Toric degenerations, Newton–Okounkov polytopes and tropical geometry

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Motivation

Toric varieties: Y an affine variety of dimension d is *toric* if it contains a dense torus $(k^*)^d$ whose action on itself extends to Y.

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Theorem

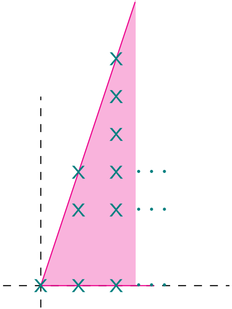
Every affine toric variety can be given in the following equivalent ways:

- *the closure of the image of* $\Phi_{\{a_1,\dots,a_s\}\subset\mathbb{Z}^n} : (k^*)^n \to k^s$ *given by* $(t_1,\dots,t_n) \mapsto (\mathrm{t}^{a_1},\dots,\mathrm{t}^{a_s})$
- *the spectrum of an affine semigroup algebra* $Y = Spec(k[S])$,
- *the vanishing set of a binomial prime ideal* $I \subset k[x_1,\dots,x_n]$,

where $S = \mathbb{N}(a_1,\dots,a_s) \subset \mathbb{Z}^n$ *and* $k[S] = k[x_1,\dots,x_n]/I$.

Example

- $\mathcal{A} = \{\binom{1}{2}, \binom{1}{3}, \binom{1}{0}\}$ then $\Phi_{\mathcal{A}} : (\mathbb{C}^*)^2 \to \mathbb{C}^3$ is given by
$$(t_1, t_2) \mapsto (t_1 t_2^2, t_1 t_2^3, t_1)$$
- $ker\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix} = \langle(-3, 2, 1)^T\rangle$ and
$$I = (y^2 z - x^3) \subset k[x, y, z]$$
- $S = \{a\binom{1}{2} + b\binom{1}{3} + c\binom{1}{0} : a, b, c \geq 0\}$ and
$$k[S] = k[t_1, t_1 t_2^2, t_1 t_2^3]$$



Motivation

Recall, an affine semigroup S can be realized as the lattice points inside a polyhedral cone $C_S = cone(S)$

properties of Y_S	combinatorics of C_S
dimension of Y_S	dimension of C_S
toric subvarieties	faces of C_S
Y_S is smooth	C_S is simplicial

Similarly, (normal) projective toric varieties are associated with polytopes, e.g. $P = conv(a_1, \dots, a_s)$ with $a_1, \dots, a_s$ inside a hyperplane and $C_P = cone(P)$, then $X_P = Proj(k[C_P \cap \mathbb{Z}^n])$.

For projective varieties we may have *toric degenerations* and hope to glean information from the toric variety (e.g. dimension, degree, Hilbert polynomial, a moment map?)

(Algebraic) Applications of toric degenerations

- Bernstein (2017): motivated by low-rank matrix completion uses toric degenerations of $Gr(2, n)$ to determine independent sets in the associated algebraic matroids
- Gross–Hacking–Keel–Kontsevich (2018): use toric degenerations of cluster varieties to prove several longstanding conjectures about cluster algebras
- Burr–Sottile–Walker (2021): use toric degenerations in a numerical homotopy continuation algorithm to solve systems of equations on an algebraic variety.
- Agostini–Fevola–Mandelshtam–Sturmfels (2021): use a toric degeneration to describe the Hirota variety that parametrizes certain tau-functions giving rise to solutions to the KP equation.
- Breiding–Michalek–Monin–Telen (in process): use Newton–Okounkov polytopes to verify the conjectured number of limit cycles in Duffing oscillators (via algebraic equations obtained from harmonic balancing)

Toric degenerations

Definition

Let X be a projective variety. A *toric degeneration* of X is a flat morphism $\xi : \mathfrak{X} \to \mathbb{A}^1$ whose *special fibre* $\xi^{-1}(0)$ is a toric variety and there is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$:

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$$\begin{tikzcd} \mathfrak{X} \setminus \xi^{-1}(0) \arrow[rr, "\sim"] \arrow[dr, "\xi"'] & & X \times (\mathbb{A}^1 \setminus \{0\}) \arrow[dl, two heads] \\ & \mathbb{A}^1 \setminus \{0\} & \end{tikzcd}$$

$\mathfrak{X}$ is a *family* of fibres, X is the *generic fibre*.

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Example: A toric degeneration can be *embedded*, for example $\mathfrak{X} = V(y^2z - x^3 + txz^2) \subset \mathbb{P}^2_{x:y:z} \times \mathbb{A}^1_t$.

Algebraic toric degenerations

An *algebraic* toric degeneration is equivalent to the data:

- a finitely generated positively graded $k[t]$-algebra $\mathfrak{R}$
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such that:

1. $\mathfrak{R}[t^{-1}] \cong R[t, t^{-1}]$ as $k[t]$-modules and graded algebras;
2. $R_0 := \mathfrak{R}/(t)$ is the algebra of a semigrup $k[S]$ where $S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$ finitely generated;
3. the action of k^* on $k[t]$ extends to $\mathfrak{R}$ respecting the grading and $k[t]$ acts as $(k^*)^d$ en R_0.

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Theorem (Kaveh–Manon–Murata arxiv 2017)

In this case there exixts a valuation $\nu : R \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$ whose image is S such that $R_0 = k[S]$ and $\mathfrak{R}$ is the Rees algebra of ν.

Toric degenerations via valuations

$R = \bigoplus_{i\geq 0} R_i$ a graded k-algebra and domain.
A map $\nu : R \setminus \{0\} \to (\mathbb{Z}^d, <)$ is a *(Krull-)valuation* if for $f, g \in R \setminus \{0\}$ and $c \in k$

$$\nu(fg) = \nu(f) + \nu(g), \quad \nu(cf) = \nu(f), \quad \nu(f+g) \geq \min{}_<\{\nu(f), \nu(g)\}$$

[1] $\Leftrightarrow R_\nu/\mathfrak{m}_\nu = k$, e.g. if ν is *full-rank*, i.e. $\mathrm{rank}(S) = \dim(R)$ using Abhyankar's inequality

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$\rightsquigarrow$ $S := \mathrm{im}(\nu)$ is a semigroup.

$\rightsquigarrow$ ν induces a filtration on R, for $m \in \mathbb{Z}^d$

$$F_m := \{f \in R : \nu(f) \leq m\} \quad \text{and} \quad F_{<m} := \{f \in R : \nu(f) < m\}.$$

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Proposition: If $\dim(F_m/F_{<m}) \leq 1$ for $m \in \mathbb{Z}^d$ [1] then

$$\mathrm{gr}_\nu(R) \cong k[S].$$

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Toric degenerations via valuations

Theorem (Anderson 2013)

Let $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ *be a full-rank Krull-valuation with* S *finitely generated.*

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Then there exists a toric degenerations of $X = Proj(R)$ with special fibre $X_0 = Proj(k[S])$ defined by the Rees algebra of ν:

$$\mathfrak{R} = \bigoplus_{i \geq 0} t^i F_{\leq i},$$

where $F_{\leq i} = \bigcup_{\pi(m) \leq i} F_m$ for a suitable projection $\pi : \mathbb{Z}^d \to \mathbb{Z}$.

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$\mathfrak{R}$ is a flat $k[t]$-algebra with

$$\mathfrak{R}/(t-1)\mathfrak{R} = R \quad \text{y} \quad \mathfrak{R}/t\mathfrak{R} = \mathrm{gr}_\nu(R).$$

Newton–Okounkov polytopes

X_0 is a projective toric variety $\Rightarrow$ exists a polytope defining its normalization $\bar{X}_0$, given by the *Newton–Okounkov polytope* of ν:

$$\Delta(R,\nu) := \operatorname{conv}\left(\bigcup_{i>0}\left\{\frac{\nu(f)}{i} : f \in R_i\right\}\right) \subset \mathbb{R}^d.$$

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Theorem (Kaveh–Khovanskii 2012, Lazarsfeld–Mustata 2009)

The number of lattice points in $\Delta(R,\nu)$ is n and the (normalized) volume of $\Delta(R,\nu)$ is the degree of $X \subset \mathbb{P}^{n-1}$.

Gröbner degenerations

Let $k = \bar{k}$ of $\mathrm{char}(k) = 0$ and $R = k[x_1, \ldots, x_n]/I$ where I is a homogeneous prime ideal.

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For $w \in \mathbb{R}^n$ we have the *initial ideal* $\mathrm{in}_w(I) := (\mathrm{in}_w(f) : f \in I)$ and a flat family

$$\xi_w : \mathfrak{X} \to \mathbb{A}^1$$

with generic fibre $X = \mathrm{Proj}(R)$ and special fibre $\mathrm{Proj}(R_w) = X_0$ where $R_w := k[x_1, \ldots, x_n]/\mathrm{in}_w(I)$.

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Example

For $I = (y^2z - x^3 + xz^2) \in k[x, y, z]$ and $w = (2, 3, 0) \in \mathbb{R}^3$ we have

$$\mathrm{in}_{(2,3,0)}(y^2z - x^3 + xz^2) = y^2z - x^3$$

which defins the flat family

$$\mathfrak{X} = \mathrm{Proj}\left(k[t][x, y, z]/(yz^2 - x^3 + txz^2)\right)$$

The Gröbner fan and the tropicalization of an ideal

Definition (Mora–Robbiano 1988)

The *Gröbner fan* $\mathrm{GF}(I)$ of the homogeneous ideal I is $\mathbb{R}^n$ with open cones defined by

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(I) = \mathrm{in}_w(I)$$

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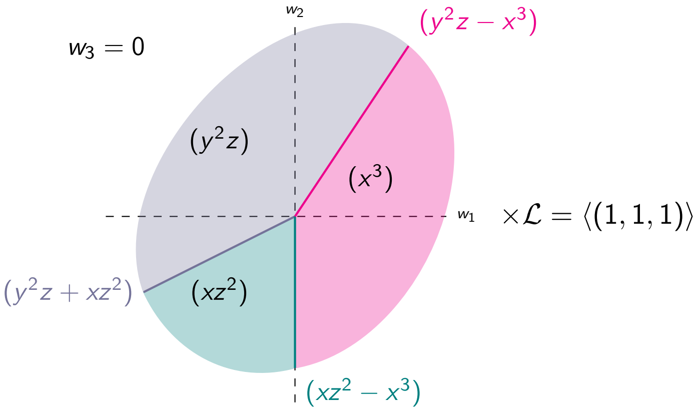
The *tropicalization* $\mathcal{T}(I)$ of I is the closed subfan of $\mathrm{GF}(I)$ consisting of $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(I)$ contains no monomials.

Example

Let $I = (y^2z - x^3 + xz^2) \subset \mathbb{C}[x, y, z]$. So, $GF(I)$ is $\mathbb{R}^3$ with the fan structure below and $\mathcal{T}(I)$ is its 1-skeleton:

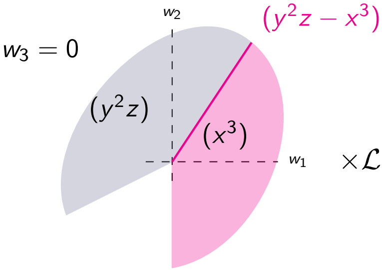


From tropicalization to Newton–Okounkov polytopes

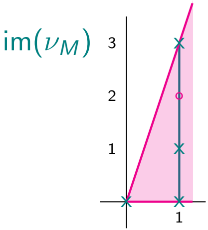
[Kaveh–Manon 2019]
Given $w \in \mathcal{T}(I)$ with $in_w(I)$ binomial and prime (i.e. $V(in_w(I))$ is toric) construct a valuation and its Newton–Okounkov polytope from $C \in \mathcal{T}(I)$ with $w \in C^\circ$:

1. $u_1, \dots, u_d$ linearly independent generators of $\langle C \rangle_{\mathbb{R}}$
2. let $M = [u_1 \dots u_d]$ then $\nu_M : k[x_1, \dots, x_n]/I \setminus \{0\} \to \mathbb{Z}^d$ given by $\nu_M(x_i) = M_i$ (i^{th} row) extends to a full-rank fin. gen. valuation with
$$\Delta(\nu_M) = \text{conv}(M_1, \dots, M_n).$$
3. Toric variety of $\Delta(\nu_M)$ is the normalization of $V(in_w(I))$.

Example



$\times \mathcal{L} = \langle (1,1,1) \rangle$



$I = (y^2z - x^3 + xz^2)$ and $in_{(2,3,0)}(I) = (y^2z - x^3)$ is toric.

Then $C = \langle (1,1,1), (2,3,0) \rangle \in \mathcal{T}(I)$ and $M := \left(\begin{smallmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{smallmatrix}\right)$.

So, $\nu_M : k[x,y,z]/I \setminus \{0\} \to \mathbb{Z}^2$ is given by

$$\nu_M(x) = (1,2),\ \nu_M(y) = (1,3),\ \nu_M(z) = (1,0)$$

From NO-polytopes to tropicalization

Theorem (B.'21, Kaveh–Manon '19)

Let R be a positively graded domain, $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ a full-rank valuation with finitely generated semigroup S.

From NO-polytopes to tropicalization

Theorem (B.'21, Kaveh–Manon '19)

Let R be a positively graded domain, $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ a full-rank valuation with finitely generated semigroup S. Then there exists an isomorphism of graded algebras

$$k[x_1, \ldots, x_n]/I \cong R$$

such that Anderson's toric variety $Proj(k[S])$ is isomorphic to the toric special fibre of a Gröbner degeneration for some $w \in \mathcal{T}(I) \subset \mathbb{R}^n$:

$$Proj(k[S]) \cong Proj(R_w).$$

Idea of the proof

Let $b_1, \ldots, b_n \in R$ be such that $\langle \nu(b_1), \ldots, \nu(b_n) \rangle = S \rightsquigarrow$ *Khovanskii basis*

Idea of the proof

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$$\pi : k[x_1, \dots, x_n] \to R, \quad x_i \mapsto b_i$$

so for $I := \ker(\pi)$ we have $R \cong k[x_1, \dots, x_n]/I$.

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so for $I := \ker(\pi)$ we have $R \cong k[x_1, \dots, x_n]/I$. For $M_\nu := (\nu(b_i)_j)_{ij} \in \mathbb{Z}^{d \times n}$ and $f = \sum \mathrm{x}^a c_a \in k[x_1, \dots, x_n]$

$$\mathrm{in}_{M_\nu}(f) := \sum_{b:\ M_\nu b = \min_{<_{lex}} \{M_\nu a : c_a \neq 0\}} \mathrm{x}^b c_b.$$

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For each $G \subset I$ with $\mathrm{in}_{M_\nu}(I) = (\mathrm{in}_{M_\nu}(g) : g \in G)$ there exists $\mathrm{pr} : \mathbb{Z}^d \to \mathbb{Z}$ such that for all $g = \sum \mathrm{x}^{a_i} c_i \in G$:

$$M_\nu a_i <_{lex} M_\nu a_j \quad \Rightarrow \quad \mathrm{pr}(M_\nu a_i) < \mathrm{pr}(M_\nu a_j).$$

Lemma: $w := \mathrm{pr}(M) \in \mathcal{T}(I)$ and $k[S] \cong k[x_1, \dots, x_n]/\mathrm{in}_w(I) = R_w$.

Outlook to differential algebra

- Aroca–Ilardi (2016): Newton's lemma for differential equations: using irrational weight vectors and initial terms define the Gröbner subdivision of a differential ideal
- Tropical Differential Algebra:
  - Aroca–Garay–Toghani (2016): Tropical fundamental theorem for differential algebraic geometry
  - Falkensteiner–Garay–Haiech–Noordman (2020): Tropical fundamental theorem for partial differential algebraic geometry
  - Hu-Gao (2021): Tropical differential Gröbner bases
  - Fink–Toghani (pre 2020): Initial forms and a notion of basis for tropical differential equations
  - Giansiracusa–Mereta (pre 2021): A general framework for tropical differential equations
- Binyamini (2017): Bezout and Bernstein–Kushnirenko–Khovanskii type theorems for systems of algebraic differential conditions over differentially closed fields.

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