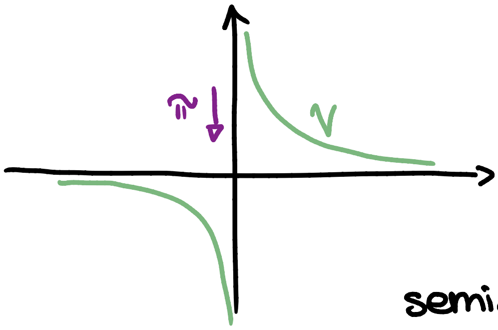
Lecture 2

Problem:



$\pi(V)$ is not an algebraic set:

$$\pi(V) = \{x \neq 0\}$$

$\rightsquigarrow$ semialgebraic set

3. Semialgebraic sets $\rightsquigarrow$ jumping between $\mathbb{R}$ and R

Def: a basic semialgebraic set $S \subset \mathbb{R}^n$ is the solution set of a system of polynomial (in)equalities:

$$S = \{x \in \mathbb{R}^n \mid f_1(x) \,\square_1\, 0, \dots, f_k(x) \,\square_k\, 0\}$$

where $f_i \in \mathbb{R}[x]$ and $\square_i \in \{<, =, >\}$ $\forall i$.

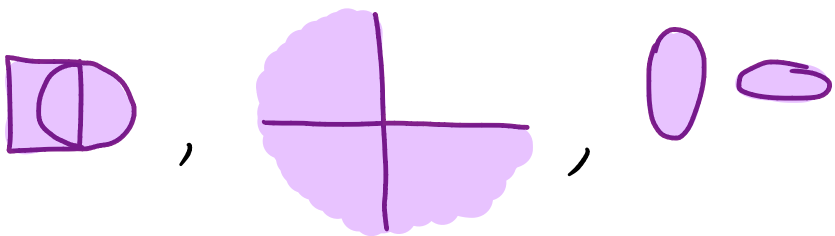
Def: a semialgebraic set is a finite union of basic semialgebraic sets.

$\Updownarrow$

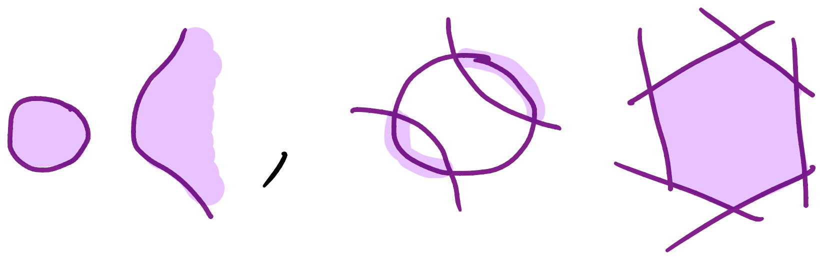
equivalently: a semialgebraic set is a Boolean combination $(\cap, \cup, \neg)$ of polynomial (in)equalities.

Examples:

- non-basic: $(\mathbb{R}^2)$



- basic:



Projection theorem: $S \subset \mathbb{R}^{n+1}$ semialgebraic. Let $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first n coordinates. Then $\pi(S)$ is semialgebraic.

$\leftarrow$ no easy proof

A few more facts about semialgebraic sets:

- the Minkowski sum of semialg. sets is semialg.
- the product of semialg. sets is semialg.
- interior, closure, boundary of a semialg. set is semialg.
- a semialg. set is the projection of an algebraic set

Def: a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is semialgebraic if

$$\text{graph}(f) \subset \mathbb{R}^n \times \mathbb{R}^m$$

is a semialgebraic set.

Examples:
- $f(x) \in \mathbb{R}[x]$
- $f(x) = \dfrac{p(x)}{q(x)}$, $p, q \in \mathbb{R}[x]$
- $f(x) = \sqrt{x}$
- $f(x) = \| x \|$
- NON-EXAMPLES: $f(x) = \cos(x)$

  $f(x) = e^x$

Some properties:

- sum of semialgebraic maps is semialgebraic
- composition of semialgebraic maps is semialgebraic
- preimage of semialg. set under semialg. map is semialg.
- image of semialg. set under semialg. map is semialg.

4. Tarski-Seidenberg

Example: $S = \{ x \geq 0,\ x^2 + y^2 - 1 \leq 0 \} \subset \mathbb{R}^2$

$S_y = \{ y^2 - 1 \leq 0 \} \subset \mathbb{R}$

Then, $\exists x$ s.t. $(x,y) \in S \iff y \in S_y$

Back to a real closed field R.

Def: a formula with coefficients in the ring A is constructed as follows:

- given $f \in A[x_1, \dots, x_n]$, $f \gtreqless 0$, $f \neq 0$ is a formula
- given the formulas ϕ, ψ also $\phi \wedge \psi$, $\phi \vee \psi$, $\neg \phi$ are formulas
- given the formula ϕ, also $\exists x_i\ \phi$, $\forall x_i\ \phi$ are formulas

quantifiers

Theorem (Quantifier elimination): ϕ formula with coefficients in the ring A contained in the real closed field R. Then there exists a quantifier-free formula ψ with coefficients in A such that for every $x \in R^n$, $\phi(x)$ is true iff $\psi(x)$ is true.

Corollary: Let ϕ be a formula with coefficients in $A \subset R$. Then $\{ x \in R^n \mid \phi(x) \text{ true} \}$ is semialgebraic.

Even stronger, the following is the Tarski-Seidenberg principle or the Transfer principle, which allows to move between different real closed fields.

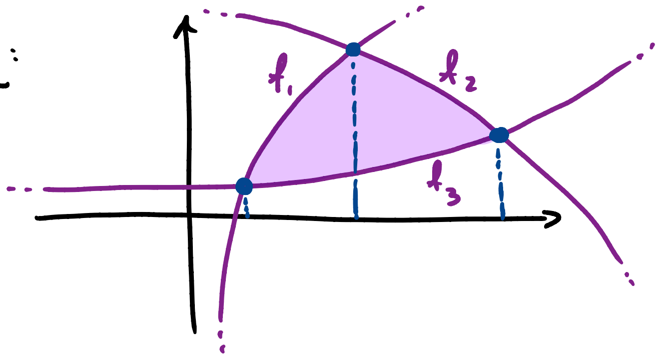
Theorem: Let $(A, \geq)$ be an ordered ring with $A \subset R_1$, $A \subset R_2$ where R_i are real closed fields extending "$\geq$". If ϕ is a formula with coefficients in A, then ϕ is true in R_1 iff it is true in R_2. ← we can always assume $A = \mathbb{Z}, \mathbb{Q}$

↳ **Another version**: Let $S \subset \mathbb{R}^{n+m}$ semialgebraic defined over $\mathbb{Z}$. Then $\exists S_y \subset \mathbb{R}^m$ semialgebraic defined over $\mathbb{Z}$ such that $\exists x \in \mathbb{R}^n \mid (x,y) \in S \iff y \in S_y$.

true (with the same S_y) for all real closed fields R

5. Cylindrical algebraic decomposition

Example:



Goal: explicit description of $\{f_1 \geq 0, f_2 \geq 0, f_3 \geq 0\}$
↓
- divide the x axis into cells
- sample one point in each cell and subdivide the vertical line into cells → this is consistent inside one cell
- check sign patterns

Def / Theorem (or stratification): a cylindrical algebraic decomposition of $\mathbb{R}^n$ (same in R) is a collection $\mathcal{C}_1, \dots, \mathcal{C}_n$ where $\mathcal{C}_i$ is a partition of $\mathbb{R}^i$ into semialgebraic sets, "cells of level i", such that

- a cell $S \in \mathcal{C}_1$ is either a point or an open interval;
- $\forall i\ \forall S \in \mathcal{C}_i$ there are finitely many continuous semialg functions $\varphi_1, \dots, \varphi_k : S \to \mathbb{R}$ such that the cylinder $S \times \mathbb{R} \subset \mathbb{R}^{i+1}$ is a disjoint union of cells of $\mathcal{C}_{i+1}$, namely:
  - either the graph of some φ_j: $\{(x,y) \in S \times \mathbb{R} \mid y = \varphi_j(x)\}$,
  - or the band between φ_j and φ_{j+1}: $\{(x,y) \in S \times \mathbb{R} \mid \varphi_j(x) < y < \varphi_{j+1}(x)\}$.

⇝ each cell of a CAD is homeomorphic to $(0,1)^i$ (here $(0,1)^0 = \{pt\}$)
↖ via a semialgebraic function

Def: a CAD adapted to a semialgebraic set $S \subset \mathbb{R}^n$ is a CAD of $\mathbb{R}^n$ such that S is a union of cells.
→ it exists $\forall S$!
same for more sets

⇝ in each cell, the polynomials defining S have constant sign (>0, <0, $=0$)

Computationally: there are algorithms to compute the CAD of a semialgebraic set $S \subset \mathbb{R}^n$

↳ implementations in Maple, Mathematica, SageMath

CylindricalDecomposition QEPCAD

↳ complexity: assume $S \subset \mathbb{R}^n$ defined by k polynomials of degree at most d. Then the complexity is bounded above by

$$(d \cdot k)^{2^{O(n)}}$$

polynomial in fixed dimension

doubly exponential in the dimension

in practice it is applicable only in very small dimension

Other interesting topic: the ordering of R induces a nice topology on R^n BUT this is not true in general for a real closed field (e.g., R Archimedean then R is totally disconnected, $[a,b]$ not compact)

- semialgebraic topology
- semialgebraic dimension