INTRODUCTION TO DETERMINISTIC AND RANDOM DISPERSIVE EQUATIONS

(CURRENTLY UNDER CONSTRUCTION)

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ABSTRACT. In this lecture series we discuss linear and nonlinear Schrödinger equations. Schrödinger equations belong to the class of dispersive equations, which is ubiquitous in mathematical physics and also includes wave equations and the Korteweg-de Vries equation.

In the first half of this lecture series, we present deterministic aspects of Schrödinger equations. In particular, we discuss Strichartz estimates and Bourgain spaces. In the second half, we present probabilistic aspects of Schrödinger equations. In particular, we prove multi-linear dispersive estimates for random initial data, which combine both dispersive and probabilistic cancellations. Furthermore, we discuss a recent random tensor estimate of Deng, Nahmod, and Yue, which has already been useful in many applications.

If time permits, we end the lecture series with a brief discussion of important open problems.

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1. Introduction

In the following four lectures, we study dispersive equations with both deterministic and random initial data. Throughout the lectures, I will focus on linear and nonlinear Schrödinger equations on the torus, which are important examples of nonlinear dispersive equations. They can be written as

$$\begin{cases} i\partial_t u + \Delta u = \sigma |u|^2 u & (t,x) \in \mathbb{R} \times \mathbb{T}^d, \\ u|_{t=0} = \phi, \end{cases} \tag{NLS}$$

where $d \geq 1$ is the spatial dimension and $\sigma \in \{-1, 0, 1\}$. The focus of the four lectures will lie on methods and techniques, and I hope to cover the following ones:

(i) Strichartz estimates
(ii) Bourgain spaces
(iii) Probabilistic multi-linear dispersive estimates
(iv) Random tensor estimates

As applications of (i)-(iv), I will discuss the following two classical theorems.

Theorem 1.1 (Deterministic local well-posedness, [Bou93])**.** *The periodic cubic nonlinear Schrödinger equation in two dimensions ($\sigma = \pm 1$, $d = 2$) is locally well-posed in $H^s_x(\mathbb{T}^2)$ for all $s > 0$.*

In Proposition 3.8, which will be proven in full detail in the lectures and tutorial sessions, we obtain a more quantitative version of Theorem 1.1.

Theorem 1.2 (Probabilistic local well-posedness, [Bou96])**.** *Let $d = 2$, let $\sigma = \pm 1$, let $(g_n)_{n \in \mathbb{Z}^d}$ be a family of independent, standard, complex-valued Gaussians, and let*

$$\phi = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle} e^{inx}. \tag{1.1}$$

Then, (NLS) *almost surely has a local solution.*

Remark 1.3. As we will see, ϕ from (1.1) lives in $H^{-\epsilon}(\mathbb{T}^2) \backslash L^2(\mathbb{T}^2)$ for all $\epsilon > 0$. This form of random initial data is relevant for proving the invariance of the Gibbs measure corresponding to (NLS), but this will not be covered in these lectures.

Due to lack of time, we won't prove Theorem 1.2 in full detail. However, we will see two of the main steps (see Proposition 5.1 and Proposition 6.1).

Remark 1.4. The two main theorems are rather classical (and chosen for simplicity), but the techniques are still current and some are quite recent. For instance, the random tensor estimate was developed in [DNY22]. In the last five years, research in this area has been very active, and there are many problems which can just now be solved.

Comment: These lecture notes were prepared for the summer school *"Summer School on PDEs and Randomness"* at the MPI in Leipzig. Since the lectures at the summer school were accompanied by tutorial sessions, the lecture notes contain plenty of exercises for the students.

Notation:

All parts of this manuscript that are colored in blue will either be mentioned verbally or skipped entirely during the actual lectures.

For $\phi : \mathbb{T}^d \to \mathbb{C}$, we define $\widehat{\phi} : \mathbb{Z}^d \to \mathbb{C}$ by

$$\widehat{\phi}(n) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathrm{d}x \, \phi(x) e^{-inx}.$$

Furthermore, for all $\phi : \mathbb{T}^d \to \mathbb{C}$, $a : \mathbb{Z}^d \to \mathbb{C}$, and $1 \leq p < \infty$, we define

$$\|\phi\|_{L^p(\mathbb{T}^d)} := \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathrm{d}x \, |\phi(x)|^p \right)^{\frac{1}{p}} \qquad \text{and} \qquad \|a\|_{\ell^p(\mathbb{Z}^d)} := \left(\sum_{n \in \mathbb{Z}^d} |a_n|^p \right)^{\frac{1}{p}}.$$

Using the above definitions, the Fourier inversion formula and Plancherell's identity are then given by

$$\phi(x) = \sum_{n \in \mathbb{Z}^d} \widehat{\phi}(n) e^{inx} \qquad \text{and} \qquad \|\phi\|_{L^2(\mathbb{T}^d)} = \|\widehat{\phi}\|_{\ell^2(\mathbb{Z}^d)}.$$

For any $n \in \mathbb{Z}^d$, we write

$$|n| := \|n\|_2 = \Big(\sum_{j=1}^d n_j^2 \Big)^{\frac{1}{2}}.$$

For all $N \in 2^{\mathbb{N}_0}$, we define the frequency-projection operators

$$P_{\leq N} \phi = \sum_{n \in \mathbb{Z}^d} \mathbf{1}\Big\{ |n| \leq N \Big\} \widehat{\phi}(n) e^{inx}. \tag{1.2}$$

Furthermore, we define

$$P_1 := P_{\leq 1} \qquad \text{and} \qquad P_N = P_{\leq N} - P_{\leq N/2} \quad \text{for all } N \geq 2.$$

While sharp frequency-projections such as (1.2) behave poorly on L^p-spaces (see e.g. [Fef71]), all of our arguments are L^2-based and this, therefore, causes no problems.

2. Strichartz estimates

In this section we study Strichartz estimates, which are estimates for solutions of linear dispersive equations. To this end, consider

$$\begin{cases} i\partial_t u + \Delta u = 0 & (t,x) \in \mathbb{R} \times \mathbb{T}^d, \\ u\big|_{t=0} = \phi. \end{cases} \tag{2.1}$$

The solution of (2.1) is given by

$$u = e^{it\Delta}\phi = \sum_{n\in\mathbb{Z}^d} e^{-it|n|^2} e^{inx}\widehat{\phi}(n), \tag{2.2}$$

where $\widehat{\phi}: \mathbb{Z}^d \to \mathbb{C}$ is the Fourier transform of ϕ. We emphasize that, in contrast to the heat flow, the Schrödinger flow preserves the size of the Fourier coefficients and is therefore not smoothing. The effect of the Schrödinger flow is more subtle, and hidden in the oscillation of $n \in \mathbb{Z}^d \mapsto e^{-it|n|^2}$.

Question: Can we use the oscillation of $n \in \mathbb{Z}^d \mapsto e^{-it|n|^2}$ to obtain space-time bounds for $e^{it\Delta}\phi$? In our estimate, we only want bounds by $\|\phi\|_{L^2_x}$ or possibly $\|\phi\|_{H^s_x}$.

In general, we do not expect any gain for a fixed time $t_0 \in \mathbb{R}$. The reason is that $\widehat{\phi}$ may oscillate like $e^{it_0|n|^2}$, and then there is no oscillation in $e^{-it_0|n|^2}\widehat{\phi}(n)$. For instance, this can happen if $\phi = e^{-it_0\Delta}\psi$ and $\psi: \mathbb{T}^d \to \mathbb{C}$ has a slowly-varying Fourier transform. However, $\widehat{\phi}$ cannot oscillate like $e^{it|n|^2}$ for every $t \in \mathbb{R}$, and thus we expect a gain after averaging in time.

Theorem 2.1 (Periodic $L^4_t L^4_x$-Strichartz estimate, [Bou93])**.** *Let $d \geq 2$ and $N \in 2^{\mathbb{N}_0}$. For all $\epsilon > 0$ and all $\phi \in L^2_x(\mathbb{T}^d)$, it holds that*

$$\big\| P_{\leq N} e^{it\Delta}\phi \big\|_{L^4_t L^4_x([0,2\pi]\times\mathbb{T}^d)} \lesssim_\epsilon N^\epsilon N^{\frac{d-2}{4}} \big\|\phi\big\|_{L^2_x(\mathbb{T}^d)}. \tag{2.3}$$

Remark 2.2. We make the following remarks regarding Theorem 2.1.

(i) Deterministic estimates, such as the Strichartz estimate (2.3), are often easier to prove on the Euclidean space $\mathbb{R}^d$ than on the torus $\mathbb{T}^d$. However, the probabilistic theory is much better understood on $\mathbb{T}^d$, which is our reason for focusing on this case (cf. Problem 7.2).

(ii) Using Sobolev embedding, we easily obtain that

$$\big\| P_{\leq N} e^{it\Delta}\phi \big\|_{L^\infty_t L^4_x([0,2\pi]\times\mathbb{T}^d)} \lesssim N^{\frac{d}{4}} \big\|\phi\big\|_{L^2_x(\mathbb{T}^d)}.$$

This controls a stronger norm, but also requires more powers of N.

(iii) Let $\widehat{\phi}(n) = N^{-d/2}\mathbf{1}\big\{|n| \leq N\big\}$, which satisfies $\big\|\phi\big\|_{L^2_x} \sim 1$. It is easy to see that

$$\big|e^{it\Delta}\phi\big|(x) \gtrsim N^{-\frac{d}{2}} N^d \mathbf{1}\big\{|t| \ll N^{-2}, |x| \ll N^{-1}\big\}.$$

As a result, it follows that

$$\big\|e^{it\Delta}\phi\big\|_{L^4_t L^4_x([0,2\pi]\times\mathbb{T}^d)} \gtrsim N^{-\frac{d}{2}} N^d \big(N^{-2-d}\big)^{\frac{1}{4}} = N^{\frac{d-2}{4}}.$$

This shows that, up to the ϵ-loss, (2.3) is optimal.

(iv) The ϵ-loss is necessary in dimension $d = 2$ (see e.g. [Dem20, Section 13]), but can be removed (in the $L^4_t L^4_x$-estimate) in dimension $d \geq 3$ (see [KV16]).

(v) The optimal range of $L_t^q L_x^p$-estimates on rational and irrational tori is still being investigated, see e.g. [BD15, DJLM23].

We split the proof of Theorem 2.1 into two steps. In the first step, we reduce the Strichartz estimate to a lattice point counting estimate. In the second step, we then prove the aforementioned lattice point counting estimate.

2.1. **Reduction.** Throughout these lectures, we will encounter multiple lattice point counting problems. Our first counting problem will be captured by the following definition.

Definition 2.3. For any $q, d \geq 1$ and $K \in 2^{\mathbb{N}_0}$, we define

$$\mathcal{M}_{q,d}(K) := \sup_{k \in \mathbb{Z}^d} \sup_{\mu \in \mathbb{Z}} \#\Big\{(k_1, k_2, \cdots, k_q) \in (\mathbb{Z}^d)^q : \max_{1 \leq j \leq q} |k_j| \leq K, \sum_{j=1}^{q} k_j = k, \sum_{j=1}^{q} |k_j|^2 = \mu\Big\}. \tag{2.4}$$

The notation "$\mathcal{M}_{q,d}$" is in line with [BDNY22, Lemma 5.4] and the "$\mathcal{M}$" stands for "molecule".

Lemma 2.4 (Reduction)**.** For all $d \geq 1$, $N \in 2^{\mathbb{N}_0}$, and $\phi \in L^2(\mathbb{T}^d)$, it holds that

$$\Big\| P_{\leq N} e^{it\Delta} \phi \Big\|_{L_t^4 L_x^4([0,2\pi] \times \mathbb{T}^d)} \lesssim \Big(\mathcal{M}_{2,d}(N)\Big)^{1/4} \|\phi\|_{L_x^2(\mathbb{T}^d)}. \tag{2.5}$$

Proof. We first introduce the set

$$\begin{aligned} S_N := \Big\{ &(n_0, n_1, n_2, n_3) \in (\mathbb{Z}^d)^4 : \max_{0 \leq j \leq 3} |n_j| \leq N, \\ &- n_0 + n_1 - n_2 + n_3 = 0,\ |n_0|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2 = 0 \Big\}. \end{aligned}$$

Then, we write

$$\begin{aligned} &\Big\| P_{\leq N} e^{it\Delta} \phi \Big\|^4_{L_t^4 L_x^4([0,2\pi] \times \mathbb{T}^d)} \\ &= \frac{1}{(2\pi)^{d+1}} \int\limits_{[0,2\pi]\times\mathbb{T}^d} \mathrm{d}t \mathrm{d}x\, \overline{P_{\leq N} e^{it\Delta}\phi} \cdot P_{\leq N} e^{it\Delta}\phi \cdot \overline{P_{\leq N} e^{it\Delta}\phi} \cdot P_{\leq N} e^{it\Delta}\phi \\ &= \frac{1}{(2\pi)^{d+1}} \sum_{\substack{n_0,n_1,n_2,n_3 \in \mathbb{Z}^d: \\ |n_j| \leq N}} \Bigg(\int\limits_{[0,2\pi]\times\mathbb{T}^d} \mathrm{d}t \mathrm{d}x \Big(e^{it(|n_0|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2)} e^{i(-n_0+n_1-n_2+n_3)x} \Big) \\ &\quad \times \overline{\widehat{\phi}(n_0)} \cdot \widehat{\phi}(n_1) \cdot \overline{\widehat{\phi}(n_2)} \cdot \widehat{\phi}(n_3) \Bigg). \end{aligned} \tag{2.6}$$

Now, since complex exponentials have mean-zero, calculating the (t,x)-integral in (2.6) yields the identity

$$\Big\| P_{\leq N} e^{it\Delta} \phi \Big\|^4_{L_t^4 L_x^4([0,2\pi] \times \mathbb{T}^d)} = \sum_{n_0,n_1,n_2,n_3 \in \mathbb{Z}^d} \mathbf{1}_{S_N} \cdot \overline{\widehat{\phi}(n_0)} \cdot \widehat{\phi}(n_1) \cdot \overline{\widehat{\phi}(n_2)} \cdot \widehat{\phi}(n_3). \tag{2.7}$$

Using Cauchy-Schwarz and Plancherell's identity, it follows that

$$\begin{aligned} (2.7) &\leq \Bigg(\sum_{n_0,n_1,n_2,n_3 \in \mathbb{Z}^d} \mathbf{1}_{S_N} \Big| \overline{\widehat{\phi}(n_0) \widehat{\phi}(n_2)} \Big|^2 \Bigg)^{\frac{1}{2}} \times \Bigg(\sum_{n_0,n_1,n_2,n_3 \in \mathbb{Z}^d} \mathbf{1}_{S_N} \Big| \widehat{\phi}(n_1) \widehat{\phi}(n_3) \Big|^2 \Bigg)^{\frac{1}{2}} \\ &\lesssim \sup_{n_0,n_2 \in \mathbb{Z}^d} \Big(\sum_{n_1,n_3 \in \mathbb{Z}^d} \mathbf{1}_{S_N} \Big)^{\frac{1}{2}} \times \sup_{n_1,n_3 \in \mathbb{Z}^d} \Big(\sum_{n_0,n_2 \in \mathbb{Z}^d} \mathbf{1}_{S_N} \Big)^{\frac{1}{2}} \times \|\phi\|^4_{L_x^2}. \end{aligned} \tag{2.8}$$

Using the definition of S_N, it easily follows that

$$\sup_{n_0,n_2\in\mathbb{Z}^d}\sum_{n_1,n_3\in\mathbb{Z}^d}\mathbf{1}_{S_N}\lesssim\mathcal{M}_{2,d}\qquad\text{and}\qquad\sup_{n_1,n_3\in\mathbb{Z}^d}\sum_{n_0,n_2\in\mathbb{Z}^d}\mathbf{1}_{S_N}\lesssim\mathcal{M}_{2,d}.\tag{2.9}$$

After inserting (2.9) into (2.8), we obtain the desired inequality (2.5). □

Remark 2.5. At a technical level, it is more convenient to replace (2.6) with the identity

$$\begin{aligned}\left\|P_{\leq N}e^{it\Delta}\phi\right\|^4_{L^4_tL^4_x([0,2\pi]\times\mathbb{T}^d)}&=\left\|(P_{\leq N}e^{it\Delta}\phi)^2\right\|^2_{L^2_tL^2_x([0,2\pi]\times\mathbb{T}^d)}\\&\sim\sum_{n\in\mathbb{Z}^d}\sum_{\mu\in\mathbb{Z}}\Bigg|\sum_{\substack{n_1,n_2\in\mathbb{Z}^d:\\|n_1|,|n_2|\leq N}}\mathbf{1}\big\{n_1+n_2=n\big\}\mathbf{1}\big\{|n_1|^2+|n_2|^2=\mu\big\}\widehat{\phi}(n_1)\widehat{\phi}(n_2)\Bigg|^2.\end{aligned}$$

However, (2.6) is closer to the arguments in Section 5, which is the reason for our presentation.

2.2. Lattice point counting estimate.

Proposition 2.6 (Sphere estimate)**.** Let $\epsilon>0$, let $d\geq 2$, and $K\in 2^{\mathbb{N}_0}$. Then, it holds that

$$\sup_{c\in\mathbb{Z}^d}\sup_{\mu\in\mathbb{Z}}\#\Big\{k\in\mathbb{Z}^d\colon|k|\lesssim K,\ |k-c|^2=\mu\Big\}\lesssim_\epsilon K^\epsilon K^{d-2}.\tag{2.10}$$

We first show that Proposition 2.6 implies the desired bound on $\mathcal{M}_{2,d}$.

Corollary 2.7 (Bound on $\mathcal{M}_{2,d}$)**.** For all $\epsilon>0$ and $K\in 2^{\mathbb{N}_0}$, it holds that

$$\mathcal{M}_{2,d}(K)\lesssim_\epsilon K^\epsilon K^{d-2}\tag{2.11}$$

Proof. Recall that

$$\mathcal{M}_{2,d}(K)=\sup_{k\in\mathbb{Z}^d}\sup_{\mu\in\mathbb{Z}}\#\Big\{(k_1,k_2)\in\mathbb{Z}^d\times\mathbb{Z}^d\colon|k_1|,|k_2|\leq K,\ k_1+k_2=k,\ |k_1|^2+|k_2|^2=\mu\Big\}.$$

Now, if $k=k_1+k_2$, then

$$|k_1|^2+|k_2|^2=|k_1|^2+|k-k_1|^2=\frac{1}{2}\big|2k_1-k\big|^2+\frac{1}{2}\big|k\big|^2.$$

Thus, using the change of variables $k_1':=2k_1$ and the sphere estimate (Proposition 2.6), it follows that

$$\begin{aligned}\mathcal{M}_{2,d}(K)&\lesssim\sup_{k\in\mathbb{Z}^d}\sup_{\mu\in\mathbb{Z}}\#\Big\{k_1\in\mathbb{Z}^d\colon|k_1|\leq K,\big|2k_1-k\big|^2=2\mu-\big|k\big|^2\Big\}\\&\leq\sup_{k\in\mathbb{Z}^d}\sup_{\mu\in\mathbb{Z}}\#\Big\{k_1'\in\mathbb{Z}^d\colon|k_1'|\leq 2K,\big|k_1'-k\big|^2=2\mu-\big|k\big|^2\Big\}\lesssim_\epsilon K^\epsilon K^{d-2}.\end{aligned}$$ □

Before proving Proposition 2.6, we need two auxiliary lemmas.

Lemma 2.8 (Divisor estimate)**.** Let $\mathcal{R}=\mathbb{Z}$ or $\mathcal{R}=\mathbb{Z}[i]$, i.e., the ring of Gaussian integers. For any $\mu\in\mathcal{R}\backslash\{0\}$, let $d(\mu)$ be the number of divisors of μ, i.e.,

$$d(\mu)=\#\big\{(a,b)\in\mathcal{R}\times\mathcal{R}\colon a\cdot b=\mu\big\}.$$

Then, it holds for all $\epsilon>0$ that

$$d(\mu)\lesssim_\epsilon|\mu|^\epsilon.$$

The proof of Lemma 2.8 is the subject of Exercise 2.12, but the main idea is simple: If, say, $\mathcal{R} = \mathbb{Z}$ and we have the prime-factorization

$$\mu = \prod_{j=1}^{m} p_j^{a_j}, \tag{2.12}$$

then the number of divisors is given explicitly by

$$d(\mu) = \prod_{j=1}^{m} (1 + a_j). \tag{2.13}$$

The combination of (2.12) and (2.13) yields a formula for the quotient $d(\mu)/|\mu|^{\epsilon}$, which can then be estimated directly.

Lemma 2.9 (Jarnik's theorem [Jar26])**.** Let $\mathcal{C}$ be a circle in $\mathbb{R}^2$ with radius R and let Γ be an arc of $\mathcal{C}$ of length $r \ll R^{1/3}$. Then, it holds that

$$\#\big(\mathbb{Z}^2 \cap \Gamma\big) \leq 2. \tag{2.14}$$

Proof. We only sketch the argument, which is best illustrated by a picture.

We argue by contradiction. Assume that $a, b, c \in \mathbb{Z}^2 \cap \Gamma$ are distinct and let $\mathcal{T}$ be the triangle spanned by a, b, c. Using linear algebra, we can write the signed area as

$$\text{area}(\mathcal{T}) = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \in \frac{1}{2}\mathbb{Z}.$$

But since Γ is convex, the area is non-zero, and therefore we must have

$$\Big|\text{area}(\mathcal{T})\Big| \geq \frac{1}{2}.$$

On the other hand, since Γ has length r and $\mathcal{C}$ has radius R, $\mathcal{T}$ is contained in a rectangle of dimensions $\sim r$ and $\sim r^2/R$. Thus, the area is bounded by

$$\Big|\text{area}(\mathcal{T})\Big| \lesssim \frac{r^3}{R}.$$

In the case $r \ll R^{\frac{1}{3}}$, the two estimates yield a contradiction. □

Proof of Proposition 2.6: We first treat the case $d = 2$, which is the main step. We then further distinguish the cases $\mu \lesssim K^6$ and $\mu \gg K^6$. In the first case, we write $(k_1 - c_1)^2 + (k_2 - c_2)^2 = \mu$ as

$$\Big(k_1 - c_1 + \mathbf{i}(k_2 - c_2)\Big)\Big(k_1 - c_1 - \mathbf{i}(k_2 - c_2)\Big) = \mu.$$

Using the divisor bound (Lemma 2.8), it then follows that

$$\#\Big\{k \in \mathbb{Z}^d \colon |k| \lesssim K,\, |k - c|^2 = \mu\Big\} \lesssim_{\epsilon} |\mu|^{\frac{\epsilon}{6}} \lesssim K^{\epsilon}.$$

In the second case, we note that the constraint $|k| \lesssim K$ enforces that all points lie on an arc of the circle $|k - c|^2 = \mu$ of length $\lesssim K$. Since the radius is $\sqrt{\mu} \gg K^3$, Jarnik's theorem (Lemma 2.9) implies that

$$\#\Big\{k \in \mathbb{Z}^d \colon |k| \lesssim K,\, |k - c|^2 = \mu\Big\} \leq 2 \lesssim_{\epsilon} K^{\epsilon}.$$

It remains to treat the case $d \geq 3$, which can be deduced from the case $d = 2$ and a slicing argument. Indeed, it holds that

$$\begin{aligned}
&\#\Big\{k \in \mathbb{Z}^d \colon |k| \lesssim K,\, |k-c|^2 = \mu\Big\} \\
\lesssim& \sum_{\substack{k_3,\cdots,k_d\colon \\ |k_3|,\cdots,|k_d| \lesssim K}} \#\Big\{(k_1,k_2) \in \mathbb{Z}^2 \colon |(k_1,k_2)| \lesssim K,\, (k_1-c_1)^2 + (k_2-c_2)^2 = \mu - \sum_{j=3}^d (k_j - c_j)^2\Big\} \\
\lesssim& K^\epsilon \sum_{\substack{k_3,\cdots,k_d\colon \\ |k_3|,\cdots,|k_d| \lesssim K}} 1 \lesssim K^\epsilon K^{d-2}. \qquad \square
\end{aligned}$$

Remark 2.10. In the proof of Corollary 2.7, the vector $k \in \mathbb{Z}^d$ satisfies $|k| \lesssim K$. As a result, the cases in the Proof of Proposition 2.6 which require Jarnick's theorem are not needed in the proof of Corollary 2.7. We therefore could have skipped Jarnick's theorem here, but it can be essential in other situations.

2.3. **Proof of Theorem 2.1 and Galilean boosts.**

Proof of Theorem 2.1: The $L_t^4 L_x^4$-Strichartz estimate (Theorem 2.1) now follows directly from Lemma 2.4 and Corollary 2.7. Indeed, we have that

$$\Big\| P_{\leq N} e^{it\Delta}\phi \Big\|_{L_t^4 L_x^4([0,2\pi]\times\mathbb{T}^d)} \lesssim \Big(\mathcal{M}_{2,d}(N)\Big)^{1/4} \|\phi\|_{L_x^2(\mathbb{T}^d)} \lesssim_\epsilon N^\epsilon N^{\frac{d-2}{4}} \|\phi\|_{L_x^2}. \qquad \square$$

We also record a direct corollary of Theorem 2.1 and the Galilean symmetry of (linear and nonlinear) Schrödinger equations.

Corollary 2.11 (Strichartz estimate and Galilean transformations)**.** Let $d \geq 2$, let $\epsilon > 0$, let $N \in 2^{\mathbb{N}_0}$, let $n_0 \in \mathbb{Z}^d$, and let $Q \coloneqq n_0 + [-N,N]^d$ be a cube of side-length N in $\mathbb{Z}^d$. Then, it holds for all $\phi \in L^2(\mathbb{T}^d)$ that

$$\Big\| P_Q e^{it\Delta}\phi \Big\|_{L_t^4 L_x^4([0,2\pi]\times\mathbb{T}^d)} \lesssim_\epsilon N^\epsilon N^{\frac{d-2}{4}} \|\phi\|_{L_x^2(\mathbb{T}^d)}.$$

Proof. This follows from Galilean symmetry and the case $n_0 = 0$ (Theorem 2.1). In this context, Galilean symmetry can be stated as

$$P_Q\Big(e^{it\Delta}\phi\Big)(x) = e^{in_0 x - i|n_0|^2 t} P_{\leq N}\Big(e^{it\Delta}\Big(e^{-in_0 x}\phi\Big)\Big)(x - 2n_0 t),$$

which can also be checked via a direct computation. $\square$

2.4. **Exercises.**

Exercise 2.12. Prove Lemma 2.8 for $\mathcal{R} = \mathbb{Z}$ and, if you are so inclined, also for $\mathcal{R} = \mathbb{Z}[i]$. Beware that the implicit constant is only allowed to depend on ϵ, but not on $m \geq 1$ from (2.12).

Exercise 2.13 (Understanding the Cauchy-Schwarz inequality in (2.8))**.** For all $d \geq 1$, $N \in 2^{\mathbb{N}_0}$, and $\phi \in L^2(\mathbb{T}^d)$, prove the inequality

$$\Big\| P_{\leq N} e^{it\Delta}\phi \Big\|_{L_t^4 L_x^4([0,2\pi]\times\mathbb{T}^d)} \lesssim \Big(\widetilde{M}(N)\Big)^{1/4} \|\phi\|_{L_x^2(\mathbb{T}^d)}, \tag{2.15}$$

where, for all $K \in 2^{\mathbb{N}_0}$,

$$\widetilde{\mathcal{M}}(K) := \sup_{k \in \mathbb{Z}^d} \sup_{\mu \in \mathbb{Z}} \#\Big\{(k_1, k_2) \in \mathbb{Z}^d \times \mathbb{Z}^d \colon |k_1|, |k_2| \leq K,\ k_1 - k_2 = k,\ |k_1|^2 - |k_2|^2 = \mu\Big\}. \tag{2.16}$$

Since

$$\widetilde{M}(K) \gtrsim \#\Big\{(k_1, k_2) \in \mathbb{Z}^d \times \mathbb{Z}^d \colon |k_1|, |k_2| \leq K,\ k_1 - k_2 = 0,\ |k_1|^2 - |k_2|^2 = 0\Big\} \gtrsim K^d,$$

the estimate (2.15) cannot directly be used for a proof of Theorem 2.1.

Exercise 2.14 (Generalization of Lemma 2.4)**.** For all $q, d \geq 1$, $N \in 2^{\mathbb{N}_0}$, and $\phi \in L^2(\mathbb{T}^d)$, prove that

$$\left\| P_{\leq N} e^{it\Delta} \phi \right\|_{L_t^{2q} L_x^{2q}([0,2\pi] \times \mathbb{T}^d)} \lesssim \Big(\mathcal{M}_{q,d}(N)\Big)^{\frac{1}{2q}} \|\phi\|_{L_x^2(\mathbb{T}^d)}.$$

Exercise 2.15 ($L_t^4 L_x^4$-estimate in $d = 1$)**.** For all $\phi \in L_x^2(\mathbb{T})$, prove that

$$\left\| e^{it\Delta} \phi \right\|_{L_t^4 L_x^4([0,2\pi] \times \mathbb{T})} \sim \|\phi\|_{L_x^2(\mathbb{T})}.$$

Exercise 2.16 ($L_t^6 L_x^6$-estimate in $d = 1$)**.** For all $\epsilon > 0$, $N \in 2^{\mathbb{N}_0}$, and $\phi \in L_x^2(\mathbb{T})$, prove the inequality

$$\left\| P_{\leq N} e^{it\Delta} \phi \right\|_{L_t^6 L_x^6([0,2\pi] \times \mathbb{T})} \lesssim_\epsilon N^\epsilon \|\phi\|_{L_x^2(\mathbb{T})}.$$

Hint: This is probably the hardest exercise of this section. Use Exercise 2.14, which implies that we only have to bound $\mathcal{M}_{3,1}$. By inserting the linear constraint $k_1 + k_2 + k_3 = k$ into the quadratic constraint, reduce the estimate of $\mathcal{M}_{3,1}$ to a counting problem of the form

$$\sup_{\mu \in \mathbb{Z}} \#\Big\{(a, b) \in \mathbb{Z}^2 \colon |a| \sim A, |b| \sim B, a^2 + 3b^2 = \mu\Big\}.$$

To solve the counting problem you may assume (without proof) that the divisor estimate holds in the ring $\mathbb{Z}[\rho]$, where $\rho = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{3}i}{2}$.

3. Bourgain spaces

3.1. **Definition and basic properties.** For any $u\colon \mathbb{R}\times\mathbb{T}^d\to\mathbb{C}$, we define its space-time Fourier transform as

$$\widetilde{u}(\xi,n) := \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}}\mathrm{d}t\int_{\mathbb{T}^d}\mathrm{d}x\, e^{-i\xi t - inx}u(t,x). \tag{3.1}$$

Definition 3.1 (Bourgain spaces)**.** For any $s\in\mathbb{R}$, $b\in\mathbb{R}$, and $u\colon\mathbb{R}_t\times\mathbb{T}^d_x\to\mathbb{C}$, we define the global norm

$$\|u\|_{X^{s,b}(\mathbb{R})} := \big\|\langle n\rangle^s\langle\lambda\rangle^b\widetilde{u}(\lambda-|n|^2,n)\big\|_{L^2_\lambda\ell^2_n(\mathbb{R}\times\mathbb{Z}^d)} = \big\|\langle n\rangle^s\langle\xi+|n|^2\rangle^b\widetilde{u}(\xi,n)\big\|_{L^2_\xi\ell^2_n(\mathbb{R}\times\mathbb{Z}^d)}.$$

For any interval $I\subseteq\mathbb{R}$ and $v\colon I\times\mathbb{T}^d_x\to\mathbb{C}$, we also define the local norm

$$\|v\|_{X^{s,b}(I)} := \inf\Big\{\|u\|_{X^{s,b}(\mathbb{R})}\colon u\big|_I = v\Big\}.$$

Remark 3.2. For $b>0$, elements of $X^{s,b}(\mathbb{R})$ are encouraged to have their space-time frequency support near the paraboloid $\{(n,-|n|^2)\colon n\in\mathbb{Z}^d\}$, and therefore are encouraged to behave like linear Schrödinger waves.

Lemma 3.3 (Basic properties of Bourgain spaces)**.** Let $s\in\mathbb{R}$, let $b,b'\in\mathbb{R}$, and let $\chi\in C^\infty_c(\mathbb{R})$. Then, we have the following estimates:

(i) (Linear evolution) For all $\phi\in H^s(\mathbb{T}^d)$, it holds that

$$\big\|\chi(t)e^{it\Delta}\phi\big\|_{X^{s,b}(\mathbb{R})}\lesssim_{b,\chi}\|\phi\|_{H^s(\mathbb{T}^d)}.$$

(ii) (Duhamel integral) If $b>\frac{1}{2}$, we have for all $F\colon\mathbb{R}\times\mathbb{T}^d\to\mathbb{C}$ that

$$\Big\|\chi(t)\int_0^t\mathrm{d}t' e^{i(t-t')\Delta}F(t')\Big\|_{X^{s,b}(\mathbb{R})}\lesssim_{b,\chi}\|F\|_{X^{s,b-1}(\mathbb{R})}.$$

(iii) (Time-localization) If $-\frac{1}{2}<b'\leq b<\frac{1}{2}$, it holds for all $0<\tau\leq 1$ and $u\in X^{s,b}([0,\tau])$ that

$$\|u\|_{X^{s,b'}([0,\tau])}\lesssim\tau^{b-b'}\|u\|_{X^{s,b}([0,\tau])}.$$

(iv) For all $u\colon\mathbb{R}\times\mathbb{T}^d\to\mathbb{C}$, it holds that

$$\|u\|_{X^{s,b}(\mathbb{R})} = \sup_{\|v\|_{X^{-s,-b}(\mathbb{R})}\leq 1}\Big|\int_{\mathbb{R}}\mathrm{d}t\int_{\mathbb{T}^d}\mathrm{d}x\, u(t,x)\overline{v(t,x)}\Big|.$$

Since the properties in Lemma 3.3 can be found in many textbooks [ET16, Tao06], we leave the proofs as an exercise (Exercise 3.9).

We briefly describe the why one should expect (i)-(iv): For (i), note that the space-time Fourier transform of $e^{it\Delta}\phi$ is a distribution supported on the paraboloid $\{(n,-|n|^2)\colon n\in\mathbb{Z}^d\}$. The multiplication with χ in time leads to a convolution with $\widehat{\chi}$ in the time-frequency variable, and thus the space-time Fourier transform of $\chi(t)e^{it\Delta}\phi$ is a function supported near the paraboloid. For (ii), this is because integrals gain a derivative, and the Duhamel integral gains a derivative with respect to the dispersive symbol $\langle\xi+|n|^2\rangle$. For (iii), this is because $X^{s,b}$ with $b<\frac{1}{2}$ does not embed into $L^\infty_t H^s_x$ (just like H^b_t does not embed into L^∞_t), and thus decreasing the size of the interval can lead to a small norm. For (iv), the reason is that, after switching to frequency-space, $X^{s,b}$ is just a weighted L^2-space.

Lemma 3.4 (Transference principle)**.** Let $A > 0$, let $\|\cdot\|_Y$ be a norm on space-time functions, and assume that, for all $\phi \in H^s_x(\mathbb{T}^d)$,

$$\sup_{\lambda\in\mathbb{R}} \big\| e^{it\lambda} e^{it\Delta}\phi \big\|_Y \leq A \big\|\phi\big\|_{H^s(\mathbb{T}^d)}. \tag{3.2}$$

For all $b > \frac{1}{2}$ and all $u \in X^{s,b}(\mathbb{R})$, it then holds that

$$\big\|u\big\|_Y \lesssim_b A \big\|u\big\|_{X^{s,b}(\mathbb{R})}. \tag{3.3}$$

Proof. Using Fourier inversion, we write

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} \mathrm{d}\xi \sum_{n\in\mathbb{Z}^d} e^{it\xi} e^{inx} \widetilde{u}(\xi,n) \\ &= \int_{\mathbb{R}} \mathrm{d}\lambda \sum_{n\in\mathbb{Z}^d} e^{it(\lambda-|n|^2)} e^{inx} \widetilde{u}(\lambda-|n|^2,n) \\ &= \int_{\mathbb{R}} \mathrm{d}\lambda \, e^{it\lambda} e^{it\Delta}\phi_\lambda, \end{aligned}$$

where $\phi_\lambda \colon \mathbb{T}^d \to \mathbb{C}$ is defined by

$$\widehat{\phi_\lambda}(n) = \widetilde{u}(\lambda-|n|^2,n).$$

Using our assumption (3.2) and the triangle inequality, it follows that

$$\big\|u\big\|_Y \leq \int_{\mathbb{R}} \mathrm{d}\lambda \, \big\|e^{it\lambda} e^{it\Delta}\phi_\lambda\big\|_Y \leq A \int_{\mathbb{R}} \mathrm{d}\lambda \, \big\|\phi_\lambda\big\|_{H^s(\mathbb{T}^d)}. \tag{3.4}$$

Using Cauchy-Schwarz, it follows that

$$(3.4) \leq A \big\| \langle\lambda\rangle^{-b} \big\|_{L^2_\lambda} \times \Big\| \langle\lambda\rangle^{b} \big\|\phi_\lambda\big\|_{H^s(\mathbb{T}^d)} \Big\|_{L^2_\lambda} \lesssim_b A \big\|u\big\|_{X^{s,b}(\mathbb{R})}.$$

□

Lemma 3.5 (Controlling $L^4_t L^4_x$ via $X^{s,b'}$)**.** Let $N \in 2^{\mathbb{N}_0}$, let Q be a cube with side-length N, and let $I \subseteq [0,2\pi]$. Then, it holds for all $u \colon \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}$ that

$$\big\|P_Q u\big\|_{L^4_t L^4_x(I\times\mathbb{T}^d)} \lesssim_\delta N^{\frac{d-2}{4}+4\delta} \big\|u\big\|_{X^{0,\frac{1}{2}-\delta}(I)}. \tag{3.5}$$

Proof. Using the definition of Bourgain-spaces, it suffices to prove that

$$\big\|P_Q u\big\|_{L^4_t L^4_x([0,2\pi]\times\mathbb{T}^d)} \lesssim_\delta N^{\frac{d-2}{4}+4\delta} \big\|u\big\|_{X^{0,\frac{1}{2}-\delta}(\mathbb{R})}. \tag{3.6}$$

Using the transference principle (Lemma 3.4) and Strichartz estimates (Theorem 2.1 and Corollary 2.11), it holds that

$$\big\|P_Q u\big\|_{L^4_t L^4_x([0,2\pi]\times\mathbb{T}^d)} \lesssim_\delta N^{\frac{d-2}{4}+\delta} \big\|u\big\|_{X^{0,\frac{1}{2}+\delta}(\mathbb{R})}. \tag{3.7}$$

Except for the exponent b-parameter in (3.7), which is $\frac{1}{2}+\delta$ rather than $\frac{1}{2}-\delta$, this already coincides with our desired estimate (3.6). In order to fix this, we will use a separate crude estimate, which has poor dependence on N but requires less of b.

Using the Hausdorff-Young inequality, it holds that

$$\big\|P_Q u\big\|_{L^4_t L^4_x([0,2\pi]\times\mathbb{T}^d)} \lesssim \big\|P_Q u\big\|_{L^4_t L^4_x(\mathbb{R}\times\mathbb{T}^d)} \lesssim \big\|\widetilde{P_Q u}\big\|_{L^{4/3}_\xi \ell^{4/3}_n(\mathbb{R}\times\mathbb{Z}^d)}.$$

Using Hölder's inequality, it holds that

$$\begin{aligned}\|\widetilde{P_Q u}\|_{L^{4/3}_\xi \ell^{4/3}_n(\mathbb{R}\times\mathbb{Z}^d)} &\lesssim \big\| \langle \xi + |n|^2 \rangle^{-1/4-\delta} \mathbf{1}_Q(n) \big\|_{L^4_\xi \ell^4_n(\mathbb{R}\times\mathbb{Z}^d)} \big\| \langle \xi + |n|^2 \rangle^{1/4+\delta} \widetilde{u}(\xi,n) \big\|_{L^2_\xi \ell^2_n(\mathbb{R}\times\mathbb{Z}^d)} \\ &\lesssim N^{\frac{d}{4}} \|u\|_{X^{0,\frac{1}{4}+\delta}(\mathbb{R})}.\end{aligned}$$

In total, it follows that

$$\|P_Q u\|_{L^4_t L^4_x([0,2\pi]\times\mathbb{T}^d)} \lesssim N^{\frac{d}{4}} \|u\|_{X^{0,\frac{1}{4}+\delta}(\mathbb{R})}. \tag{3.8}$$

By interpolating (3.7) and (3.8), we arrive at the desired estimate (3.6). □

Remark 3.6. In the proof of Lemma 3.5, we slightly adjusted a parameter in (3.7) by supplementing it with the "easy" second inequality (3.8), which is a pretty general trick in PDE. This generally only fails if there is a good reason for it. For example, in deterministic dispersive equations it is almost never possible to slightly decrease the power of the highest frequency-scale, since there is no smoothing.

3.2. **Nonlinear estimates.**

Lemma 3.7 (Trilinear estimate)**.** Let $d = 2$. For $s \geq 20\delta$, $b = \frac{1}{2} + \delta$, $b' := \frac{1}{2} - \delta$, and $I \subseteq [0, 2\pi]$, it holds that

$$\|u_1 \overline{u_2} u_3\|_{X^{s,b-1}(I)} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b'}(I)}. \tag{3.9}$$

Proof. Using the definition of Bourgain-spaces, it suffices to prove the estimate with I replaced by $[0, 2\pi]$. Using Exercise 3.10, it suffices to prove for all $N_0, N_1, N_2, N_3 \in 2^{\mathbb{N}_0}$ that

$$\Big\| P_{N_0}\big(P_{N_1} u_1 \overline{P_{N_2} u_2} P_{N_3} u_3 \big) \Big\|_{X^{0,b-1}([0,2\pi])} \lesssim (N^{(2)})^{16\delta} \prod_{j=1}^{3} \|P_{N_j} u_j\|_{X^{0,b'}(\mathbb{R})}. \tag{3.10}$$

where $N^{(0)} \geq N^{(1)} \geq N^{(2)} \geq N^{(3)}$ is the non-increasing rearrangement of N_0, N_1, N_2, and N_3. Using duality (Lemma 3.3), $b - 1 = -b'$, and the definition of Bourgain-spaces, it suffices to prove that, for all $u_0 \colon \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$,

$$\Big| \int\limits_{[0,2\pi]\times\mathbb{T}^2} \mathrm{d}t\mathrm{d}x\, \overline{P_{N_0} u_0} P_{N_1} u_1 \overline{P_{N_2} u_2} P_{N_3} u_3 \Big| \lesssim (N^{(2)})^{16\delta} \prod_{j=0}^{3} \|P_{N_j} u_j\|_{X^{0,b'}(\mathbb{R})}, \tag{3.11}$$

Since the other cases are similar, we only treat the case $N_0 \geq N_1 \geq N_2 \geq N_3$. Note that, in order for (3.11) to be non-zero, it then holds that $N_0 \sim N_1$.

Side note: We want to use the $L^4_t L^4_x$-estimate for u_0, u_1, u_2, and u_3. However, doing this directly would cost $(N_0 N_1 N_2 N_3)^{4\delta}$. This would cost powers of the highest frequency scales N_0 and N_1 which, if N_2 is much smaller than N_0 and N_1, we cannot gain back. To avoid this loss, we use an essential technique called *box localization*.

We let $\mathcal{Q} = \mathcal{Q}(N_0, N_1, N_2)$ be a collection of cubes that have side-length $\sim N_2$, cover the cube centered at the origin with side-length $\sim N_0$, and have finite overlap. It then follows from frequency-support considerations that

$$\begin{aligned}
&\int\limits_{[0,2\pi]\times\mathbb{T}^2} \mathrm{d}t\mathrm{d}x\, \overline{P_{N_0} u_0} P_{N_1} u_1 \overline{P_{N_2} u_2} P_{N_3} u_3 \\
&= \sum_{\substack{Q_0,Q_1\in\mathcal{Q}:\\ d(Q_0,Q_1)\lesssim N_2}} \int\limits_{[0,2\pi]\times\mathbb{T}^2} \mathrm{d}t\mathrm{d}x\, \overline{P_{Q_0} P_{N_0} u_0} \cdot P_{Q_1} P_{N_1} u_1 \cdot \overline{P_{N_2} u_2} \cdot P_{N_3} u_3.
\end{aligned} \tag{3.12}$$

Using Hölder's inequality and Lemma 3.5, it then follows that

$$\begin{aligned}
\big|(3.12)\big| &\lesssim \Bigg(\sum_{\substack{Q_0,Q_1\in\mathcal{Q}:\\ d(Q_0,Q_1)\lesssim N_2}} \big\| P_{Q_0} P_{N_0} u_0 \big\|_{L_t^4 L_x^4([0,2\pi]\times\mathbb{T}^2)} \big\| P_{Q_1} P_{N_1} u_1 \big\|_{L_t^4 L_x^4([0,2\pi]\times\mathbb{T}^2)} \Bigg) \\
&\quad \times \big\| P_{N_2} u_2 \big\|_{L_t^4 L_x^4([0,2\pi]\times\mathbb{T}^2)} \big\| P_{N_3} u_3 \big\|_{L_t^4 L_x^4([0,2\pi]\times\mathbb{T}^2)} \\
&\lesssim (N^{(2)})^{16\delta} \Bigg(\sum_{\substack{Q_0,Q_1\in\mathcal{Q}:\\ d(Q_0,Q_1)\lesssim N_2}} \big\| P_{Q_0} P_{N_0} u_0 \big\|_{X^{0,b'}(\mathbb{R})} \big\| P_{Q_1} P_{N_1} u_1 \big\|_{X^{0,b'}(\mathbb{R})} \Bigg) \\
&\quad \times \big\| P_{N_2} u_2 \big\|_{X^{0,b'}(\mathbb{R})} \big\| P_{N_3} u_3 \big\|_{X^{0,b'}(\mathbb{R})}.
\end{aligned} \tag{3.13}$$

In order to obtain the desired estimate, it remains to treat the sum over Q_0 and Q_1 in (3.13). Using Cauchy-Schwarz, using that $\mathcal{Q}$ is finitely overlapping, and using orthogonality, it holds that

$$\begin{aligned}
&\sum_{\substack{Q_0,Q_1\in\mathcal{Q}:\\ d(Q_0,Q_1)\lesssim N_2}} \big\| P_{Q_0} P_{N_0} u_0 \big\|_{X^{0,b'}(\mathbb{R})} \big\| P_{Q_1} P_{N_1} u_1 \big\|_{X^{0,b'}(\mathbb{R})} \\
&\lesssim \Bigg(\sum_{\substack{Q_0,Q_1\in\mathcal{Q}:\\ d(Q_0,Q_1)\lesssim N_2}} \big\| P_{Q_0} P_{N_0} u_0 \big\|_{X^{0,b'}(\mathbb{R})}^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{Q_0,Q_1\in\mathcal{Q}:\\ d(Q_0,Q_1)\lesssim N_2}} \big\| P_{Q_1} P_{N_1} u_1 \big\|_{X^{0,b'}(\mathbb{R})}^2 \Bigg)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{Q_0\in\mathcal{Q}} \big\| P_{Q_0} P_{N_0} u_0 \big\|_{X^{0,b'}(\mathbb{R})}^2 \Big)^{\frac{1}{2}} \Big(\sum_{Q_1\in\mathcal{Q}} \big\| P_{Q_1} P_{N_1} u_1 \big\|_{X^{0,b'}(\mathbb{R})}^2 \Big)^{\frac{1}{2}} \\
&\lesssim \big\| P_{N_0} u_0 \big\|_{X^{0,b'}(\mathbb{R})} \big\| P_{N_1} u_1 \big\|_{X^{0,b'}(\mathbb{R})}
\end{aligned}$$

This completes the proof of (3.11). □

3.3. **Local well-posedness.**

Proposition 3.8 (Local well-posedness in $X^{s,b}$-spaces)**.** Let $d = 2$, let $s > 0$, let $D = D_s \geq 1$ be sufficiently large, let $\delta = \delta_s > 0$ be sufficiently small, and let $b := \frac{1}{2} + \delta$. Let $R \geq 1$ be arbitrary, let B_R be the ball of radius R in $H_x^s(\mathbb{T}^2)$, and let $0 < \tau \leq D^{-1} R^{-D}$. Then, for all $\phi \in B_R$, there exists a unique local solution of the integral formulation of (NLS) in $X^{s,b}([0,\tau])$. Furthermore, the data-to-solution map

$$B_R \to X^{s,b}([0,\tau]),\ \phi \mapsto u$$

is Lipschitz continuous.

We have all the necessary estimates available, and it only remains to setup the contraction-mapping argument. This is left as an exercise (Exercise 3.11).

3.4. **Exercises.**

Exercise 3.9. Prove Lemma 3.3.
Hint: (i) and (iv) are rather straightforward. For (ii) and (iii), one may want to look up the proofs from [Tao06, Lemma 2.11 and Proposition 2.12] or [ET16, Lemma 3.12 and Lemma 3.11].

Exercise 3.10 (A step in the proof of Lemma 3.7)**.** Using the estimate (3.10), which is satisfied for all $N_0, N_1, N_2, N_3 \in 2^{\mathbb{N}_0}$ and all $u_1, u_2, u_3 \colon \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$, prove the estimate (3.9) for all $s \geq 20\delta$.

Exercise 3.11 (Contraction-mapping argument)**.** Using the estimates of this section and a contraction-mapping argument, prove Proposition 3.8.

4. Probability theory and Bourgain's trick

4.1. Probability theory.

Definition 4.1 (Massive Gaussian free field)**.** Let $d \geq 1$. Then, a random distribution $\phi\colon \mathbb{T}^d \to \mathbb{C}$ is called a d-dimensional (massive) Gaussian free field if it can be represented as

$$\phi = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle} e^{inx}, \tag{4.1}$$

where $(g_n)_{n\in\mathbb{Z}^d}$ are independent, standard, complex-valued Gaussians. We denote the unique corresponding measure by $\mathfrak{g} = \mathfrak{g}_d$, i.e., we define

$$\mathfrak{g}_d := \mathrm{Law}\Big(\sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle} e^{inx} \Big). \tag{4.2}$$

Remark 4.2. We make the following remarks regarding Definition 4.1.

(i) Our definition corresponds to a unit mass $m = 1$. For a general mass $m > 0$, one would replace $\langle n \rangle = \sqrt{1 + |n|^2}$ by $\sqrt{m^2 + |n|^2}$. In the following, we often omit writing "massive".

(ii) The Gaussian free field is a natural model from probability, and we refer to [She07] for a survey of its properties.

(iii) The Gibbs measure of (NLS) is given by the complex-valued Φ^4_d-measure. In dimension $d = 1, 2$, Φ^4_d is absolutely continuous with respect to the Gaussian free field. Thus, in dimension $d = 1, 2$, if we want to prove the almost-sure local well-posedness of (NLS) with respect to the Gibbs measure, it suffices to prove it with respect to the Gaussian free field.

In order to work with the Gaussian free field, we later need the following standard lemma, which will not be proven here.

Lemma 4.3 (Wick's theorem)**.** Let $k \geq 1$ and let $X_1, X_2, \cdots, X_{2k}$ be mean-zero complex-valued Gaussian random variables. Then, it holds that

$$\mathbb{E}\Big[\prod_{l=1}^{2k} X_l \Big] = \sum_{\mathcal{P}} \prod_{\{i,j\} \in \mathcal{P}} \mathbb{E}\big[X_i X_j \big]. \tag{4.3}$$

The sum in (4.3) is over all pairings $\mathcal{P}$ of $\{1, 2, \cdots, 2k\}$, i.e., all partitions of $\{1, 2, \cdots, 2k\}$ into two-element subsets.

For example, if $k = 2$, then Lemma 4.3 implies that

$$\mathbb{E}\big[X_1X_2X_3X_4\big] = \mathbb{E}\big[X_1X_2\big] \times \mathbb{E}\big[X_3X_4\big] + \mathbb{E}\big[X_1X_3\big] \times \mathbb{E}\big[X_2X_4\big] + \mathbb{E}\big[X_1X_4\big] \times \mathbb{E}\big[X_2X_3\big].$$

Lemma 4.4 (Regularity of GFF)**.** Let $\phi\colon \mathbb{T}^d \to \mathbb{C}$ be a Gaussian free field. Then, it holds that

$$\mathbb{E}\|\phi\|^2_{H^s_x(\mathbb{T}^d)} < \infty \qquad \Longleftrightarrow \qquad s < 1 - \frac{d}{2}.$$

Remark 4.5. In dimension $d = 2$, it follows that the Gaussian free field has regularity $0-$. Thus, it is barely outside the scope of Theorem 1.1.

Proof. It holds that

$$\mathbb{E}\|\phi\|^2_{H^s_x(\mathbb{T}^d)} = \mathbb{E} \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} \frac{|g_n|^2}{\langle n \rangle^2} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2(s-1)} \mathbb{E}|g_n|^2 = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2(s-1)}.$$

The latter sum is finite if and only if $2(s-1) < -d$, which yields the desired claim. □

4.2. **Wick-ordered nonlinear Schrödinger equation.** For reasons that will become clear later, we now study the Wick-ordered nonlinear Schrödinger equation, which is given by

$$\text{(WNLS)} \qquad \begin{cases} i\partial_t u + \Delta u = \sigma\big(|u|^2 - 2\|u\|_{L^2}^2\big)\, u & (t,x) \in \mathbb{R} \times \mathbb{T}^d, \\ u|_{t=0} = \phi. \end{cases}$$

There exists an invertible gauge-transformation $\mathcal{G}\colon L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ which transforms smooth solutions of (NLS) into smooths solutions of (WNLS), and thus (NLS) and (WNLS) are equivalent for regular initial data (see Exercise 4.8). For initial data below L^2, however, this equivalence breaks down. The additional term $-2\|u\|_{L^2}^2$ acts as a renormalization, which is a standard step in the treatment of many random PDEs.

Using Fourier analysis, we obtain the following decomposition of the Wick-ordered nonlinearity.

Lemma 4.6 (Decomposition)**.** For all smooth $u\colon \mathbb{T}^d \to \mathbb{C}$, it holds that

$$\big(|u|^2 - 2\|u\|_{L^2}^2\big)\, u = \mathcal{N}(u,u,u) + \mathcal{N}^{\mathfrak{r}}(u,u,u),$$

where

$$\mathcal{N}(u_1,u_2,u_3) := \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^d: \\ n_2 \neq n_1, n_3}} \widehat{u}_1(n_1)\overline{\widehat{u}_2(n_2)}\widehat{u}_3(n_3) e^{i(n_1-n_2+n_3)x},$$

$$\mathcal{N}^{\mathfrak{r}}(u_1,u_2,u_3) := \sum_{n \in \mathbb{Z}^d} \widehat{u}_1(n)\overline{\widehat{u}_2(n)}\widehat{u}_3(n) e^{inx}.$$

The simple proof of Lemma 4.6 is left as an exercise (Exercise 4.9).

Remark 4.7. We make the following remarks regarding Lemma 4.6.

(i) The nonlinearity $\mathcal{N}(u,u,u)$ is the main term on the right-hand side of (WNLS). The condition $n_2 \neq n_1, n_3$ is due to Wick-ordering and will be crucial later.

(ii) The superscript in $\mathcal{N}^{\mathfrak{r}}$ stands for (doubly) resonant. This term is harmless (at least in dimension $d=2$), and we will ignore it almost entirely.

4.3. **Bourgain's trick.** In the following, we specialize to $d=2$ (as in Theorem 1.2) and our goal is to solve

$$\begin{cases} i\partial_t u + \Delta u = \sigma \cdot \big(\mathcal{N}(u,u,u) + \mathcal{N}^{\mathfrak{r}}(u,u,u)\big) & (t,x) \in \mathbb{R} \times \mathbb{T}^d, \\ u|_{t=0} = \phi, \end{cases} \tag{4.4}$$

where ϕ is a sample of the Gaussian free field.

Problem: ϕ only has regularity $0-$ and (4.4) cannot be solved in $X^{0-,b}$.
We use the linear-nonlinear decomposition

$$u = e^{it\Delta}\phi + v. \tag{4.5}$$

The nonlinear remainder v then must satisfy $v(0)=0$ and the evolution equation

$$i\partial_t v + \Delta v = \sigma \sum_{\substack{w_1,w_2,w_3 \\ \in \{e^{it\Delta}\phi, v\}}} \big(\mathcal{N}(w_1,w_2,w_3) + \mathcal{N}^{\mathfrak{r}}(w_1,w_2,w_3)\big). \tag{4.6}$$

Goal: Since $v(0) = 0$ is smooth, we want to solve (4.6) in $X^{s,b}$ for certain $s > 0$.

Since $s > 0$, the $\mathcal{N}(v,v,v)$-term can be treated deterministically. Among others, however, we have to address the following two issues:

(I) Can we control self-interactions of $e^{it\Delta}\phi$, i.e., $\mathcal{N}(e^{it\Delta}\phi, e^{it\Delta}\phi, e^{it\Delta}\phi)$ in $X^{s,b-1}$? This will be addressed in Section 5.

(II) Can we control interactions between $e^{it\Delta}\phi$ and v, such as $\mathcal{N}(e^{it\Delta}\phi, e^{it\Delta}\phi, v)$, in $X^{s,b-1}$? This will be addressed in Section 6.

In both (I) and (II), we will need to uncover a nonlinear smoothing effect.

4.4. **Exercises.**

Exercise 4.8 (Gauge-transformation)**.** Define the gauge-transform $\mathcal{G}\colon L_t^\infty L_x^2 \to L_t^\infty L_x^2$ by

$$\mathcal{G}(u)(t,x) := e^{2i\sigma\|u\|_{L^2}^2 t} u(t,x).$$

Show that $\mathcal{G}$ is invertible and compute its inverse. Furthermore, show that for any smooth function $u\colon \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}$, u is a classical solution of (NLS) if and only if $\mathcal{G}u$ is a classical solution of (WNLS).

Exercise 4.9 (Decomposition)**.** Using the Fourier expansion of u into $(|u|^2 - 2\|u\|_{L^2}^2)\,u$, prove Lemma 4.6.

Exercise 4.10 (Invariance of Gaussian free field under linear Schrödinger equation)**.** Let $t \in \mathbb{R}$, let $d \geq 1$, and let $\phi\colon \mathbb{T}^d \to \mathbb{C}$ be a d-dimensional Gaussian free field. Prove that $e^{it\Delta}\phi$ is also a d-dimensional Gaussian free field, i.e., prove that $e^{it\Delta}\phi$ and ϕ have the same law.

5. The cubic stochastic object

In the main proposition of this section, we control the first Picard iterate for the random initial data from (1.1).

Proposition 5.1. Let $s := \frac{1}{2} - \delta_1$ and $b := \frac{1}{2} + \delta_2$, where $0 < \delta_2 \ll \delta_1$, and let ϕ be as in (1.1). Then, it holds that

$$\mathbb{E}\Big[\Big\|\mathcal{N}\big(e^{it\Delta}\phi, e^{it\Delta}\phi, e^{it\Delta}\phi\big)\Big\|^2_{X^{s,b-1}([0,2\pi])}\Big] \lesssim_{\delta_1,\delta_2} 1. \tag{5.1}$$

Proposition 5.1 is a direct consequence of Lemma 5.3, which uses probabilistic cancellations, and Lemma 5.4, which relies on dispersive effects.

5.1. **Reduction.** We first introduce a different counting problem than in Section 2.

Definition 5.2. Let $d \geq 2$, let $N_0, N_1, N_2, N_3 \in 2^{\mathbb{N}_0}$, and let $(\iota_0, \iota_1, \iota_2, \iota_3) = (-1, 1, -1, 1)$. We then define

$$\begin{aligned} &\mathcal{M}_{4,d}(N_j, \iota_j : 0 \leq j \leq 3) \\ &:= \sup_{\mu \in \mathbb{Z}} \#\Big\{(n_0, n_1, n_2, n_3) \in (\mathbb{Z}^d)^4 : |n_j| \sim N_j \quad \text{for all } 0 \leq j \leq 3, \\ &\qquad (\iota_j, n_j) \neq (\iota_k, n_k) \quad \text{for all } 0 \leq j \neq k \leq 3, \sum_{j=0}^{3} \iota_j n_j = 0, \quad \sum_{j=0}^{3} \iota_j |n_j|^2 = \mu \Big\}. \end{aligned} \tag{5.2}$$

In the following, we often write

$$N_{\max} := \max(N_0, N_1, N_2, N_3).$$

Lemma 5.3 (Probabilistic cancellations/Reduction)**.** Let $s \in \mathbb{R}$ and $b := \frac{1}{2} + \delta_2$ be as in Proposition 5.1. Then, it holds that

$$\begin{aligned} &\mathbb{E}\Big[\Big\|\mathcal{N}\big(e^{it\Delta}\phi, e^{it\Delta}\phi, e^{it\Delta}\phi\big)\Big\|^2_{X^{s,b-1}([0,2\pi])}\Big] \\ &\lesssim \sup_{\substack{N_0,N_1,N_2,\\ N_3 \in 2^{\mathbb{N}_0}}} N_{\max}^{8\delta_2} N_0^{2s} (N_1 N_2 N_3)^{-2} \mathcal{M}_{4,2}(N_j, \iota_j : 0 \leq j \leq 3), \end{aligned} \tag{5.3}$$

where $(\iota_0, \iota_1, \iota_2, \iota_3) = (-1, +1, -1, +1)$.

Proof. It suffices to prove the following estimate:

$$\begin{aligned} &\mathbb{E}\Big[\Big\|P_{N_0}\mathcal{N}\big(P_{N_1}e^{it\Delta}\phi, P_{N_2}e^{it\Delta}\phi, P_{N_3}e^{it\Delta}\phi\big)\Big\|^2_{X^{s,b-1}([0,2\pi])}\Big] \\ &\lesssim N_{\max}^{4\delta_2} N_0^{2s} (N_1 N_2 N_3)^{-2} \mathcal{M}_{4,2}(N_j, \iota_j : 0 \leq j \leq 3). \end{aligned} \tag{5.4}$$

Indeed, since we gave ourselves an $N_{\max}^{-4\delta_2}$-factor of room, we can easily prove (5.3) by summing (5.4) over dyadic scales. To simplify the notation, we introduce the phase function

$$\Omega = \Omega(n_0, n_1, n_2, n_3) = -|n_0|^2 + |n_1|^2 - |n_2|^2 + |n_3|^2.$$

To prove (5.4), we first note that

$$
\begin{aligned}
&e^{-it\Delta}P_{N_0}\mathcal{N}\big(P_{N_1}e^{it\Delta}\phi, P_{N_2}e^{it\Delta}\phi, P_{N_3}e^{it\Delta}\phi\big)\\
&=\sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^2:\\ n_0=n_1-n_2+n_3,\\ n_2\neq n_1,n_3}}\bigg(e^{-it\Omega}e^{in_0x}\Big(\prod_{j=0}^{3}\mathbf{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-1}g_{n_1}\overline{g_{n_2}}g_{n_3}\bigg).
\end{aligned}
\tag{5.5}
$$

Using the statement from Exercise 5.8, it follows that

$$
\begin{aligned}
&\Big\|P_{N_0}\mathcal{N}\big(P_{N_1}e^{it\Delta}\phi, P_{N_2}e^{it\Delta}\phi, P_{N_3}e^{it\Delta}\phi\big)\Big\|^2_{X^{s,b-1}([0,2\pi])}\\
&\lesssim\sum_{\mu\in\mathbb{Z}}\sum_{n_0\in\mathbb{Z}^2}\langle\mu\rangle^{2(b-1)}\langle n_0\rangle^{2s}\bigg|\sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2:\\ n_0=n_1-n_2+n_3,\\ n_2\neq n_1,n_3,\\ \Omega=\mu.}}\Big(\prod_{j=0}^{3}\mathbf{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-1}g_{n_1}\overline{g_{n_2}}g_{n_3}\bigg|^2.
\end{aligned}
\tag{5.6}
$$

We now claim that the family $(g_{n_1}\overline{g_{n_2}}g_{n_3})_{n_2\neq n_1,n_3}$ is, up to permutations, orthogonal in $L^2(\mathbb{P})$. Indeed, for any $n_1,n_2,n_3\in\mathbb{Z}^2$ satisfying $n_2\neq n_1,n_3$ and $n_1',n_2',n_3'\in\mathbb{Z}^2$ also satisfying $n_2'\neq n_1',n_3'$, Wick's theorem (Lemma 4.3) implies that

$$
\begin{aligned}
&\mathbb{E}\big[g_{n_1}\overline{g_{n_2}}g_{n_3}\overline{g_{n_1'}\overline{g_{n_2'}}g_{n_3'}}\big]\\
&=\mathbf{1}\Big\{(n_1,n_2,n_3)=(n_1',n_2',n_3')\Big\}+\mathbf{1}\Big\{(n_1,n_2,n_3)=(n_3',n_2',n_1')\Big\}.
\end{aligned}
\tag{5.7}
$$

Using orthogonality up to permutations, it follows that

$$
\begin{aligned}
\mathbb{E}(5.6)&\leq\sum_{\mu\in\mathbb{Z}}\sum_{n_0\in\mathbb{Z}^2}\langle\mu\rangle^{2(b-1)}\langle n_0\rangle^{2s}\mathbb{E}\bigg|\sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2:\\ n_0=n_1-n_2+n_3,\\ n_2\neq n_1,n_3,\\ \Omega=\mu.}}\Big(\prod_{j=0}^{3}\mathbf{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-1}g_{n_1}\overline{g_{n_2}}g_{n_3}\bigg|^2\\
&\lesssim\sum_{\mu\in\mathbb{Z}}\sum_{n_0\in\mathbb{Z}^2}\langle\mu\rangle^{2(b-1)}\langle n_0\rangle^{2s}\sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2:\\ n_0=n_1-n_2+n_3,\\ n_2\neq n_1,n_3,\\ \Omega=\mu.}}\Big(\prod_{j=0}^{3}\mathbf{1}_{N_j}(n_j)\Big)\langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\langle n_3\rangle^{-2}.
\end{aligned}
\tag{5.8}
$$

Side note: We emphasize that the bound (5.8) heavily relies on the randomness in $g_{n_1}\overline{g_{n_2}}g_{n_3}$ and is significantly better than any bound we could obtain if $g_{n_1}\overline{g_{n_2}}g_{n_3}$ is replaced by 1.

Using that the summand is non-trivial only for $|\mu|\lesssim N_{\max}^2$, $2(b-1)=-1+2\delta_2$, and Definition 5.2, it follows that

$$
\begin{aligned}
(5.8)&\lesssim N_{\max}^{4\delta_2}N_0^{2s}N_1^{-2}N_2^{-2}N_3^{-2}\times\sup_{\mu\in\mathbb{Z}}\sum_{\substack{n_0,n_1,n_2,n_3\in\mathbb{Z}^2:\\ n_0=n_1-n_2+n_3,\\ n_2\neq n_1,n_3,\\ \Omega=\mu.}}\Big(\prod_{j=0}^{3}\mathbf{1}_{N_j}(n_j)\Big)\\
&\lesssim N_{\max}^{4\delta_2}N_0^{2s}N_1^{-2}N_2^{-2}N_3^{-2}\mathcal{M}_{4,2}(N_j,\iota_j\colon 0\leq j\leq 3).
\end{aligned}
$$

This completes the proof of (5.4). □

5.2. **Lattice point counting estimates.**

Lemma 5.4. Let $d \geq 2$, let $N_0, N_1, N_2, N_3 \in 2^{\mathbb{N}_0}$, and let $(\iota_0, \iota_1, \iota_2, \iota_3) = (-1, 1, -1, 1)$. For all $\epsilon > 0$, it then holds that

$$\mathcal{M}_{4,d}(N_j, \iota_j : 0 \leq j \leq 3) \lesssim_\epsilon (N^{(1)} N^{(2)})^{d-1+\epsilon} (N^{(3)})^d, \tag{5.9}$$

where $N^{(0)} \geq N^{(1)} \geq N^{(2)} \geq N^{(3)}$ is the non-decreasing rearrangement of the frequency-scales.

Remark 5.5. The statement of the Lemma 5.4 is technical, but it quantitatively answers a natural question: For how many frequencies $n_1, n_2, n_3 \in \mathbb{Z}^d$ do the corresponding plane waves $e^{-it|n_j|^2} e^{i n_j x}$, where $j = 1, 2, 3$, interact strongly?

The proof is the subject of Exercise 5.10, but I will describe the general idea and make an additional comment. For simplicity, let us assume that

$$N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3. \tag{5.10}$$

In that case, we would write the linear constraint as $n_0 = n_1 - n_2 + n_3$ and insert the formula for n_0 into the quadratic constraint. After algebraic manipulations, the quadratic constraint then reads as

$$2(n_1 - n_2) \cdot (n_3 - n_2) = -\mu. \tag{5.11}$$

According to our lemma, we are supposed to have the estimate

$$\begin{aligned} &\#\Big\{ (n_1, n_2, n_3) \in (\mathbb{Z}^d)^3 : |n_j| \sim N_j \text{ for } 1 \leq j \leq 3,\, n_1, n_3 \neq n_2,\, 2(n_1 - n_2) \cdot (n_3 - n_2) = -\mu \Big\} \\ &\lesssim_\epsilon (N_1 N_2)^{d-1+\epsilon} N_3^d. \end{aligned} \tag{5.12}$$

The estimate (5.12) can be proven using the sphere and divisor estimate (Proposition 2.6 and Lemma 2.8). Note that, up to the ϵ-loss, (5.12) is optimal. The reason is that

$$\sum_{\mu \in \mathbb{Z}} \#\{\cdots\} \sim (N_1 N_2 N_3)^d$$

and that the bilinear form $(n_1 - n_2)(n_3 - n_2)$ takes at most $\sim N_1 N_2$ different values. Thus, one cannot do better in (5.12) than

$$\frac{(N_1 N_2 N_3)^d}{N_1 N_2}.$$

Remark 5.6. Note that the condition $n_1, n_3 \neq n_2$ is crucial. Otherwise, for $\mu = 0$ and $N_2 = N_3$, any tuple (n_1, n_2, n_3) with $n_2 = n_3$ satisfies the constraint, and thus we obtain a lower bound by

$$\sim N_1^d N_2^d.$$

If $N_0 \sim N_1$ and $N_2 \sim N_3 \sim 1$, this would be much worse than the upper bound in (5.12).

5.3. **Proof of Proposition 5.1:**

Proof. Using Lemma 5.3 and using that $N^{(0)} \sim N^{(1)}$ (otherwise, the linear constraint cannot be satisfied), it follows that

$$
\begin{aligned}
&\mathbb{E}\Big[\Big\| \mathcal{N}\big(e^{it\Delta}\phi, e^{it\Delta}\phi, e^{it\Delta}\phi\big)\Big\|^2_{X^{s,b-1}([0,2\pi])}\Big] \\
&\lesssim \sup_{\substack{N_0,N_1,N_2,\\ N_3\in 2^{\mathbb{N}_0}}} N_{\max}^{8\delta_2} N_0^{2s} (N_1N_2N_3)^{-2} \mathcal{M}_{4,2}(N_j,\iota_j\colon 0\le j\le 3) \\
&\lesssim \sup_{\substack{N_0,N_1,N_2,\\ N_3\in 2^{\mathbb{N}_0}}} (N^{(1)})^{2s+8\delta_2-2} (N^{(2)}N^{(3)})^{-2} \mathcal{M}_{4,2}(N_j,\iota_j\colon 0\le j\le 3).
\end{aligned}
\tag{5.13}
$$

Using our counting estimate (Lemma 5.4), it follows for all $\epsilon > 0$ that

$$
\begin{aligned}
(5.13) &\lesssim \sup_{\substack{N_0,N_1,N_2,\\ N_3\in 2^{\mathbb{N}_0}}} \Big((N^{(1)})^{2s+8\delta_2-2} (N^{(2)}N^{(3)})^{-2} \times (N^{(1)}N^{(2)})^{1+2\epsilon} (N^{(3)})^2 \Big) \\
&\lesssim \sup_{\substack{N_0,N_1,N_2,\\ N_3\in 2^{\mathbb{N}_0}}} (N^{(1)})^{2s-1+8\delta_2+2\epsilon} (N^{(2)})^{-1+2\epsilon}.
\end{aligned}
\tag{5.14}
$$

This is acceptable as long as

$$
s < \frac{1}{2} - 4\delta_2 - \epsilon,
$$

which can be satisfied (by choosing $\epsilon = \delta_2$) in our case. □

Remark 5.7. In the above argument, we make no use of the additional $(N^{(2)})^{-1+2\epsilon}$-factor in (5.14). Whether or not this factor can be used to increase the regularity s depends on the specific interaction:

(i) For high×low×low-interactions, it holds that $N^{(2)} \sim 1$, and thus we cannot increase s.
(ii) For high×high×low-interactions, however, $N^{(2)} \sim N^{(1)}$, and thus we can increase the regularity from $s = \frac{1}{2} - \delta_1$ to $s = 1 - \delta_1$.

This observation is crucial for many recent developments, see e.g. [Bri21, DNY19].

5.4. **Exercises.**

Exercise 5.8. Let $d \ge 1$, let $k \ge 1$, and let $\iota_1, \cdots, \iota_k \in \{-1, +1\}$. Furthermore, let $a\colon (\mathbb{Z}^d)^k \to \mathbb{C}$, let $\Omega\colon (\mathbb{Z}^d)^k \to \mathbb{Z}$, and let $u\colon \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$ be given by

$$
e^{-it\Delta} u = \sum_{n_1,\cdots,n_k\in\mathbb{Z}^d} a(n_1,\cdots,n_k) e^{i(\iota_1 n_1 + \cdots + \iota_k n_k)x} e^{-it\Omega}.
$$

Then, it holds for all $s \in \mathbb{R}$ and $b \in \mathbb{R}$ that

$$
\|u\|^2_{X^{s,b}([0,2\pi])} \lesssim \sum_{n_0\in\mathbb{Z}^d} \sum_{\mu\in\mathbb{Z}} \langle n_0\rangle^{2s} \langle \mu\rangle^{2b} \Big| \sum_{\substack{n_1,\cdots,n_k\in\mathbb{Z}^d:\\ \iota_1 n_1+\cdots+\iota_k n_k = n_0,\\ \Omega=\mu}} a(n_1,\cdots,n_k) \Big|^2.
$$

Exercise 5.9. For all $d \ge 2$, $M, N \in 2^{\mathbb{N}_0}$, and $\epsilon > 0$, prove that

$$
\sup_{k,\ell\in\mathbb{Z}^d} \sup_{\mu\in\mathbb{Z}} \#\Big\{ (m,n) \in \mathbb{Z}^d \times \mathbb{Z}^d \colon m,n \ne 0,\ |m-k| \le M,\ |n-\ell| \le N,\ m\cdot n = \mu \Big\} \lesssim (MN)^{d-1+\epsilon}.
\tag{5.15}
$$

For this exercise, you may use (without proof) that the following strengthened divisor estimate is satisfied: For all $A, B \geq 1$, $a_0, b_0 \in \mathbb{Z}$, and $\mu \in \mathbb{Z}$, it holds that

$$\#\Big\{(a,b) \in \mathbb{Z} \times \mathbb{Z} \colon a, b \neq 0,\, |a - a_0| \leq A,\, |b - b_0| \leq B,\, a \cdot b = \mu\Big\} \lesssim_\epsilon (AB)^\epsilon. \tag{5.16}$$

For a proof of (5.16), see [DNY19, Lemma 4.3.(1)].

Hint/Note: Apply (5.16) to the entries of m and n. While it is not a hard exercise, it is not completely trivial either. The reason is that even if $m, n \in \mathbb{Z}^d$ satisfy $m, n \neq 0$, they may still have zero entries.

Exercise 5.10. Using Exercise 5.9 (and potentially Proposition 2.6), prove Lemma 5.4.

6. Random tensor estimates

As described in Subsection 4.3, solving the evolution equation for the nonlinear remainder v requires control over interactions between $e^{it\Delta}\phi$ and v. One of these interactions is the subject of the next definition.

Proposition 6.1. Let $s := \frac{1}{2} - \delta_1$, let $b := \frac{1}{2} + \delta_2$, let $b' := \frac{1}{2} - \delta_2$, where $0 \ll \delta_2 \ll \delta_1 \ll 1$. Furthermore, let ϕ be as in (1.1). Then, it holds that

$$\mathbb{E}\bigg[\Big\| v \mapsto \mathcal{N}\big(e^{it\Delta}\phi, e^{it\Delta}\phi, v\big) \Big\|^2_{X^{s,b'}([0,2\pi]) \to X^{s,b-1}([0,2\pi])} \bigg] \lesssim_{\delta_1,\delta_2} 1. \tag{6.1}$$

Compared to Proposition 5.1, the new aspect of Proposition 6.1 is that it concerns a random operator, and not an explicit random distribution. Unfortunately, I will not have time to present a full proof of Proposition 6.1. Instead, I will focus on general random tensor estimates, which are a new technique [DNY22] and can be used to prove Proposition 6.1.

6.1. **Abstract random tensor estimate.**

Definition 6.2 (Tensors)**.** A tensor is a map $h\colon (\mathbb{Z}^d)^J \to \mathbb{C}$, where J is a finite index set. We often write $h = h_{n_J}$ and, if $J = \{j_1, j_2, \cdots, j_k\}$, we often also write $h = h_{n_{j_1} \cdots j_k}$.

Example 6.3. For Proposition 6.1, the most important tensor is

$$\begin{aligned} h_{n_0n_1n_2n_3} := &\Big(\prod_{j=0}^{3} \mathbf{1}_{N_j}(n_j) \Big) \times \mathbf{1}\{n_1, n_3 \neq n_2\} \times \mathbf{1}\big\{ -n_0 + n_1 - n_2 + n_3 = 0 \big\} \\ &\times \mathbf{1}\big\{ -|n_0|^2 + |n_1|^2 - |n_2|^2 + |n_3|^2 = \mu \big\}, \end{aligned} \tag{6.2}$$

where $\mu \in \mathbb{Z}$.

Definition 6.4 (Tensor norms)**.** Let $h = h_{n_J}$ be a tensor and let X and Y form a partition of J, i.e., $X, Y \subseteq J$ and $X \uplus Y = J$. Then, we define

$$\|h\|_{n_X \to n_Y} := \sup\Big\{ \sum_{n_Y} \Big| \sum_{n_X} h_{n_J} v_{n_X} \Big|^2 : \sum_{n_X \in (\mathbb{Z}^d)^X} |v_{n_X}|^2 \leq 1 \Big\}. \tag{6.3}$$

After viewing h as a linear operator from $\ell^2_{n_X}$ to $\ell^2_{n_Y}$, the tensor norm $\|h\|_{n_X \to n_Y}$ coincides with the usual operator norm. We emphasize that, even for a fixed set J, there are many different choices of X and Y and thus many different tensor norms.

Proposition 6.5 (Abstract random tensor estimate [DNY22])**.** Let $h = h_{n_1n_2n_An_B}$ be a tensor, where A and B are finite index sets. Let $N \in 2^{\mathbb{N}_0}$ and assume that, on the support of h, it holds that

$$|n_1|, |n_2|, \max_{a\in A} |n_a|, \max_{b \in B} |n_b| \lesssim N. \tag{6.4}$$

Furthermore, let $(g_n)_{n\in\mathbb{Z}^d}$ be a sequence of independent, standard complex-valued Gaussians and let $H = H_{n_An_B}$ be he random tensor defined as

$$H_{n_An_B} := \sum_{n_1,n_2 \in \mathbb{Z}^d} h_{n_1n_2n_An_B}\big(g_{n_1}\overline{g_{n_2}} - \delta_{n_1n_2} \big). \tag{6.5}$$

Then, it holds for all $\epsilon > 0$ and $p \geq 1$ that

$$\mathbb{E}\Big[\|H\|_{n_A \to n_B}^p\Big]^{\frac{1}{p}} \lesssim_\epsilon pN^\epsilon \max_{X \uplus Y = \{1,2\}} \|h\|_{n_A n_X \to n_B n_Y}. \tag{6.6}$$

Remark 6.6. We make the following remarks regarding Proposition 6.5.

(i) The estimate was stated in the simplest possible form for our application, but can be generalized. For any $m \geq 1$, a similar estimate holds for products of m Gaussians.

(ii) The estimate has found many applications in the recent literature, see e.g. [Bri20, BDNY22]. In these applications, one first uses Schur's test, which yields

$$\|h\|_{n_A n_X \to n_B n_Y}^2 \lesssim \Big(\sup_{n_A n_X} \sum_{n_B n_Y} |h_{n_X n_Y n_A n_B}|\Big) \times \Big(\sup_{n_B n_Y} \sum_{n_A n_X} |h_{n_X n_Y n_A n_B}|\Big).$$

The right-hand side is then controlled through further counting problems.

(iii) The estimate is closely related to operator bounds for structured random matrices. In fact, it can be proven using the methods in [vH17].

6.2. **Proof of abstract random tensor estimate.** We present a slightly different proof of the abstract tensor estimate than in [DNY22], which is more modular. It relies on the following three ingredients:

(i) Tensor merging estimate (Lemma 6.7),
(ii) Probabilistic decoupling (Lemma 6.8),
(iii) Gaussian case (Lemma 6.9).

Lemma 6.7 (Merging estimate)**.** Let A_1, A_2, B_1, B_2, and C be disjoint finite index sets and let $h^{(1)} = h^{(1)}_{n_{A_1} n_{B_1} n_C}$ and $h^{(2)} = h^{(2)}_{n_{A_2} n_{B_2} n_C}$ be two different tensors. Then, it holds that

$$\Big\| \sum_{n_C} h^{(1)}_{n_{A_1} n_{B_1} n_C}\, h^{(2)}_{n_{A_2} n_{B_2} n_C} \Big\|_{n_{A_1} n_{A_2} \to n_{B_1} n_{B_2}} \leq \big\| h^{(1)}_{n_{A_1} n_{B_1} n_C} \big\|_{n_{A_1} \to n_{B_1} n_C} \big\| h^{(2)}_{n_{A_2} n_{B_2} n_C} \big\|_{n_{A_2} n_C \to n_{B_2}}. \tag{6.7}$$

Proof. Let $z = z_{n_{A_1} n_{A_2}}$ be arbitrary. By first using the tensor estimate for $h^{(2)}$, we obtain that

$$\begin{aligned}
&\sum_{n_{B_1} n_{B_2}} \Big| \sum_{n_{A_1} n_{A_2} n_C} h^{(1)}_{n_{A_1} n_{B_1} n_C}\, h^{(2)}_{n_{A_2} n_{B_2} n_C} z_{n_{A_1} n_{A_2}} \Big|^2 \\
&= \sum_{n_{B_1}} \sum_{n_{B_2}} \Big| \sum_{n_{A_2} n_C} h^{(2)}_{n_{A_2} n_{B_2} n_C} \Big(\sum_{n_{A_1}} h^{(1)}_{n_{A_1} n_{B_1} n_C} z_{n_{A_1} n_{A_2}} \Big) \Big|^2 \\
&\leq \big\| h^{(2)}_{n_{A_2} n_{B_2} n_C} \big\|^2_{n_{A_2} n_C \to n_{B_2}} \sum_{n_{B_1}} \sum_{n_{A_2}, n_C} \Big| \sum_{n_{A_1}} h^{(1)}_{n_{A_1} n_{B_1} n_C} z_{n_{A_1} n_{A_2}} \Big|^2.
\end{aligned}$$

By using the tensor bound for $h^{(1)}$, we also obtain that

$$\sum_{n_{A_2}} \sum_{n_{B_1}, n_C} \Big| \sum_{n_{A_1}} h^{(1)}_{n_{A_1} n_{B_1} n_C} z_{n_{A_1} n_{A_2}} \Big|^2 \leq \big\| h^{(1)}_{n_{A_1} n_{B_1} n_C} \big\|^2_{n_{A_1} \to n_{B_1} n_C} \sum_{n_{A_2}} \sum_{n_{A_1}} |z_{n_{A_1} n_{A_2}}|^2.$$

This implies the desired merging estimate (6.7). ☐

Lemma 6.8 (Probabilistic decoupling)**.** Let h and $g = (g_n)_{n \in \mathbb{Z}^d}$ be as in Proposition 6.5 and let $g' = (g'_n)_{n \in \mathbb{Z}^d}$ be an independent copy of g. For all $p \geq 1$, it then holds that

$$\begin{aligned} &\mathbb{E}\Big[\Big\| \sum_{n_1,n_2 \in \mathbb{Z}^d} h_{n_1 n_2 n_A n_B} \big(g_{n_1} \overline{g_{n_2}} - \delta_{n_1 n_2} \big) \Big\|_{n_A \to n_B}^p \Big]^{1/p} \\ &\leq \pi \cdot \mathbb{E}\Big[\Big\| \sum_{n_1,n_2 \in \mathbb{Z}^d} h_{n_1 n_2 n_A n_B}\, g_{n_1} \overline{g'_{n_2}} \Big\|_{n_A \to n_B}^p \Big]^{1/p}. \end{aligned} \tag{6.8}$$

Proof. For expository purposes, we write $\mathbb{E}_g$ and $\mathbb{E}_{g'}$ for the expectations taken over g and g', respectively. Furthermore, for any $a = (a_{n_1 n_2})_{n_1,n_2 \in \mathbb{Z}^d}$, we write

$$F(a_{n_1 n_2}) := \Big\| \sum_{n_1,n_2 \in \mathbb{Z}^d} h_{n_1 n_2 n_A n_B} a_{n_1 n_2} \Big\|_{n_A \to n_B}, \tag{6.9}$$

which is convex and one-homogeneous. It holds that

$$\begin{aligned} &\mathbb{E}_g\Big[\Big\| \sum_{n_1,n_2 \in \mathbb{Z}^d} h_{n_1 n_2 n_A n_B} \big(g_{n_1} \overline{g_{n_2}} - \delta_{n_1 n_2} \big) \Big\|_{n_A \to n_B}^p \Big]^{1/p} \\ &= \mathbb{E}_g\Big[F\big(g_{n_1} \overline{g_{n_2}} - \delta_{n_1 n_2} \big)^p \Big]^{1/p} \\ &= \mathbb{E}_g\Big[F\big(\mathbb{E}_{g'}\big[g_{n_1} \overline{g_{n_2}} - g'_{n_1} \overline{g'_{n_2}} \big] \big)^p \Big]^{1/p} \\ &\leq \mathbb{E}_g \mathbb{E}_{g'}\Big[F\big(g_{n_1} \overline{g_{n_2}} - g'_{n_1} \overline{g'_{n_2}} \big)^p \Big]^{1/p}. \end{aligned} \tag{6.10}$$

In the last line, we used Jensen's inequality. For any $\varphi \in [0, \frac{\pi}{2}]$, we define

$$g(\varphi) := \sin(\varphi) g + \cos(\varphi) g'. \tag{6.11}$$

By definition, it holds that $g(0) = g'$ and $g(\frac{\pi}{2}) = g$, and thus $g(\varphi)$ interpolates between g and g'. We note that

$$\partial_\varphi g(\varphi) = \cos(\varphi) g - \sin(\varphi) g'.$$

Since $(\sin(\varphi), \cos(\varphi))$ and $(\cos(\varphi), -\sin(\varphi))$ are orthonormal in $\mathbb{R}^2$ for all $\varphi \in [0, \frac{\pi}{2}]$, rotation-invariance of Gaussians implies for all $\varphi \in [0, \frac{\pi}{2}]$ that

$$\Big(g(\varphi), \partial_\varphi g(\varphi) \Big) \overset{d}{=} \Big(\partial_\varphi g(\varphi), g(\varphi) \Big) \overset{d}{=} \Big(g, g' \Big). \tag{6.12}$$

By using (6.11) and the triangle-inequality, it follows that

$$\begin{aligned} &\mathbb{E}_g \mathbb{E}_{g'}\Big[F\big(g_{n_1} \overline{g_{n_2}} - g'_{n_1} \overline{g'_{n_2}} \big)^p \Big]^{1/p} \\ &= \mathbb{E}_g \mathbb{E}_{g'}\Big[F\Big(\int_0^{\frac{\pi}{2}} \partial_\varphi \big(g_{n_1}(\varphi) \overline{g_{n_2}(\varphi)} \big) \Big)^p \Big]^{1/p} \\ &\leq \int_0^{\frac{\pi}{2}} \mathbb{E}_g \mathbb{E}_{g'}\Big[F\Big(\partial_\varphi \big(g_{n_1}(\varphi) \overline{g_{n_2}(\varphi)} \big) \Big)^p \Big]^{1/p} \\ &\leq \int_0^{\frac{\pi}{2}} \mathbb{E}_g \mathbb{E}_{g'}\Big[F\Big(\partial_\varphi g_{n_1}(\varphi)\, \overline{g_{n_2}(\varphi)} \Big)^p \Big]^{1/p} + \int_0^{\frac{\pi}{2}} \mathbb{E}_g \mathbb{E}_{g'}\Big[F\Big(g_{n_1}(\varphi)\, \overline{\partial_\varphi g_{n_2}(\varphi)} \Big)^p \Big]^{1/p}. \end{aligned} \tag{6.13}$$

By using (6.12), the integrands in (6.13) are given by

$$\mathbb{E}_g \mathbb{E}_{g'}\Big[F\Big(\partial_\varphi g_{n_1}(\varphi)\, \overline{g_{n_2}(\varphi)} \Big)^p \Big]^{1/p} = \mathbb{E}_g \mathbb{E}_{g'}\Big[F\Big(g_{n_1}(\varphi)\, \overline{\partial_\varphi g_{n_2}(\varphi)} \Big)^p \Big]^{1/p} = \mathbb{E}_g \mathbb{E}_{g'}\Big[F\Big(g_{n_1}\, \overline{g'_{n_2}} \Big)^p \Big]^{1/p}.$$

As a result, it follows that

$$(6.13) = \pi \cdot \mathbb{E}_g \mathbb{E}_{g'} \Big[F\Big(g_{n_1} \overline{g'_{n_2}} \Big)^p \Big]^{1/p}.$$

After recalling the definition of F, this implies the desired inequality (6.8). $\square$

Lemma 6.9 (Gaussian case)**.** Let $h = h_{n_0 n_A n_B}$ be a tensor and assume that, on the support of h, it holds that

$$|n_0|, \max_{a \in A} |n_a|, \max_{b \in B} |n_b| \lesssim N.$$

Then, it holds for all $\epsilon > 0$ and $p \geq 1$ that

$$\mathbb{E}\Big[\Big\| \sum_{n_0} h_{n_0 n_A n_B} g_{n_0} \Big\|_{n_A \to n_B}^p \Big]^{\frac{1}{p}} \lesssim \sqrt{p} N^\epsilon \max\Big(\big\| h_{n_0 n_A n_B} \big\|_{n_0 n_A \to n_B}, \big\| h_{n_0 n_A n_B} \big\|_{n_A \to n_0 n_B} \Big). \tag{6.14}$$

The main ingredient in the proof is the non-commutative Khintchine inequality. For a precise statement and elegant proof, we refer to [vH17].

Proof. For each $n_0 \in \mathbb{Z}^d$, we let $\mathcal{L}_{n_0} \colon \ell^2_{n_A} \to \ell^2_{n_B}$ be the linear operator defined as

$$(\mathcal{L}_{n_0} z)_{n_B} = \sum_{n_A} h_{n_0 n_A n_B} z_{n_A}.$$

Using the definition of $\mathcal{L}_{n_0}$, we can write

$$\Big\| \sum_{n_0} h_{n_0 n_A n_B} g_{n_0} \Big\|_{n_A \to n_B} = \Big\| \sum_{n_0} g_{n_0} \mathcal{L}_{n_0} \Big\|_{\mathrm{op}},$$

where $\| \cdot \|_{\mathrm{op}}$ is the usual operator norm. Using the non-commutative Khintchine inequality [vH17, Theorem 3.3] for non-symmetric matrices (see [vH17, p.6]), it follows that

$$\mathbb{E}\Big[\Big\| \sum_{n_0} g_{n_0} \mathcal{L}_{n_0} \Big\|_{\mathrm{op}}^p \Big]^{\frac{1}{p}} \lesssim \sqrt{p} N^\epsilon \max\Big(\Big\| \sum_{n_0} \mathcal{L}^*_{n_0} \mathcal{L}_{n_0} \Big\|_{\mathrm{op}}^{\frac{1}{2}}, \Big\| \sum_{n_0} \mathcal{L}_{n_0} \mathcal{L}^*_{n_0} \Big\|_{\mathrm{op}}^{\frac{1}{2}} \Big). \tag{6.15}$$

We now estimate the two arguments in (6.15) separately. Using the definition of $\mathcal{L}_{n_0}$, we can write the entries in the first argument as

$$\Big(\sum_{n_0} \mathcal{L}^*_{n_0} \mathcal{L}_{n_0} \Big)_{n'_A n_A} = \sum_{n_0, n_B} \overline{h_{n_0 n'_A n_B}} h_{n_0 n_A n_B}.$$

Using the merging estimate (Lemma 6.7), it follows that

$$\begin{aligned} \Big\| \sum_{n_0} \mathcal{L}^*_{n_0} \mathcal{L}_{n_0} \Big\|_{\mathrm{op}} &= \Big\| \sum_{n_0, n_B} \overline{h_{n_0 n'_A n_B}} h_{n_0 n_A n_B} \Big\|_{n_A \to n'_A} \\ &\leq \big\| h_{n_0 n_A n_B} \big\|_{n_A \to n_0 n_B} \big\| \overline{h_{n_0 n'_A n_B}} \big\|_{n_0 n_B \to n'_A} \\ &= \big\| h_{n_0 n_A n_B} \big\|_{n_A \to n_0 n_B}^2. \end{aligned}$$

Arguing similarly, it also follows that

$$\Big\| \sum_{n_0} \mathcal{L}_{n_0} \mathcal{L}^*_{n_0} \Big\|_{\mathrm{op}} \leq \big\| h_{n_0 n_A n_B} \big\|_{n_0 n_A \to n_B}^2$$

This yields the desired estimate (6.14). $\square$

Equipped with Lemma 6.8 and Lemma 6.9, the proof of Proposition 6.5 is now rather simple.

Proof of Proposition 6.5: Using Lemma 6.8, it sufffices to control

$$\Big\| \sum_{n_1,n_2} h_{n_1 n_2 n_A n_B} g_{n_1} \overline{g'}_{n_2} \Big\|_{n_A \to n_B}.$$

Since $(g_n)_{n\in\mathbb{Z}^d}$ and $(g'_n)_{n\in\mathbb{Z}^d}$ are probabilistically independent, this can be done by applying Lemma 6.9 twice. □

6.3. **What is left to be done?** In order to complete the proof of Proposition 6.1, we still have to do the following:

(i) Using Schur's test and counting estimates, control the tensor norms of

$$\begin{aligned} h_{n_0 n_1 n_2 n_3} :=& \Big(\prod_{j=0}^{3} \mathbf{1}_{N_j}(n_j) \Big) \times \mathbf{1}\{n_1, n_3 \neq n_2\} \times \mathbf{1}\Big\{ -n_0 + n_1 - n_2 + n_3 = 0 \Big\} \\ &\times \mathbf{1}\Big\{ -|n_0|^2 + |n_1|^2 - |n_2|^2 + |n_3|^2 = \mu \Big\}. \end{aligned}$$

This is the subject of Exercise 6.11.

(ii) Deal with a (technical) problem due to the ϵ-loss in Proposition 6.5, which is an issue in Proposition 6.1 when v enters at much higher frequencies than both factors of $e^{it\Delta}\phi$ (see e.g. [DNY19, Claim 5.2] and [BR23, Proof of Proposition 5.1]).

In order to complete the proof of Theorem 1.2, we then still need to solve the equation for the nonlinear remainder v, i.e., we need to solve

$$i\partial_t v + \Delta v = \sigma \sum_{\substack{w_1,w_2,w_3 \\ \in\{e^{it\Delta}\phi, v\}}} \Big(\mathcal{N}(w_1,w_2,w_3) + \mathcal{N}^{\mathfrak{r}}(w_1,w_2,w_3) \Big). \tag{6.16}$$

In Proposition 5.1 and Proposition 6.1, we have obtained estimates for

$$\mathcal{N}(e^{it\Delta}\phi, e^{it\Delta}\phi, e^{it\Delta}\phi) \qquad \text{and} \qquad \mathcal{N}(e^{it\Delta}\phi, e^{it\Delta}\phi, v),$$

respectively. While the remaining terms have not been treated here, their estimates are simpler (or similar).

6.4. **Exercises.**

Exercise 6.10. Let $M \geq 2$ and let $(g_m)_{m=1}^M$ be independent, standard, complex-valued Gaussians. Show that

$$\mathbb{E}\Big[\max_{m=1,\cdots,M} |g_m| \Big] \sim \sqrt{\log(M)}. \tag{6.17}$$

Furthermore, let $G \in \mathbb{C}^{M\times M}$ be the diagonal matrix with diagonal entries $g_1, \ldots, g_M$. Using (6.17), prove that

$$\mathbb{E}\Big[\|G\|_{\mathrm{op}} \Big] \sim \sqrt{\log(M)}.$$

Note: It follows that, without additional information on h, at least a logarithmic dimension-dependent loss in Proposition 6.5 is necessary.

Exercise 6.11 (Estimate of tensor norms)**.** Let $d \geq 2$ and let $h = h_{n_0 n_1 n_2 n_3}$ be as in (6.2). Using Schur's test and counting estimates, prove the following three estimates:

$$\|h\|^2_{n_1 n_2 n_3 \to n_0} \lesssim \operatorname{med}(N_1, N_2, N_3)^{d-1+\epsilon} \min(N_1, N_2, N_3)^{d-1+\epsilon}, \tag{6.18}$$

$$\|h\|^2_{n_2 n_3 \to n_0 n_1} \lesssim \min(N_0, N_1)^{d-1} \min(N_2, N_3)^{d-1}. \tag{6.19}$$

$$\|h\|^2_{n_1 n_3 \to n_0 n_2} \lesssim \min(N_0, N_2)^{d-2+\epsilon} \min(N_1, N_3)^{d-2+\epsilon}. \tag{6.20}$$

We remark that the second estimate (6.19) is rather crude, but potential improvements require further number-theoretic restrictions and/or considerations.

7. Open problems

In this section, I list some of the main open problems in random dispersive equations. All of the problems concern the invariance of Gibbs measures, but the core of the problems is primarily about probabilistic well-posedness. The open problems listed here are currently driving research in this field, but most likely are not suitable first projects for a beginning graduate student. If you are looking for a problem of the latter kind, feel free to talk to me in person.

Problem 7.1. Prove the invariance of the Gibbs measure for the defocusing cubic nonlinear Schrödinger equation on $\mathbb{T}^3$, i.e., in dimension $d = 3$.

This is the extension of [Bou96], which we mostly proved, from $d = 2$ to $d = 3$. It is really difficult because it is critical with respect to probabilistic scaling [DNY22].

Problem 7.2. Prove the invariance of the Gibbs measure for the defocusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2$.

Since the Φ^4_d-measures are translation invariant, random distributions $\phi\colon \mathbb{R}^d \to \mathbb{C}$ drawn from the Φ^4_d-measure exhibit no spatial decay. Together with the infinite speed of propagation of Schrödinger equations, this makes Problem 7.2 quite challenging. The one-dimensional case was treated by Bourgain in [Bou00].

Problem 7.3. Consider the two-dimensional sine-Gordon equation

$$\big(\partial_t^2 + 1 - \Delta\big)u + \sin\big(\beta u\big) = 0 \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \tag{sG}$$

where $\beta \in \mathbb{R}$. What are the optimal conditions on β which guarantee the invariance of the corresponding Gibbs measure under (sG)?

One of the most interesting aspects of Problem 7.3 is the non-polynomial nonlinearity, which creates a challenge for frequency-based methods. Partial progress towards this problem has been made in [ORSW21].

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