MEMBRANE MODELS:
VARIATIONAL ANALYSIS AND LARGE DEVIATIONS PRINCIPLE

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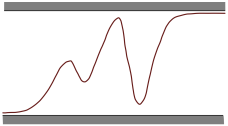
Deterministic model

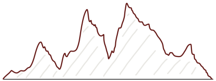
- Bending energy: $|\Delta|^2$-term
- Two attractive walls



Goal: describe the asymptotic behaviour of the energy when attraction to walls is much stronger than the bending rigidity of a membrane

Stochastic model

- Trajectory of a Gaussian process
- Covariance = Green function of Δ^2



Goal: prove Large Deviations principle and compute corresponding rate functional

Part I:

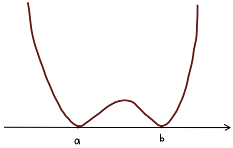
DETERMINISTIC MODEL

MODEL SETUP

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega \left(\frac{W(u(x))}{\varepsilon} + \varepsilon^3 |\Delta u(x)|^2 \right) dx & \text{if } u \in W^{2,2}(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus W^{2,2}(\Omega) \end{cases}$$

- $\Omega \subseteq \mathbb{R}^n$ open, bounded, Lipschitz domain
- $u : \Omega \to \mathbb{R}$ height profile
- $W : \mathbb{R} \to [0, \infty)$ continuous such that

$(H1)\ \ W(u) = 0 \ \Leftrightarrow\ u \in \{a, b\}$
$(H2)\ \ u > R \ \Rightarrow W(u) \ \geq Cu^2$



What is the suitable limit as $\varepsilon \to 0$ such that the variational structure is preserved **???**

- Family of minimum problems:

$$\min\{F_\varepsilon(u) : u \in X\}, \quad \varepsilon > 0$$

- Would like to have a simpler 'effective problem'

$$\min\{F(u) : u \in X\}$$

  such that it captures relevant behaviour of minimizers

- Ennio De Giorgi, early 70s

Definition

Let (X,d) be a metric space and let $F_n : X \to [-\infty,+\infty]$ be a family of functionals. We say that (F_n) $\Gamma-$*converges* to $F_\infty : X \to [-\infty,+\infty]$ if the following is satisfied

(i) "$\liminf$-inequality"

$$\forall x \in X \ \ \forall (x_n) \subseteq X \text{ such that } x_n \to x \text{ it holds } F_\infty(x) \leq \liminf_{n\to\infty} F_n(x_n)$$

(ii) "$\limsup$-inequality"

$$\forall x \in X \ \ \exists (x_n) \subseteq X \text{ such that } x_n \to x \text{ and } F_\infty(x) \geq \limsup_{n\to\infty} F_n(x_n).$$

We denote this by $F_\infty = \Gamma - \lim F_n$ or $F_n \xrightarrow{\Gamma} F_\infty$.

"$\liminf$-inequality"

$$\forall x \in X \;\; \forall (x_n) \subseteq X \text{ such that } x_n \to x \text{ it holds } F_\infty(x) \le \liminf_{n\to\infty} F_n(x_n)$$

"$\limsup$-inequality"

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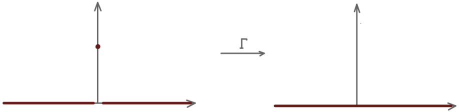
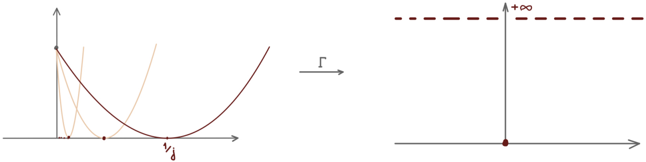




(!) Under additional compactness assumptions $\Gamma-$convergence implies *convergence of minimizers.*

Theorem

Let $\Omega \subset \mathbb{R}^n$ *open, bounded, Lipschitz domain. Let* $a, b \in \mathbb{R}$ *and* $W : \mathbb{R} \to [0, +\infty]$ *continuous such that*

(H1) $W(u) = 0 \;\Leftrightarrow\; u \in \{a, b\}$

(H2) $u > R \;\Rightarrow W(u) \;\geq Cu^2$.

Consider the family $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ *given by*

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^3 |\Delta u|^2 \right) dx & \text{if } u \in W^{2,2}(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus W^{2,2}(\Omega) \end{cases}$$

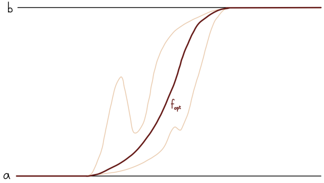
Then it holds

$$\Gamma(L^1) - \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u) = \mathcal{F}(u) := \begin{cases} \mathbf{m} \mathit{Per}_\Omega(\{u = a\}) & \text{if } u \in BV(\Omega; \{a, b\}) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\mathbf{m} = \min \left\{ \int_\mathbb{R} W(f) + |f''|^2 dt : f \in W^{2,2}_{loc}(\mathbb{R}), \lim_{t \to +\infty} f(t) = b, \lim_{t \to -\infty} f(t) = a \right\}$$

Constant **m** is defined as the smallest energy needed for a membrane to cross from level a to level b:

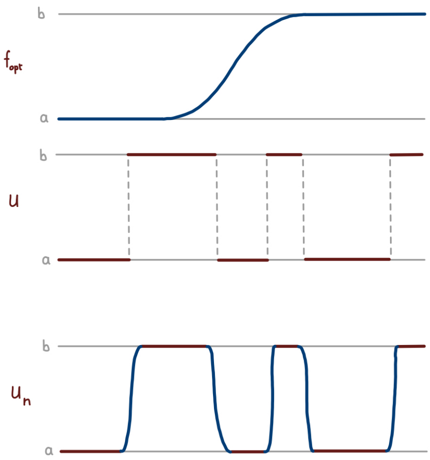


More precisely:

$$\mathcal{A} := \left\{ f \in W^{2,2}_{\text{loc}}(\mathbb{R}) : f(t) = b \text{ if } t > C, f(t) = a \text{ if } t < -C, \text{ for some } C > 0 \right\}$$

$$\mathbf{m} := \inf \left\{ \int_{-\infty}^{+\infty} W(f) + (f''(t))^2 dt : f \in \mathcal{A} \right\} \tag{2.1}$$

- Proof for dimension $n = 1$ follows closely [Fonseca & Mantegazza, 2001]
- $\liminf$-inequality in higher dimensions uses blow-up type argument as in [Hilhorst, Peletier & Schätzle, 2002] and [Burger, Esposito & Zeppieri, 2015]
- $\limsup$-inequality requires a recovery sequence constructed using optimal profile



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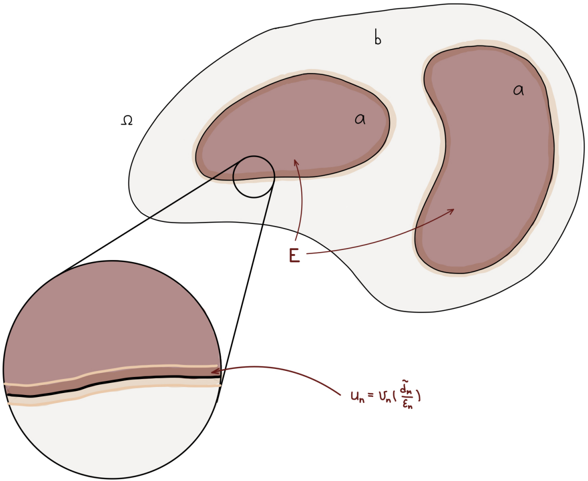
b
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$u_n = v_n(\frac{\tilde{d}_n}{\varepsilon_n})$

Proposition

Let $(u_\varepsilon) \subseteq W^{2,2}(\Omega)$ *with* $\frac{\partial u_\varepsilon}{\partial \nu} = 0$ *on* $\partial\Omega$ *such that*

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty.$$

Then there exists a subsequence satisfying

$$u_{\varepsilon_n} \xrightarrow{L^1} u \in BV(\Omega; \{a, b\}).$$

- Follows from the bound introduced in **[Fonseca et.al. 2016]**

$$\|\nabla^2 u\|_{L^2}^2 \leq 3\|\Delta u\|_{L^2}^2 + c(\Omega)\|u\|_{L^2}^2$$

  and compactness result from **[Fonseca, Mantegazza, 2000]**
- Remark: Neumann-boundary conditions can be omitted!

- $\Gamma-$limit for the functional modeling the membrane between two attractive walls of the perimeter type
- Compactness (with and without Neumann-boundary conditions)

$\Rightarrow$ Convergence of minimizers, description with optimal profile

- In the meantime: also obtained $\Gamma-$convergence result on a smaller space

$$\{u \in W^{2,2}(\Omega) : \frac{\partial u}{\partial \nu} = 0, \text{ a.e. on } \partial\Omega\}$$

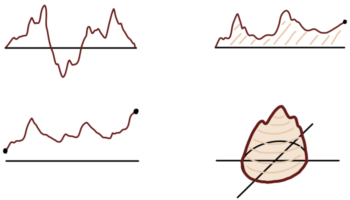
(useful for a model for pattern formation in biomembranes)

Part II:

STOCHASTIC MODEL

MODEL DESCRIPTION

- Starting point: Gaussian process with covariance being the Green function of Δ^2-operator
- Several cases and constraints:



- Asymptotic behaviour of a scaled family of such processes?

Definition

We say that $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a *Large Deviations Principle* with a rate functional I if

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(U) \geq -\inf_{x\in U} I(x), \quad \forall U \subseteq \mathcal{X} \text{ open}$$
$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \leq -\inf_{x\in G} I(x), \quad \forall G \subseteq \mathcal{X} \text{ closed.}$$

(!) Notice that then μ_ε *concentrates around minimizers* of the rate functional I as $\varepsilon \to 0$.

OVERVIEW OF THE RESULTS

LDP was obtained for all cases with the rate functional of the general form

$$I(\varphi) = \frac{1}{2}\int_0^1 (\Delta\varphi)^2 dt - \inf_{\phi\in\mathcal{A}} \left(\frac{1}{2}\int_0^1 (\Delta\phi)^2 dx \right)$$

where $\mathcal{A}$ denotes the set of trajectories with imposed boundary / integral constraints.

OVERVIEW OF THE RESULTS

Membrane	Modeling process	I	LDP derivation
Zero bdry cond. on left side	$\psi_L(t) = \int_0^t \beta(s)ds$	$I(\varphi) = \frac{1}{2}\int_0^1 (\varphi'')^2$ for $\varphi \in W^{2,2}(0,1)$ $\varphi(0) = \varphi'(0) = 0$	Legendre transform or Schilder Thm. and Contraction pr.
Zero bdry cond. on both sides	$\tilde{\psi}_L(t) = \psi_L(t) - ((3\frac{t}{L})^2 - 2(\frac{t}{L})^3)\psi_L(L)$ $-\frac{t^2}{L}(\frac{t}{L} - 1)\beta(L)$	$\tilde{I}(\varphi) = \frac{1}{2}\int_0^1 (\varphi'')^2$ for $\varphi \in W^{2,2}(0,1)$ $\varphi(0) = \varphi'(0) = \varphi(1) = \varphi'(1) = 0$	Legandre transform or conditioning on bdry values
Fixed $\neq 0$ bdry cond. on right	$\tilde{\psi}_{h,g,L}(t) = \psi_L(t) - ((3\frac{t}{L})^2 - 2(\frac{t}{L})^3)\psi_L(L)$ $-\frac{t^2}{L}(\frac{t}{L} - 1)\beta(L) + (3h - gL)(\frac{t}{L})^2 + (gL - 2h)(\frac{t}{L})^3$	$\tilde{I}^{h,g}(\varphi) = I(\varphi) - \inf_{A_0^h \cap B_0^g} I$ $= I(\varphi) - 6h^2 + 6hg - 2g^2$ for $\varphi \in C^{1,\alpha}_{h,g}([0,1])$	Conditioning on bdry values
Volume constraint and bdry constr.	$\mu_L^{h,g,V}(\cdot) = \mathbb{P}\{\tilde{\psi}^{(L)} \in \cdot \mid \sigma(Y)\}$	$\tilde{I}^{g,h,U}(\varphi) = I(\varphi) - \inf_{A_0^h \cap B_0^g \cap D_0^U} I$ for $\varphi \in C^{1,\alpha}_{h,g}([0,1]) \cap C_0^V$ and $\varphi^*(x) =$ $= \frac{3}{2}(g - 8h = 20V)x^2 - 4(g - 7h + 15V)x^3 + \frac{5}{2}(g - 6h + 12V)x^4$	Conditioning on bdry values and integral constraint
Quadratic integral constraint and bdry constr.	$\mu_L^{h,g,V}(\cdot) = \mathbb{P}\{\tilde{\psi}^{(L)} \in \cdot \mid \sigma(Y)\}$	$\tilde{I}^{g,h,U}(\varphi) = I(\varphi) - \inf_{A_0^h \cap B_0^g \cap D_0^U} I$ for $\varphi \in C^{1,\alpha}_{h,g}([0,1])$	Conditioning on bdry values and integral constraint
2D membrane, zero-bdry cond.	$\varphi =$ centered Gaussian process, covariance = Green fct. of Δ^2	$I(\varphi) = \frac{1}{2}\int_D (\Delta\varphi(t))^2 dt$ for $\varphi \in W^{2,2}(D)$ and $\varphi\vert_{\partial D} = 0$	Legendre transform or conditioning on bdry values
2D membrane, volume constraint	$\mu_L^V(\cdot) := \mathbb{P}\{\varphi \in \cdot \mid \sigma(Y)\}$	$I^V(\varphi) = I(\varphi) = \inf_A I$ if $\varphi \in \mathcal{A}$	Conditioning on bdry values and integral constraint
1D membrane, tilted measure	$d\mu^{(L),F} := \frac{1}{Z^{(L),F}} e^{LF(\varphi^{(L)})} d\mu^{(L)}$ for $F(\varphi) := -\frac{1}{2}\int_0^1 W(\varphi(t))dt$	$I^W(\varphi) = \frac{1}{2}\int_0^1 (\varphi'')^2 + W(\varphi)dt - \inf_{\psi \in C^{1,\alpha}([0,1])}[\frac{1}{2}\int_0^1 (\psi'')^2 + W(\psi)dt]$	Varadhan's Lemma

THANK YOU FOR YOUR ATTENTION!

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