The Marked Length Spectrum of Holomorphic Quadratic Differential metrics

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Furthermore, we get

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Let Λ be a subset of metrics on M. We say λ is spectrally rigid if $\mathcal{L}|_\Lambda$ is injective.

Standard results

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Theorem ($9g-9$)

If M is a closed surface of genus $g \geq 2$, then $\exists$ $9g-9$ closed curves such that

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Theorem (Otal-Croke, 90)

The set of negatively curved metrics on surface is spectrally rigid.

Higher dimensional case

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Theorem (Hamenstädt, 99)

Suppose $\dim M \geq 3$. Two negatively curved metrics g and g_0 have the same marked length spectrum, g_0 is locally symmetric. Then $g \cong g_0$.

Metrics with singularities

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Theorem (Hersonsky-Paulin, 97)

The set of negatively curved cone metrics on surface is spectrally rigid.

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The set of flat cone metrics on surface is spectrally rigid.

Let $S_r(p)$ be the circle around p of radius r. The cone angle at p is defined to be

$$\lim_{r \to 0} \frac{\ell(S_r(p))}{r}$$

Flat cone surface

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E.g. Glue an octagon.

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> $f(0) \neq 0$: let $\xi = \int \sqrt{f} dz$, $q = d\xi^2$.
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> $f(0) = 0$: suppose 0 is of order k, then $q = w^k dw^2$. Let $\eta = \frac{2}{k+2} w^{\frac{k+2}{2}}$, then $q = d\eta^2$ and the cone angle is $(k+2)\pi$.

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Proposition

A flat cone metric comes from a holomorphic quadratic differential if and only if

(1) every cone angle is of the form $k\pi, k \geq 3, k \in \mathbb{Z}$

(2) $\mathrm{Hol}(S) = \pm \mathrm{Id}$

Main result

Theorem

Let (S, σ) be a closed oriented flat cone surface of genus at least 2. If its marked length spectrum is integral, simple and recurrent, then σ comes from a holomorphic quadratic differential.

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Liouville current

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$\partial_\infty S$: visual boundary of the universal cover of S

$\mathcal{G}(\tilde{S}) = (S^1_\infty \times S^1_\infty \setminus \Delta)/(x, y) \sim (y, x)$

$\mathcal{G}^*(\tilde{S})$: closure of endpoints of nonsingular geodesics

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$\mathrm{supp}\, L = \mathcal{G}^*(\tilde{S})$

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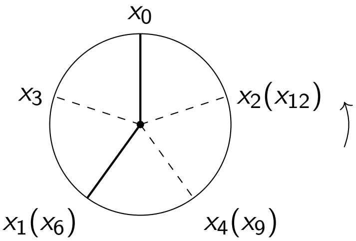

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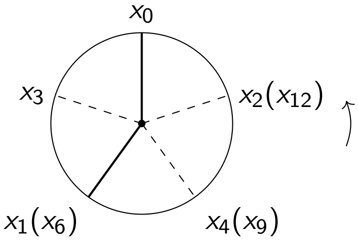

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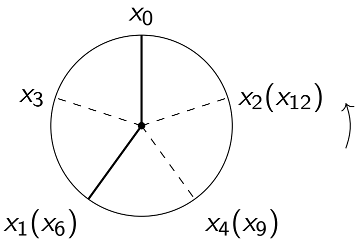

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Integral: all the cone angles are integer multiples of π

A special leaf

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Simple: $\forall (x, y) \in \operatorname{supp} L$, $\forall g \in \pi_1(S)$, (x, y) and $(g.x, g.y)$ don´t link

Recurrent: $\forall (x, y) \in \operatorname{supp} L$, the closure of its $\pi_1(S)$-orbit does not admit a proper closed $\pi_1(S)-$invariant subset

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Lemma

For a flat cone surface (S, σ), if the Liouville current is simple and recurrent, then there exists a nonsingular noncompact geodesic h in S such that every leaf in the closure of h contains at most one cone point.

Dense leaf

Theorem

If the Liouville current is simple and recurrent, then there exists a nonsingular simple geodesic whose closure is the entire surface S.

Holonomy

$v, w \in T^1(S)$, $v \sim w$ if they form an angle π. Each equivalent class is called a direction at p.

Idea: use the dense leaf to define a field of directions, invariant under parallel transport.

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