## Tropical geometry, p-adics, probability, and applications

Ngoc Mai Tran

https://web.ma.utexas.edu/users/ntran/, August, 2021

## I. Tropical vs Tropicalization

## Tropical mathematics

Tropical mathematics is mathematics over the max-plus algebra

$$
a \oplus b:=\max (a, b) \quad a \odot b:=a+b
$$

Friends: min-plus, max-times.
Example. Matrix-vector multiplication

$$
\left(\begin{array}{cc}
0 & -1 \\
3 & 0
\end{array}\right) \odot\binom{1}{3}=
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Naïve recipe to do tropical mathematics:

1. Take a classical object (polynomial, polytope, hyperplanes, etc)
2. Replace + with $\oplus, \times$ with $\odot$
3. Ask: does this still make sense? Do I get an interesting object? Are there tropical analogues of major theorems? Are there applications?

## Examples: tropical polytopes and hyperplanes



A tropical polytope (left) and the associated arrangement of tropical hyperplanes (right).
Source: X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig, Tropicalizing the Simplex Algorithm SIAM J. Discrete Math., 29(2), 751-795.

## Example: tropical polynomials

As an example we consider the general quadratic polynomial

$$
p(x, y)=a \odot x^{2} \oplus b \odot x y \oplus c \odot y^{2} \oplus d \odot y \oplus e \oplus f \odot x
$$

Suppose that the coefficients $a, b, c, d, e, f \in \mathbb{R}$ satisfy the inequalities

$$
2 b<a+c, 2 d<a+f, 2 e<c+f
$$

Then the graph of $p: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is the lower envelope of six planes in $\mathbb{R}^{3}$. This is shown in Figure 1.3 .2 , where each linear piece of the graph is labeled by the corresponding linear function. Below this "tent" lies the tropical quadratic curve $V(p) \subset \mathbb{R}^{2}$. This curve has four vertices, three bounded edges and six half-rays (two northern, two eastern and two southwestern).


## Why tropical mathematics? Answer \#1: applications

Many objects in applications are tropical. (Lecture 4)
Example: deep ReLU networks.


A cartoon drawing of a biological neuron (left) and its mathematical model (right).

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f(x)=\max \left\{\mathbf{w}^{\top} \mathbf{x}+b, c\right\}=b \odot \mathbf{x}^{\odot} \mathbf{w} \oplus c .
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One layer ReLU network = a max-plus affine monomial

## Deep ReLU networks are tropical rational functions



One layer = affine monomial

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$$

Stack $D$ layers =

## Deep ReLU networks are tropical rational functions



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$$
f(x)=\max \left\{\mathbf{a}^{\top} \mathbf{x}+b, c\right\}=b \odot \mathbf{x}^{\odot} \cdot \mathbf{a} \oplus c .
$$

Stack $D$ layers $=$ compose monomials $=$ polynomial.
$f=f_{1} \circ f_{2} \circ \cdots \circ f_{D}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$.

## Deep ReLU networks are tropical rational functions

Theorem. (Zhang, Naitzat, Lim, PMLR 2018)
Over the max-plus algebra, deep ReLU neural networks are ratios of polynomials.

Research direction: use tropical algebraic geometry to answer hard questions on deep ReLUs.

Researchers: Guido Montufar, Lek-Heng Lim, Yue Ren, Vasileios Charisopoulos, Petros Maragos, and co-authors.

## Why tropical mathematics? Answer \#2: tropicalization

Algebraic varieties can be logarithmically degenerated to tropical varieties, which are polyhedral complexes satisfying certain combinatorial properties. At any intermediate stage before reaching its final limit, the degeneration process yields a structure known as an amoeba. Both tropical varieties and amoebas are dual to subdivisions of polytopes, and this leads to an infusion of combinatorial methods into algebraic, complex-analytic, and nonArchimedean geometry.


Figure 1.3: The graph and the curve defined by a quadratic polynomial

## Valuations \& tropicalization

Let $K$ be a field. We denote by $K^{*}$ the nonzero elements of $K$. A valuation on $K$ is a function val: $K \rightarrow \mathbb{R} \cup\{\infty\}$ satisfying the following three axioms:
(1) $\operatorname{val}(a)=\infty$ if and only if $a=0$,
(2) $\operatorname{val}(a b)=\operatorname{val}(a)+\operatorname{val}(b)$ and
(3) $\operatorname{val}(a+b) \geq \min \{\operatorname{val}(a), \operatorname{val}(b)\}$ for all $a, b \in K$.

Lemma 2.1.1. If $\operatorname{val}(a) \neq \operatorname{val}(b)$ then $\operatorname{val}(a+b)=\min (\operatorname{val}(a), \operatorname{val}(b))$.
Tropicalization map

$$
\text { trop : }\left(k^{*}\right)^{n} \rightarrow \mathbb{R}^{n},\left(a_{1}, \ldots, a_{n}\right) \mapsto\left(\operatorname{val}\left(a_{1}\right), \ldots, \operatorname{val}\left(a_{n}\right)\right)
$$

turns $\left(k^{*},+, \times\right)$ to $\left(\Gamma_{v a l}, \min ,+\right)$.

## Examples of fields with valuations

## 1. $\mathbb{Q}_{p}$ and the $p$-adic valuation

Examples 1.2.2. (a) The field $\mathbb{k}=\mathbb{Q}_{p}$ of $p$-adic numbers has the $p$-adic valuation $v_{p}$ and the $p$-adic norm $|a|_{p}=p^{-v_{p}(a)}$, with respect to which it is complete. We sometimes write $\mathbb{Q}_{\infty}$ for $\mathbb{R}$. Every variety defined over the rationals therefore has a $p$-adic amoeba for each $p \leqslant \infty$, where $p=\infty$ corresponds to the complex amoeba.

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## 2. Field of Puiseux series over $\mathbb{C}$.

$$
\begin{gathered}
k=\mathbb{C}\{\{t\}\}=\left\{\sum_{i=\ell}^{\infty} a_{i} t^{i / n}: a_{i} \in \mathbb{C}, \ell \in \mathbb{Z}, n \in \mathbb{N}\right\}, \operatorname{val}(x)=\min (\operatorname{Supp}(x)) \\
c(t)=\frac{4 t^{2}-7 t^{3}+9 t^{5}}{6+11 t^{4}}=\frac{2}{3} t^{2}-\frac{7}{6} t^{3}+\frac{3}{2} t^{5}+\cdots \quad \text { has } \operatorname{val}(c(t))=2 \\
\tilde{c}(t)=\frac{14 t+3 t^{2}}{7 t^{4}+3 t^{7}+8 t^{8}}=2 t^{-3}+\frac{3}{7} t^{-2}+\cdots \quad \text { has } \operatorname{val}(\tilde{c}(t))=-3 \\
\pi=3.1415926535897932385 \ldots \quad \operatorname{has} \operatorname{val}(\pi)=0
\end{gathered}
$$

3. If $k$ is a field with non-archimedean norm $|\cdot|$ (ie:
$|a+b| \leq \max (|a|,|b|)$, then $v a l: a \mapsto-\log (|a|)$ is a valuation.

## Kapranov's theorem: $\operatorname{trop}(V(f))=V(\operatorname{trop}(f))$

2.1. Amoebas and tropical varieties of polynomials. Let

$$
f(\mathbf{x})=\sum_{\mathbf{n} \in \mathbb{Z}^{d}} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}, \quad \mathbf{n}=\left(n_{1}, \ldots, n_{d}\right), \quad \mathbf{x}^{\mathbf{n}}=x_{1}^{n_{1}} \ldots x_{d}^{n_{d}}
$$

be a Laurent polynomial with coefficients $a_{\mathbf{n}} \in \mathbb{k}$, and let $X=X_{f} \subset \mathbb{G}_{\mathrm{m}}^{d}$ be the hypersurface $\{f=0\}$. We denote $\mathcal{T}\left(X_{f}\right)$ by $\mathcal{T}(f)$. For $\mathbf{u} \in \mathbb{R}^{d}$ define

$$
\begin{equation*}
f^{\tau}(\mathbf{u})=\min _{\mathbf{n} \in \mathbb{Z}^{d}}\left\{v\left(a_{\mathbf{n}}\right)+\mathbf{u} \cdot \mathbf{n}\right\} . \tag{2.1.1}
\end{equation*}
$$

Then $f^{\tau}$ is a convex piecewise-linear function on $\mathbb{R}^{d}$ known as the tropicalization of $f$ (see [20] and [23] for background). Note that for almost all $\mathbf{n}$ we have $a_{\mathbf{n}}=0$, so $v\left(a_{\mathbf{n}}\right)=+\infty$. Therefore $f^{\tau}$ is the minimum of finitely many affine-linear functions.

Theorem 2.1.1. If $f \neq 0$ then $\mathcal{T}(f)$ is equal to the non-differentiability locus of $f^{\tau}$. In particular, $\mathcal{T}(f)$ is either empty (when $f$ is a monomial), or is a rational polyhedral set of pure dimension $d-1$, or is all of $\mathbb{R}^{d}$ (when $f=0$ ).

## Beyond hypersurfaces: tropical varieties, tropical basis

Remark 2.2.8. Let $I$ be the ideal in $\mathbb{k}\left[x_{1}^{ \pm 1}, \ldots, x_{d}^{ \pm 1}\right]$ defining $X$. Then trivially $\mathcal{T}(X) \subset \mathcal{T}(f)$ for every $f \in I$. Speyer and Sturmfels [23, Thm. 2.1] have shown that

$$
\mathcal{T}(X)=\bigcap_{f \in I} \mathcal{T}(f)
$$

Furthermore, they describe in [23, Cor. 2.3] that the intersection can be taken over just those $f$ in a (finite) universal Gröbner basis for $I$. Hence a tropical variety is always the intersection of a finite number of tropical hypersurfaces, each of which has an explicit description as a $\Gamma$-rational polyhedral set from Theorem 2.1.1 Their approach can be developed into an alternative proof of Theorem 2.2.5

See Exercises for more references and open questions.

## Deformations to complex varieties

## Tropical geometry and amoebas, Alain Yger, 2012.

The archimedean triangle inequality $|a-b| \leq|a \pm b| \leq|a|+|b|$, is known to be far more difficult to handle in geometric problems than the ultrametric triangle inequality $|a+b| \leq \max (|a|,|b|)$. On the other hand, classical questions in complex algebraic or analytic geometry arise in the archimedean context, not in the ultrametric one as in section 1.3. Therefore, it is important to extend Definitions 1.10 (completed by Proposition 1.2) or 1.14 to the archimedean context. We will also profit in such a context from the fact that the logarithmic map Log $\mid{ }_{\text {arch }}$ is paired with an "argument" multivalued map arg.

ENUMERATIVE TROPICAL ALGEBRAIC GEOMETRY IN $\mathbb{R}^{2}$

## GRIGORY MIKHALKIN


#### Abstract

The paper establishes a formula for enumeration of curves of arbitrary genus in toric surfaces. It turns out that such curves can be counted by means of certain lattice paths in the Newton polygon. The formula was announced earlier in [18].

The result is established with the help of the so-called tropical algebraic geometry. This geometry allows one to replace complex toric varieties with the real space $\mathbb{R}^{n}$ and holomorphic curves with certain piecewise-linear graphs there.


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II. The hunt for tropical Gaussians



## How to get a 'tropical Gaussian'?

- Tropicalize the Gaussian measure on local fields ${ }^{1}$

Other approaches

- Do classical probability on the tropical affine space $\mathbb{T} \mathbb{P}^{d-1} \simeq \mathbb{R}^{d} \backslash \mathbb{R} \cdot(1, \ldots, 1)$.
- Decision calculus (Measure theory in max-plus) ${ }^{2}$
- Brownian motion on tropical curves ${ }^{3}$

[^0]
## Anatomy of a Gaussian

Density of a standard Gaussian in $\mathbb{R}^{n}$

$$
f\left(x_{1}, \ldots, x_{n}\right)=\frac{1}{\sqrt{2 \pi}} \exp \left(-\frac{1}{2} \sum_{i=1}^{n} x_{i}^{2}\right)=\frac{1}{\sqrt{2 \pi}} \exp \left(-\frac{1}{2}\|x\|_{2}^{2}\right)
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Ingredients: a norm.

## Not all norms lead to Gaussians!

## Laplace distribution

$$
\begin{aligned}
& f(x)= \frac{1}{2} \exp \left(-\|x\|_{1}\right) \\
& \times 10^{-3}
\end{aligned}
$$




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Laplace distribution

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What makes the Gaussian special?

## Gaussians play well with linear algebra and independence

- Linear transformations of Gaussians are Gaussians:

$$
X \sim \mathbb{N}(0, \Sigma) \Rightarrow D X \sim \mathbb{N}\left(0, D \Sigma D^{\top}\right)
$$

- If $D$ is orthonormal ( $D^{\top} D=D D^{\top}=I$ ) and $\Sigma=I$, then $D X \stackrel{d}{=} X$.
- Theorem. (Maxwell) Let $X_{1}, \ldots, X_{d}$ be independent random variables on $\mathbb{R}$ with the same distribution. Then the distribution of $\left(X_{1}, \ldots, X_{d}\right) \in \mathbb{R}^{d}$ is spherically symmetric iff $X_{i}$ 's are centered Gaussians. (Need: orthogonality)
- $\Sigma=I \Longleftrightarrow$ coordinates of $X$ are independent $\Longleftrightarrow$ they are uncorrelated. Fundamental to statistical applications (eg: PCA)



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SOUNDS LIKE THE CLASS HELPED.


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p-adic Gaussians satisfy the first three.It is derived from Maxwell's characterization.


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- $\Sigma=I \Longleftrightarrow$ coordinates of $X$ are independent $\Longleftrightarrow$ they are uncorrelated. Fundamental to statistical applications (eg: PCA)
p-adic Gaussians satisfy the first three.It is derived from Maxwell's characterization. Open Problem. PCA on $p$-adics. What do we get if we tropicalize? (tropical PCA in the sense of Ruriko Yoshida et al?)


## Tropicalization of p-adic Gaussians

Switch to board.

# III. Random tropical polynomials and varieties 

## Overview

Goal: study functionals of random tropical objects and their intersections.

Examples: dimensions of varieties, \# vertices of polytopes, number of zeros of a system of random polynomials,

## Motivations

- "Typical" varieties have "common" properties
- Randomness is natural in applications
- New proofs for classical results via tropicalization (?)


## Example 1: systems of random p-adic polynomials

Steve Evans, 2006, the expected number of zeros of a random system of p-adic polynomials

## Abstract

We study the simultaneous zeros of a random family of $d$ polynomials in $d$ variables over the $p$ -adic numbers. For a family of natural models, we obtain an explicit constant for the expected number of zeros that lie in the d-fold Cartesian product of the p-adic integers. Considering models in which the maximum degree that each variable appears is $N$, this expected value is

$$
\mathrm{p}^{\mathrm{d}\left(\log _{\mathrm{p}} \mathrm{~N} \mathrm{~J}\right.}\left(1+\mathrm{p}^{-1}+\mathrm{p}^{-2}+\cdots+\mathrm{p}^{-\mathrm{d}}\right)^{-1}
$$

for the simplest such model.
Open problem. Is there a tropical proof?
See exercises sheet. See also paper: Avinash Kulkarni, Antonio Lerario, p-adic integral geometry, 2019.

## Example 2: random tropical polynomials in one variable

Baccelli and T., 2014. Zeros of random tropical polynomials, random polytopes and stick-breaking.

Consider the tropical min-plus algebra $(\mathbb{R}, \odot, \oplus), a \odot b=a+b, a \oplus b=\min (a, b)$. A tropical polynomial $\mathcal{T}: \mathbb{R} \rightarrow \mathbb{R}$ of degree $n$ has the general form

$$
\begin{equation*}
\mathcal{T} f(x)=\bigoplus_{i=0}^{n}\left(C_{i} \odot x^{i}\right)=\min _{i=0, \ldots, n}\left(C_{i}+i x\right), \tag{1}
\end{equation*}
$$

for coefficients $C_{i} \in \mathbb{R}$. The zeros of $\mathcal{T}$ f are points in $\mathbb{R}$ where the minimum in (1) is achieved at least twice. When the coefficients $C_{i}$ 's are random, the zeros of $\mathcal{T f}$ form a collection of random points in $\mathbb{R}$. A natural model of randomness is one where the $C_{i}$ 's independent and identically distributed (i.i.d.) according to some distribution $F$, called the atom distribution [35]. In this paper, we derive the asymptotic distribution for the number of zeros as $n \rightarrow \infty$, under various atom distributions.

Theorem 1. Let $F$ be a continuous distribution, supported on $(0, \infty)$. Assume $F(y) \sim$ $C y^{a}+o\left(y^{a}\right)$ as $y \rightarrow 0$ for some constants $C, a>0$. Let $Z_{n}$ be the number of zeros of $\mathcal{T} f$ in (1). Then as $n \rightarrow \infty$,

$$
\begin{equation*}
\frac{Z_{n}-\frac{2 a+2}{2 a+1} \log (n)}{\sqrt{\frac{2 a(a+1)\left(2 a^{2}+2 a+1\right)}{(2 a+1)^{3}}} \log (n)} \stackrel{d}{\rightarrow} \mathcal{N}(0,1) . \tag{2}
\end{equation*}
$$

Open problem. Two or More variables?
See exercises sheet

## The case of one variable

Random min-plus tropical polynomial $\mathcal{T f}: \mathbb{R} \rightarrow \mathbb{R}$

$$
\mathcal{T} f(x)=\bigoplus_{i=1}^{n}\left(C_{i} \odot x^{i}\right)=\min _{i=1, \ldots, n}\left(C_{i}+i x\right), \quad C_{i} \stackrel{i . i . d}{\sim} F
$$

Example: $\mathcal{T} f(x)=a \odot x^{\odot 2} \oplus b \odot x \oplus c$.


Zeros of $\mathcal{T}$ f: where the minimum is achieved at least twice.

## Classical analogue: real zeroes of random polynomials

Kac polynomial: $f=\sum_{i} C_{i} x^{i}, C_{i} \stackrel{i . i . d}{\sim} F$.

Kac '40s: for $F=\mathcal{N}(0,1)$,

$$
\mathrm{E}\left(R_{n}\right) \sim\left(\frac{2}{\pi}+o(1)\right) \log (n)
$$

Ibragimov and Maslova ('70s): for $F$ mean 0 , variance 1 , no mass at 0 ,

$$
\operatorname{Var}\left(R_{n}\right) \sim\left(\frac{4}{\pi}\left(1-\frac{2}{\pi}\right)+o(1)\right) \log (n)
$$

Tao and Vu (2013): local universality of zeroes of random polynomials

## Counting zeros of $\mathcal{T} f$

\# roots of $\mathcal{T} f=\#$ lower faces in the convex hull of $\left(i, C_{i}\right)$


Points are roots
Slopes $i$, intercept $C_{i}$

Legendre transform of -f


Slopes are roots
Points (i, $C_{i}$ )

## Proof sketch: $a=1, F=\operatorname{exponential}(1)$

\# roots of $\mathcal{T} f \approx$ \# lower faces of $n$ uniform points in $[0,1]^{2}$ $\approx \#$ lower faces of Poisson $(n)$ points in $[0,1]^{2}$

100 points, 9 vertices


In general, sample $n$ points uniformly at random from a convex $r$-gon. Let $V_{n}$ be \# vertices (equivalently, faces).

Groeneboom (1988) showed that

$$
\frac{V_{n}-\frac{2}{3} r \log (n)}{\sqrt{\frac{10}{27} r \log (n)}} \xrightarrow{d} N(0,1) .
$$

Francois and T. 2014 has a simple stick-breaking proof, which generalizes to non-exponential $F$.

## Proof Step 1: The vertex process

Sufficient to study the "lower left" corner $(r=1)$.


Key Lemma. The $y$-coordinates of consecutive vertices form a Beta(2,1) stick-breaking sequence.

## Proof. Step 2: Stick-breaking in y



Given $Y_{0}=y_{0}$, , independent of the slope,

$$
\mathbb{P}\left(Y_{1} \in d y\right)=\frac{\ell(y)}{y_{0} \ell(0) / 2}=\frac{2}{y_{0}} \frac{y_{0}-y}{y_{0}} .
$$

$\Rightarrow Y_{1}=Y_{0} B_{1}$, for $B_{1} \stackrel{d}{=} \operatorname{Beta}(2,1)$, independent of $Y_{0}$.

Let $B_{i}$ be i.i.d $\operatorname{Beta}(2,1)$. Then

- $Y_{0}$ is $\operatorname{Uniform}(0,1)$
- $Y_{1}=Y_{0} B_{1}$
- $Y_{2}=Y_{0} B_{1} B_{2}$
- $Y_{k}=Y_{0} \prod_{i=1}^{k} B_{i}$
$Y_{i}$ is a stick-breaking process!



## Proof. Step 3: approximate stopping time

The above recursion works up to $Y_{\min }$, the minimum $y$-coordinate of points in the square.
$Y_{\text {min }}=O\left(\frac{1}{n}\right), \Rightarrow$ stop when remaining stick $<1 / n$.

Approximation Lemma. Let $Y_{i}$ be the $\operatorname{Beta}(2,1)$ stick-breaking sequence. Let $J_{n}=\inf \left\{i \geq 0: Y_{i} \leq n^{-1}\right\}$. Then

$$
\left|V_{n}-J_{n}\right|=\mathcal{O}_{P}(1)
$$

By classic renewal theory:

$$
\mathrm{E}\left(J_{n}\right)=\frac{2}{3} \log (n), \quad \operatorname{Var}\left(J_{n}\right)=\frac{10}{27} \log (n)
$$

## General F: proof idea



## What happens for tropicalized p-adic Gaussians

$F \sim \exp (1)$ (left) for $n=100$ vs $F \sim \operatorname{geometric}(1 / 2)$ (right) for $n=50$


Guess: need to generate $p$-adic polynomials with i.i.d coefficients but in a different basis (not as $\sum_{i} c_{i} x^{i}$ but $\left.\sum_{k} c_{k} f_{k}(x)\right)$. eg: $f_{k}(x)=\binom{x}{k}$ (Mahler basis).

## What happens when $d=2$



## What happens when $d=2$

Sarah Brodsky, Michael Joswig, Ralph Morrison and Bernd Sturmfels, Moduli of Tropical Plane Curves (2014)


Figure 1: Unimodular triangulation, tropical quartic, and skeleton
Some concrete open questions: see Exercises.

## IV. Applications

## The key idea of tropical geometry in applications

Tropical geometry translates geometric problems on piecewise-linear functions to questions on discrete convex geometry and combinatorics.


Geometry of [neural networks] $\rightarrow$ geometry of tropical polynomials $\rightarrow$ discrete convex geometry.

## Example 1. Number of linear regions in a ReLU network

A max-plus tropical polynomial $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is convex and piecewise-linear

$$
f(x)=\bigoplus_{a \in A \subset \mathbb{Z}^{d}} c_{a} \odot x^{\odot a}=\max _{a}\left(c_{a}+\langle a, x\rangle\right)
$$

The graph of its convex conjugate $f^{*}: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is the lower convex hull of $\left\{\left(a,-c_{a}\right): a \in A\right\}$

$$
f^{*}(w)=\sup _{x}(\langle w, x\rangle-f(x)) .
$$

(Visualize: upper hull of $\left(a, c_{a}\right)$. Regular subdivision $\Delta_{c}$ ) \# linear regions of $f=\#$ cells in $\Delta_{c}$.


## Example 1. Number of linear regions in a ReLU network

A max-plus tropical polynomial $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is convex and piecewise-linear

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f(x)=\bigoplus_{a \in A \subset \mathbb{Z}^{d}} c_{a} \odot x^{\odot a}=\max _{a}\left(c_{a}+\langle a, x\rangle\right)
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\# linear regions = 'expressiveness' of the network


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Deep $\gg$ shallow networks.

Example 2. Auction theory

## Combinatorial auctions

Many objects in auction theory and game theory are tropical.


The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2020 was awarded jointly to Paul R. Milgrom and Robert B. Wilson "for improvements to auction theory and inventions of new auction formats."

Applications:

- Radio spectrum, highway lanes, Bank of England crisis 2008
- Stable coalition, stable matching (eg: kidney donations, medical residency matching)
- Online auctions (Amazon, Google, etc)


## Single-item auction



## Multi-unit auction



- Utility $u^{j}: A^{j} \subset \mathbb{Z}^{n} \rightarrow \mathbb{R}, u^{j}(a)=$ bid for bundle $a$.


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- Demand $D_{u j}(p)=\arg \max _{a \in A^{i}}\left\{u^{j}(a)-p \cdot a\right\}$
- Aggregrated demand $D_{U}(p)=\left\{\sum_{j=1}^{J} a^{j}: a^{j} \in D_{u^{i}}(p)\right\}$.


## The problem: can we make everyone happy?



- Given $\left\{u^{j}, j=1, \ldots, J\right\}, a^{*} \in \mathbb{Z}^{n}$.
- Does there exist $p$ s.t. $a^{*} \in D_{U}(p)$ ?
- If Yes, say that we have competitive equilibrium at $a^{*}$.


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- NP-Hard in general (subset sum). Auction design = put conditions on $\left\{u^{j}\right\}$ and pricing rules $p$ to guarantee CE.


## Combinatorial auction and Tropical Geometry

Utility, profit and demand has tropical meanings

- $a \oplus b=\max \{a, b\}, a \odot b=a+b$
- $u: A \subset \mathbb{Z}^{n} \rightarrow \mathbb{R}$ defines tropical polynomial $f_{u}$

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f_{u}(-p)=\oplus_{a \in A} u(a) \odot(-p)^{\odot a}=\max _{a \in A}\{u(a)-a \cdot p\}
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- $T\left(f_{u}\right) \stackrel{\text { dual }}{\longleftrightarrow}$ regular subdivision $\Delta_{u}$ of $A$ :

$$
\Delta_{u}=\left\{D_{u}(p): p \in \mathbb{R}^{n}\right\}=\text { all possible demand sets. }
$$

## Example



## Example



## Example



## Example



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## Example



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## Example



## Multiple agents = product of tropical polynomials

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CE at $a^{*} \Leftrightarrow a^{*} \in D_{U}(p)$ for some $p \Leftrightarrow a^{*}$ is lifted by $U$ $\Leftrightarrow a^{*}$ is a marked point of $\Delta_{u}$.
$\mathrm{CE}($ everywhere $) \Leftrightarrow D_{U}(p)=\operatorname{conv}\left(D_{U}(p)\right) \cap \mathbb{Z}^{n}$
$\Leftrightarrow U$ is concave: $U=\operatorname{conv}(U)$ on $\operatorname{conv}(A) \cap \mathbb{Z}^{n}$

## Example (cont)

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5 not marked in $\Delta_{u} \Rightarrow$ no CE at $a^{*}=5$.

## Combinatorial (aka. product-mix) auctions in higher dimensions


$(1,1)$ is not marked in $\Delta_{U} \Rightarrow$ no CE at $a^{*}=(1,1)$.

## More interesting: $n \geq 2$ types of items

Active research directions

- Characterize equilibria
- Generalize to auctions with non-linear pricing
- Find counter-examples for certain classes of auctions ( ${ }^{*}$ )


## Some main theorems obtained by tropical geometry

Unimodularity Theorem. Baldwin \& Klemperer (2015), Danilov, Koshevoy and Murota (2001), Howard (2007). Exposition \& connections to Oda Conjecture: T. and Yu, 2015.

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Edin Husić, Georg Loho, Ben Smith, László A. Végh


#### Abstract

We characterize a rich class of valuated matroids, called R-minor valuated matroids that includes the indicator functions of matroids, and is closed under operations such as taking minors, duality, and induction by network. We refute the refinement of a 2003 conjecture by Frank, exhibiting valuated matroids that are not R-minor. The family of counterexamples is based on sparse paving matroids. Valuated matroids are inherently related to gross substitute valuations in mathematical economics. By the same token we refute the Matroid Based Valuation Conjecture by Ostrovsky and Paes Leme (Theoretical Economics 2015) asserting that every gross substitute valuation arises from weighted matroid rank functions by repeated applications of merge and endowment operations. Our result also has implications in the context of Lorentzian polynomials: it reveals the limitations of known construction operations.


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## CE for nonlinear pricings: Brandenburg, Haase, T.; 2021

In a multi-unit combinatorial auction, also known as product-mix auction [TY19, BK19], multiple agents can make simultaneous bids on multiple subsets of indivisible goods of distinct types. The running example in this paper is the Cutlery Auction of three items, a fork, a knife and a spoon, among three agents, Fruit, Spaghetti and Steak. Each is willing to pay at most 1 dollar for their favorite combination and no other: for Fruit, it is (knife, spoon), for Spaghetti, it is (fork, spoon), and for Steak, it is (fork, knife).

## Another application: computational complexity

## Log-barrier interior point methods are not strongly polynomial

Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, Michael Joswig

We prove that primal-dual log-barrier interior point methods are not strongly polynomial, by constructing a family of linear programs with $3 r+1$ inequalities in dimension $2 r$ for which the number of iterations performed is in $\Omega\left(2^{r}\right)$. The total curvature of the central path of these linear programs is also exponential in $r$, disproving a continuous analogue of the Hirsch conjecture proposed by Deza, Terlaky and Zinchenko. Our method is to tropicalize the central path in linear programming. The tropical central path is the piecewise-linear limit of the central paths of parameterized families of classical linear programs viewed through logarithmic glasses. This allows us to provide combinatorial lower bounds for the number of iterations and the total curvature, in a general setting.

(2017)

## Another application: phylogenetics

## Tropical Data Science

## Ruriko Yoshida

Phylogenomics is a new field which applies to tools in phylogenetics to genome data. Due to a new technology and increasing amount of data, we face new challenges to analyze them over a space of phylogenetic trees. Because a space of phylogenetic trees with a fixed set of labels on leaves is not Euclidean, we cannot simply apply tools in data science. In this paper we survey some new developments of machine learning models using tropical geometry to analyze a set of phylogenetic trees over a tree space.
(2020)



[^0]:    ${ }^{1}$ Steve Evans, Local fields, Gaussian measures, and Brownian motions, 2000
    ${ }^{2}$ Marianne Akian, Jean-Pierre Quadrat, Michel Viot, Bellman
    Processes,2000; and references therein
    ${ }^{3}$ T., Tropical Gaussians, a brief survey. arXiv:1808.10843

