz = 0\} \cap \{z - 4x^3 + 3x = 0\}$$



Consider two ellipsoids given by the zero loci of $q_1 = 4x^2 + 4y^2 + z^2 - 4$ and $q_2 = (x-1)^2 + 2y^2 + z^2 - 2$. Let C be the curve obtained as their intersection:

$$C = \{q_1 = 0\} \cap \{q_2 = 0\}.$$



The pink curve C is a quartic elliptic space curve.

Consider the pencil of quadrics given by $q_1 + tq_2$.
Let Q_1, Q_2 be the 4×4 symmetric matrices associated to q_1, q_2. The univariate polynomial $f(t) = \det(Q_1 + tQ_2)$ has generically 4 distinct roots $t_1, \ldots, t_4$.

Each of these values corresponds to a singular quadric $q_1 + t_i q_2$ of the pencil.

FACT:

The edge surface is the union of the 4 singular quadratic surfaces $\mathcal{V}(q_1 + t_i q_2)$.

In practice the equation of the edge surface can be computed as

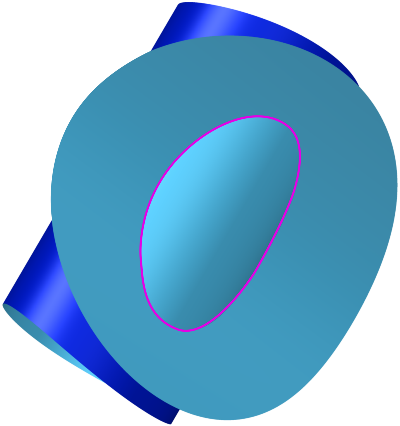
$$\text{resultant}_t(f(t), (q_1 + tq_2)(x, y, z)).$$

In our case we obtain the polynomial

$$\begin{gathered}
-192x^8 - 320x^6y^2 - 320x^4y^4 - 128x^2y^6 + 144x^6z^2 + 32x^2y^4z^2 + 64y^6z^2 \\
-108x^4z^4 + 24x^2y^2z^4 + 64y^4z^4 + 42x^2z^6 + 28y^2z^6 - 896x^7 - 640x^5y^2 \\
-512x^3y^4 - 256xy^6 + 864x^5z^2 - 128x^3y^2z^2 - 320xy^4z^2 - 480x^3z^4 - 208xy^2z^4 \\
+28xz^6 - 1280x^6 + 704x^4y^2 + 896x^2y^4 + 128y^6 + 1424x^4z^2 + 512x^2y^2z^2 \\
-32y^4z^2 - 56x^2z^4 - 24y^2z^4 - 42z^6 + 128x^5 + 768x^3y^2 + 512xy^4 - 832x^3z^2 \\
+128xy^2z^2 + 480xz^4 + 1920x^4 - 704x^2y^2 - 320y^4 - 1424x^2z^2 - 108z^4 \\
-128x^3 - 640xy^2 + 864xz^2 - 1280x^2 + 320y^2 - 144z^2 + 896x - 192
\end{gathered}$$

Only two components contribute to the actual convex hull:

$$2x^2 - z^2 + 4x - 2 = 0 \qquad 3x^2 + 2y^2 + 2x - 3 = 0$$



Outline of the algorithm:

Consider the corresponding complex projective curve $\overline{C} \subset \mathbb{P}^3_{\mathbb{C}}$, whose points can be expressed as

$$[F_0(x_0, x_1), F_1(x_0, x_1), F_2(x_0, x_1), F_3(x_0, x_1)].$$

The bisecant line between $p = (p_0, p_1)$ and $q = (q_0, q_1)$ is stationary if the determinant of the following matrix vanishes:

$$\begin{pmatrix} \frac{\partial F_0}{\partial x_0}(p) & \frac{\partial F_1}{\partial x_0}(p) & \frac{\partial F_2}{\partial x_0}(p) & \frac{\partial F_3}{\partial x_0}(p) \\ \frac{\partial F_0}{\partial x_1}(p) & \frac{\partial F_1}{\partial x_1}(p) & \frac{\partial F_2}{\partial x_1}(p) & \frac{\partial F_3}{\partial x_1}(p) \\ \frac{\partial F_0}{\partial x_0}(q) & \frac{\partial F_1}{\partial x_0}(q) & \frac{\partial F_2}{\partial x_0}(q) & \frac{\partial F_3}{\partial x_0}(q) \\ \frac{\partial F_0}{\partial x_1}(q) & \frac{\partial F_1}{\partial x_1}(q) & \frac{\partial F_2}{\partial x_1}(q) & \frac{\partial F_3}{\partial x_1}(q) \end{pmatrix}$$

Outline of the algorithm:

$$\mathbb{P}^3_{\mathbb{C}} \text{ with coordinates } p_0, p_1, q_0, q_1$$
$$\downarrow$$
$$\mathbb{P}^2_{\mathbb{C}}$$
$$\downarrow$$
$$\mathsf{Gr}(2,4)$$
$$\downarrow$$
$$\text{our } \mathbb{P}^3_{\mathbb{C}}$$

Kristian Ranestad and Bernd Sturmfels,
The Convex Hull of a Space Curve,
Advances in Geometry, 12, 157-178 (2012)

A trigonometric space curve of degree d is a curve in $\mathbb{R}^3$ parametrized by three degree d trigonometric polynomials of the form

$$\sum_{i=1}^{d/2} a_i \cos(i\theta) + \sum_{i=1}^{d/2} b_i \sin(i\theta) + c.$$

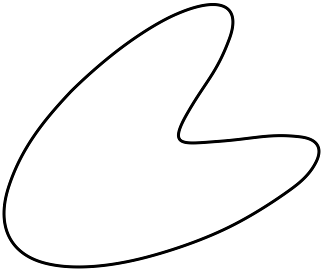
Using the change of coordinates

$$\cos(\theta) = \frac{x_0^2 - x_1^2}{x_0^2 + x_1^2} \qquad \sin(\theta) = \frac{2x_0x_1}{x_0^2 + x_1^2}$$

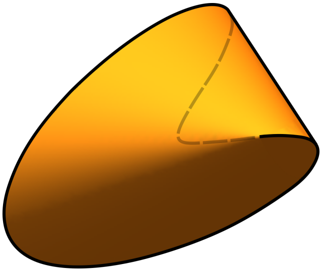
and clearing denominators, we obtain a polynomial parametrization of $\overline{C}$, which is rational and, for generic a_i, b_i, c, smooth.

We can use the algorithm!

$$C = \left\{ (\cos(\theta), \sin(\theta) + \cos(2\theta), \sin(2\theta)) \in \mathbb{R}^3 \mid \theta \in [0, 2\pi] \right\}$$



$$C = \left\{ (\cos(\theta), \sin(\theta) + \cos(2\theta), \sin(2\theta)) \in \mathbb{R}^3 \mid \theta \in [0, 2\pi] \right\}$$



The edge surface is the irreducible sextic $16x^6 - 32x^4y^2 + 16x^2y^4 - 96x^5z - 160x^3y^2z + 192x^4z^2 + 16x^2y^2z^2 - 128x^3z^3 + 216x^4y + 48x^2y^3 - 8y^5 + 72x^3yz + 88xy^3z - 72x^2yz^2 - 8y^3z^2 + 72xyz^3 - 207x^4 - 138x^2y^2 - 23y^4 + 180x^3z + 60xy^2z - 126x^2z^2 - 54y^2z^2 + 108xz^3 - 27z^4 - 36x^2y + 4y^3 - 36xyz + 108x^2 + 36y^2 - 108xz + 27z^2 = 0$.

Theorem

Let C be a general smooth space curve of degree d and genus g. The degree of its edge surface is $2(d-3)(d+g-1)$.

Sanity check with previous examples:

curve	degree	genus	edge surface
elliptic space curve	4	1	8
trigonometric space curve	4	0	6

A plane H is a tritangent plane of $\overline{C} \subset \mathbb{P}^3_{\mathbb{C}}$ if it is tangent to $\overline{C}$ at three or more points.

Theorem

Let C be a general smooth space curve of degree d and genus g. The number of tritangent planes is

$$8\binom{d+g-1}{3} - 8(d+g-4)(d+2g-2) + 8g - 8.$$

Sanity check with previous examples:

curve	degree	genus	tritangent planes
elliptic space curve	4	1	0
trigonometric space curve	4	0	0

Let's try to generalize! Goal: $\partial_a\left(\mathrm{conv}(X)\right)$

Let X be a variety in $\mathbb{R}^n$, that spans the whole space. Fix $k \in \mathbb{N}$ and define a variety $X^{[k]}$ of $(\mathbb{P}^n_{\mathbb{C}})^*$ as follows: it is the Zariski closure of the set of hyperplanes that are tangent to X at k regular points which span a $(k-1)$–plane in $\mathbb{P}^n_{\mathbb{C}}$.

Remark: $X^{[1]} = X^*$

Theorem

Under reasonable assumptions,

$$\partial_a\left(\mathrm{conv}(X)\right) \subset \bigcup_{k=1}^{n} \left(X^{[k]}\right)^*$$

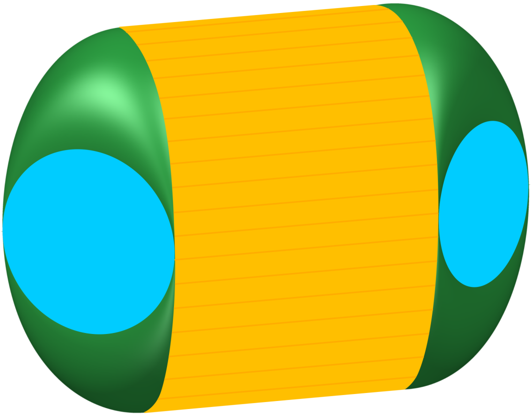
Do you want to know more?
Take a look at

Kristian Ranestad and Bernd Sturmfels,
The Convex Hull of a Variety,
"Notions of Positivity and the Geometry of Polynomials", Trends in Mathematics, Springer Verlag, Basel, pp. 331-344 (2011)

Rainer Sinn,
Algebraic Boundaries of Convex Semi-algebraic Sets,
PhD thesis, Universität Konstanz (2014)

The notion of patches

f–vector $\rightsquigarrow$ patches



Let $K \subset \mathbb{R}^n$, hence $\dim \partial K = n - 1$ and its patches are $(n - 1 - k)$–dimensional families of k–faces (+ some technical conditions).
Key word: normal cycle

Do you want to know more?
Take a look at

Daniel Plaumann, Rainer Sinn and Jannik Lennart Wesner,
Families of Faces and the Normal Cycle of a Convex Semi-algebraic Set,
Preprint, arXiv:2104.13306 (2021)

Thank you

and see you at

the exercise session!