Lecture 1

1. What are the real numbers?

$\mathbb{R}$ is a field and its elements are ordered.

reflexive
antisymmetric
transitive

Def: an ORDERING of a field R is a relation "$\leq$" that is compatible with addition and multiplication:

(i) $x \leq y \Rightarrow x+z \leq y+z$,

(ii) $0 \leq x,\ 0 \leq y \Rightarrow 0 \leq xy$.

An ORDERED FIELD $(R, \leq)$ is a field R equipped with an ordering $\leq$.

Properties:
- $-1 < 0$ in $(R, \leq)$
- only one of the following holds for $a \in (R, \leq)$: $a < 0$, $a = 0$, $a > 0$
- $\operatorname{char}(R, \leq) = 0$

Examples:
- $\mathbb{R}$, $\mathbb{Q}$ are ordered fields
- $\mathbb{R}[x]$ admits orderings
- $\mathbb{C}$ does NOT admit orderings

How to detect if a field R can be ordered?

Def: a CONE of a field R is a subset $P \subset R$ such that

(i) $x, y \in P \Rightarrow \begin{array}{l} x+y \in P \\ x \cdot y \in P \end{array}$

(ii) $x \in F \Rightarrow x^2 \in P$

P is called PROPER (or PREORDERING) if $-1 \notin P$.

The set of sums of squares of R is a cone, denoted ΣR^2.
Note that ΣR^2 is contained in every cone.

If $(R, \geq)$ ordered, then $P = \{x \in R \mid x \geq 0\}$ is a proper cone.

"positive cone"

Theorem: Let R be a field. The following are equivalent:

(i) R can be ordered

(ii) R has a proper cone

(iii) $-1 \notin \sum R^2$

(iv) $\forall\, x_1,\dots,x_n \in R$, $\quad x_1^2+\dots+x_n^2=0 \implies x_1=\dots=x_n=0$

Proof: (i)$\Rightarrow$(ii): the positive cone.
(ii) $\Rightarrow$ (iii): P proper cone, $P \supset \sum R^2$. Since $1 \notin P$ then $1 \notin \sum R^2$.
(iii) $\Rightarrow$ (i): $\sum R^2$ is a proper cone $\Rightarrow$ FACT: it is contained in the positive cone of an ordering $\Leftarrow$ R has an ordering
(i) $\Rightarrow$ (iv): assume $\sum x_i^2 = 0$ and $x_1 \neq 0$. Then $\sum x_i^2 \geq x_1^2 > 0$.
(iv) $\Rightarrow$ (iii): assume $-1 = \sum x_i^2$. Then $1 + \sum x_i^2 = 0$ implies $1 = 0$. //

$\rightsquigarrow$ easier now to check that $\mathbb{C}$ cannot be ordered.

Def: a field R is REAL if it satisfies any of the properties in the previous theorem.

We need a bit more: if we consider a field extension of R, can we extend also the ordering of R?

↳ $a \in R$ not a square $\Rightarrow$ we can extend the ordering to $R(\sqrt{a})$ in the case $a>0$

↳ L field extension of R of odd degree $\Rightarrow$ any ordering extends to L

How much can we extend the ordering?

Theorem (Artin-Schreier) The following are equivalent:

(i) R is a real field and no proper algebraic extension of R is real

(ii) $\sum R^2$ is the positive cone of an ordering and every polynomial of odd degree in one variable over R has a root in R,

(iii) $-1 \in R$ is not a square and $R(\sqrt{-1})$ is algebraically closed.

Def: a field R is REAL CLOSED if it satisfies any of the properties in the previous theorem.

Examples:
- $\mathbb{R}$ is real closed
- $\mathbb{Q}$ is NOT real closed $\rightsquigarrow$ there is a concept of REAL CLOSURE: $\mathbb{R}_{alg}$ is the real closure of $\mathbb{Q}$

FACT: a real closed field has a unique ordering

2. Real algebraic varieties

Let $V \subset \mathbb{C}^n$ be an irreducible algebraic variety.
We can view V in $\mathbb{R}^{2n}$ and we have that

- V connected
- V unbounded
- locally the (real) dim at every smooth point of V is $2d$, where $d = \dim_{\mathbb{C}} V$

These statements do not hold for real algebraic sets

$\rightarrow$ see §3.1 [BCR]

R real closed field, $C = \overline{R}$ algebraic closure.

Let $I \subset R[x_1, \dots, x_n]$ be an ideal, then

$$V = V_C(I) = \{ x \in C^n \mid f(x) = 0 \ \forall f \in I \}$$

is an affine $\underbrace{R}_{\text{= defined over } R}$-variety.

$\rightarrow$ complex

The set of real points of V is $V(R) = V_R(I) = V \cap R^n$

We can also define an R-ideal of a set $S \subset \mathbb{C}^n$:

$$I_{\mathbb{R}}(S) = \{ f \in \mathbb{R}[x_1,\dots,x_n] \mid f(x)=0 \ \forall x \in S \}.$$

Then, if V is an $\mathbb{R}$-variety, we define its COORDINATE RING

$$\mathbb{R}[V] = \mathbb{R}[x_1,\dots,x_n] / I_{\mathbb{R}}(V)$$

two polynomials are in the same equivalence class iff they coincide over V

How to move between varieties and ideals?

Hilbert's Nullstellensatz: $I \subset \mathbb{R}[x_1,\dots,x_n]$ ideal, then

(i) $V_{\mathbb{C}}(I) = \emptyset \iff 1 \in I$

(ii) $I_{\mathbb{R}}(V_{\mathbb{C}}(I)) = \sqrt{I} = \{ f \in \mathbb{R}[x_1,\dots,x_n] \mid f^r \in I \text{ for some } r \in \mathbb{N} \}$

$I \subset$

But what about the real points?

(i) : $I = \langle f \rangle$ for $f = x^2+y^2+1$
then $V_{\mathbb{R}}(I) = \emptyset$ BUT $1 \notin I$

(ii) : $I = \langle f \rangle$ for $f = x^2+y^2$
then $V_{\mathbb{R}}(I) = \{(0,0)\} \implies I_{\mathbb{R}}(V_{\mathbb{R}}(I)) = \langle x, y \rangle$

what is the relation?

Def: $I \subset \mathbb{R}[x_1,\dots,x_n]$ ideal is called REAL if $\forall f_1,\dots,f_k \in \mathbb{R}[x_1,\dots,x_n]$

$$f_1^2 + \dots + f_k^2 \in I \implies f_1,\dots,f_k \in I$$

Examples:
- $\langle x^2+y^2 \rangle$ is NOT real
- $\langle y - x^2 \rangle$ is real

Real Nullstellensatz: $I \subset \mathbb{R}[x_1,\dots,x_n]$ ideal, then

(i) $V_{\mathbb{R}}(I) = \emptyset \iff \exists\, f_1,\dots,f_k \in \mathbb{R}[x_1,\dots,x_n]$ s.t. $1+f_1^2+\dots+f_k^2 \in I$

(ii) $I = I_{\mathbb{R}}(V_{\mathbb{R}}(I)) \iff I$ is real

(iii) $I_{\mathbb{R}}(V_{\mathbb{R}}(I)) = \sqrt[\text{re}]{I} := \{ g \in \mathbb{R}[x_1,\dots,x_n] \mid g^{2m}+f_1^2+\dots+f_k^2 \in I$ for some $m \in \mathbb{N}$ and $f_1,\dots,f_k \in \mathbb{R}[x_1,\dots,x_n] \}$

not clear that it is an ideal

$\Rightarrow$ • $\mathbb{R}$-Zariski topology on $\mathbb{C}^n$:

$$\overline{S}^{\mathbb{R}} = V_{\mathbb{C}}(I_{\mathbb{R}}(S))$$

$\hookrightarrow$ $\{a\}$ is closed if $a \in \mathbb{R}$

$\{i\}$ is NOT closed $\rightsquigarrow$ $\overline{\{i\}}^{\mathbb{R}} = \{i,-i\}$

Def: $V \subset \mathbb{C}^n$ affine $\mathbb{R}$-variety is REAL if $V(\mathbb{R})$ is dense in V w.r.t. the $\mathbb{R}$-Zariski topology

$\hookrightarrow$ $I \subset \mathbb{R}[x_1,\dots,x_n]$ ideal. $V_{\mathbb{C}}(I)$ real $\iff \sqrt{I}$ real

$\hookrightarrow$ $V \subset \mathbb{C}^n$ irreducible real $\mathbb{R}$-variety. Then, $f \in \mathbb{R}(V)$ satisfies $f(x) \geq 0 \;\forall\, x \in V(\mathbb{R}) \iff f =$ s.o.s. in $\mathbb{R}(V)$.

Hilbert 17^{th} problem for irreducible varieties $\to$ see lecture 3

Later: Raluca $\to$ different point of view