Optimal Hardy Inequality for Fractional Laplacians on the Integers

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The *classical Hardy inequality* may be written as

$$\langle \Delta \ \varphi, \varphi \rangle \geq \langle w \ \varphi, \varphi \rangle, \quad \varphi \in C_c(\mathbb{N})$$

where Δ is the standard Laplacian on $\mathbb{N}_0$ and

$$w \ (x) = \frac{1}{4x^2}$$

is the classical Hardy weight.

Introduction

The *fractional Hardy inequality* may be written as

$$\langle \Delta^{\sigma} \varphi, \varphi \rangle \geq \langle w_{\sigma} \varphi, \varphi \rangle, \quad \varphi \in C_c(\mathbb{Z})$$

where Δ^{σ} is the fractional Laplacian on $\mathbb{Z}$ and

$$w_{\sigma}(x) = \frac{C_{\sigma}}{|x|^{2\sigma}} + HOT$$

is the fractional Hardy weight.

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Ciaurri and Roncal showed that

$$w_\sigma(x) = 4^\sigma \frac{\Gamma(\frac{1+2\sigma}{4})^2}{\Gamma(\frac{1-2\sigma}{4})^2} \cdot \frac{\Gamma(|x| + \frac{1-2\sigma}{4})\Gamma(|x| + \frac{3-2\sigma}{4})}{\Gamma(|x| + \frac{3+2\sigma}{4})\Gamma(|x| + \frac{1+2\sigma}{4})} = \frac{C_\sigma}{|x|^{2\sigma}} + HOT$$

is a Hardy weight for Δ^σ, $0 < \sigma < 1/2$.

Set-Up – The Fractional Laplacian

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where

$$\kappa_\sigma(x) = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty e^{-t\Delta} 1_0(x) \frac{dt}{t^{1+\sigma}}$$

if $|x| > \sigma$ and 0 else.

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if $|x| > \sigma$ and 0 else. The integral also converges if $-1/2 < \sigma \leq 0$ allowing us to define κ_σ for $-1/2 < \sigma < 1$ and $\kappa_0 = 1_0$.

A non-zero function $w : \mathbb{Z} \longrightarrow [0, \infty)$ is called a *Hardy weight* if

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Theorem

The Hardy weight w_σ obtained by Ciaurri and Roncal is optimal.

Recall that we have defined the kernels κ_β for $-1/2 < \beta < 1$ as

$$\kappa_\beta(x) = \frac{1}{|\Gamma(-\beta)|} \int_0^\infty e^{-t\Delta} 1_0(x) \frac{dt}{t^{1+\beta}}, \quad |x| > \beta,$$

and 0 otherwise and $\kappa_0 = 1_0$.

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For $0 < \sigma \leq \alpha < 1/2$, *we have*

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A family of Hardy weights for Δ^σ is given by

$$w_{\sigma,\alpha} = \frac{\Delta^\sigma \kappa_{-\alpha}}{\kappa_{-\alpha}}.$$

Proof of Optimality

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Let $0 < \sigma \leq \alpha < 1/2$ *and* $w_{\sigma,\alpha} = \Delta^\sigma \kappa_{-\alpha}/\kappa_{-\alpha}$. *We furthermore set* $\alpha_0 = (1+2\sigma)/4$.

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b) *We have $\kappa_{-\alpha} \in \ell^2(\mathbb{Z}, w_{\sigma,\alpha})$ if and only if $\alpha < \alpha_0$.*

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One can show $\kappa_{-\alpha} \in O(x^{-1+2\alpha})$ and $w_{\sigma,\alpha} \in O(x^{-2\sigma})$. Hence,

$$\sum \kappa_{-\alpha}^2 w_{\sigma,\alpha} \asymp \sum x^{2(-1+2\alpha)-2\sigma} = \sum x^{(4\alpha-1-2\sigma)-1}.$$

$\square$

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For the “in particular” statement observe that $\kappa_{-\alpha}$ is harmonic for $(\Delta^\sigma - w_{\sigma,\alpha})$ and is thus the Agmon ground state. □

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Proof.

A computation shows $w_\sigma = w_{\sigma,\alpha}$ for $\alpha = \alpha_0$. The theorem above yields the claim. □