Viro's Patchworking Technique

Let's get ℝeal

Raluca Vlad

June 2023

Real Algebraic Curves

Definition

A *real plane algebraic curve* C is the vanishing locus of a homogeneous polynomial $f \in \mathbb{R}[x, y, z]$ in $\mathbb{RP}^2$:

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- the *degree* of the curve is $d := \deg(f)$;

- the *genus* of the curve is $g := \frac{(d-1)(d-2)}{2}$.

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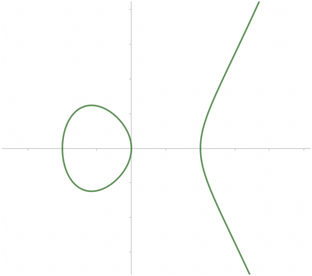
To plot $C = V(f) \subset \mathbb{RP}^2$, we look in the affine chart $\{z = 1\} \cong \mathbb{R}^2$.



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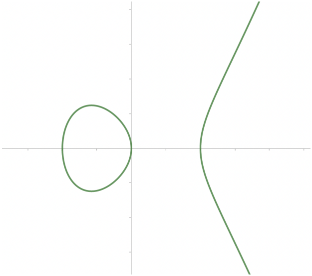
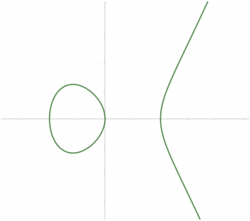


Question: How do the components of a curve look like?

Components of a Curve



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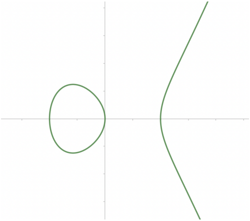

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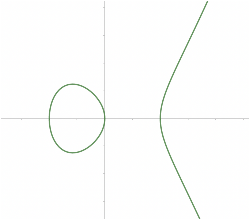

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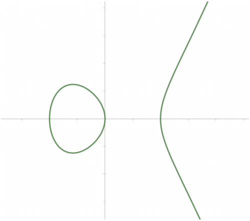

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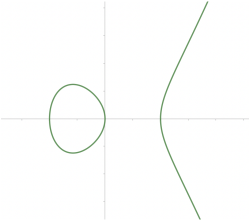

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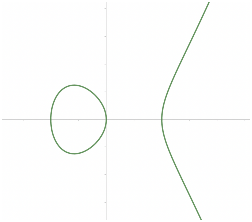

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- $\#$ connected components $\leq g + 1$.

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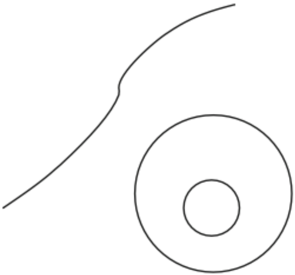

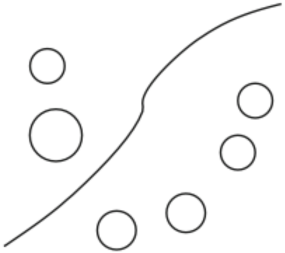

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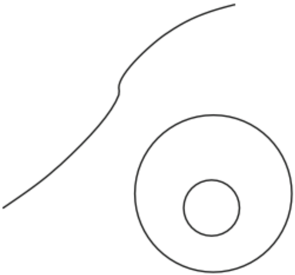
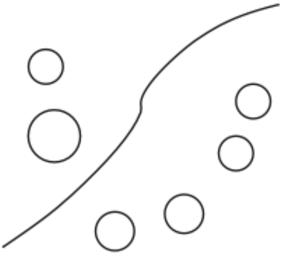
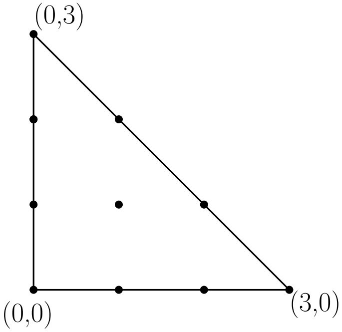




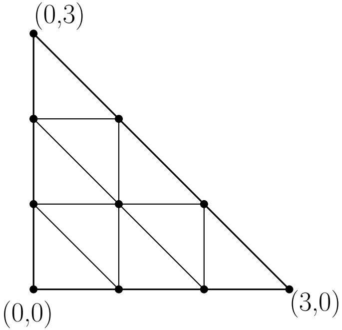
Viro's patchworking is a combinatorial process that gives us a way to generate curves with a certain prescribed topology.

- Start with the triangle T with vertices $(0,0),(0,d),(d,0)$.



Viro's Patchworking

- Triangulate T. Triangulation should be:
  - *unimodular* – all triangles have area $1/2$;
  - *regular* – obtained as the lower convex hull of a height function $h : V(T) \to \mathbb{R}$, where $V(T)$ are the integer points in T.



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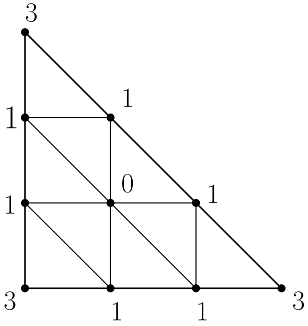
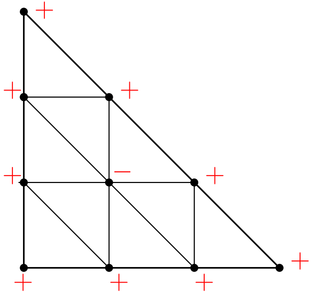


- Give signs to the integer points of T,

$$\text{sgn} : V(T) \to \{\pm\}.$$



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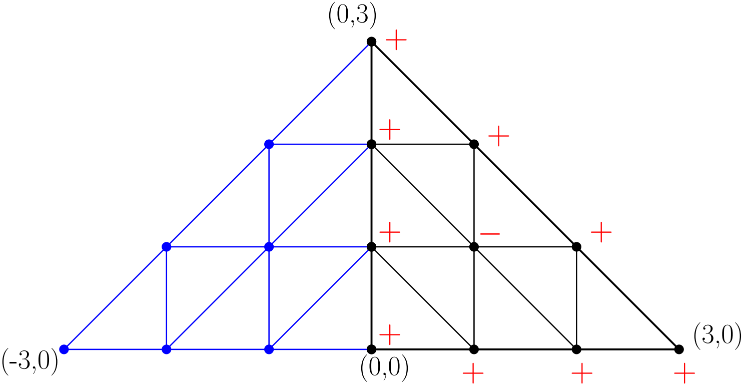
- Reflect triangulation about y-axis.
- The signs "even distance away" from the axis of reflection stay the same as their mirror images.
- Switch the signs that are "odd distance away."



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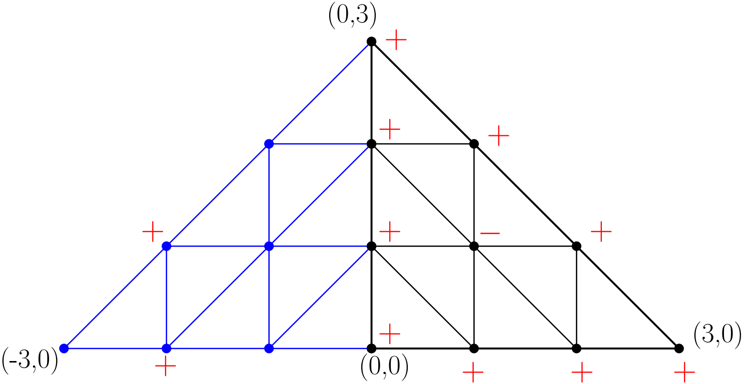

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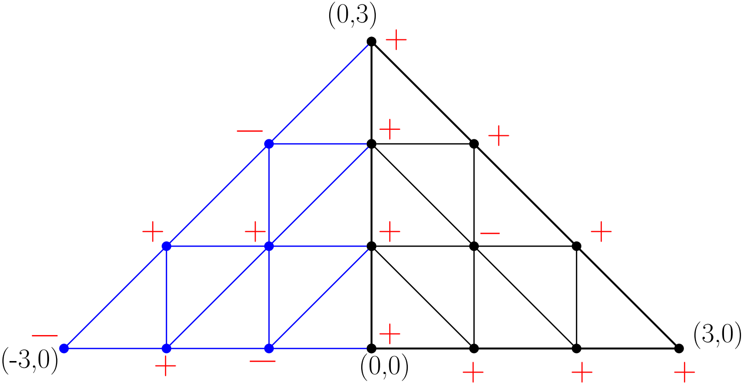

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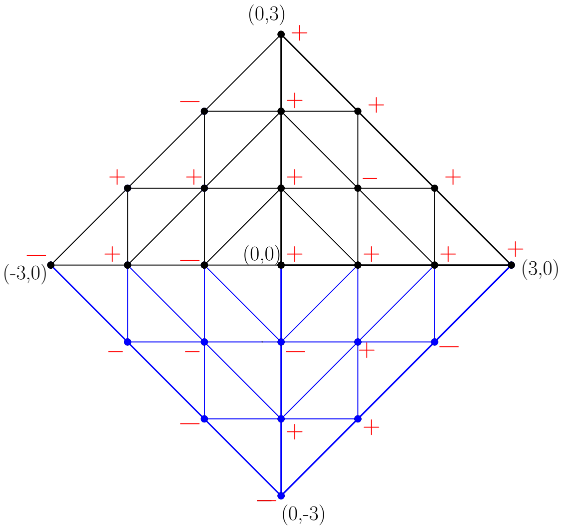
- Do analogous reflection about the x-axis.



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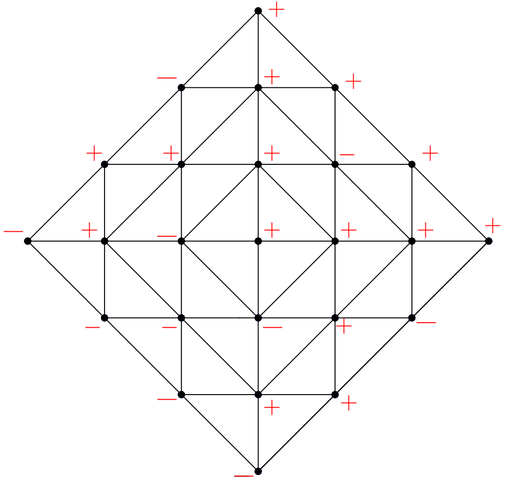
- The square we obtain represents a model of $\mathbb{RP}^2$. (Imagine its opposite edges identified.)



Viro's Patchworking

- We draw a "curve contour" inside of $\mathbb{RP}^2$ as follows:
  - we add a blue dot in the midpoint of every edge with different signs at endpoints;
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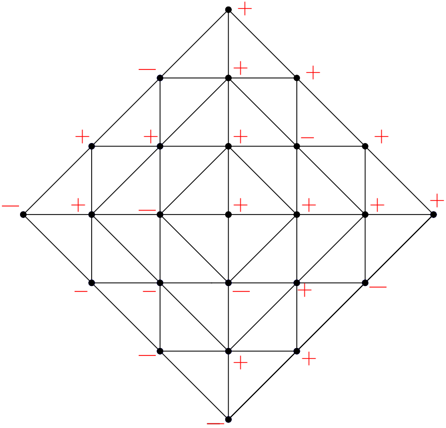

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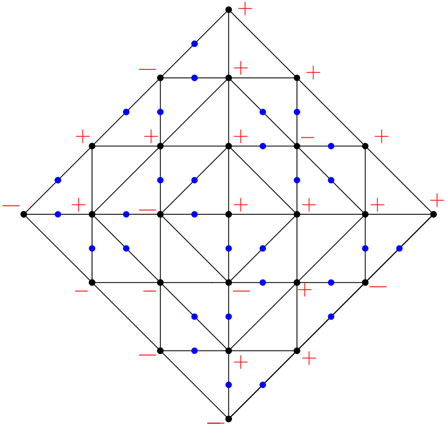

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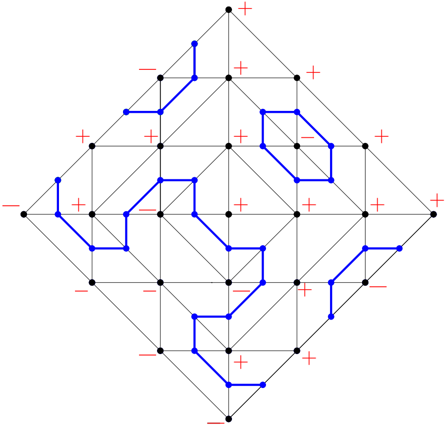

Theorem (Viro, 1979)

Fix some initial data, consisting of a weight function $w : V(T) \to \mathbb{R}$ *and a sign function* $sgn : V(T) \to \{\pm\}$*. Carry out the previous process to obtain a topological contour* L *inside the square* $\Diamond$*.*

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Consider the polynomial

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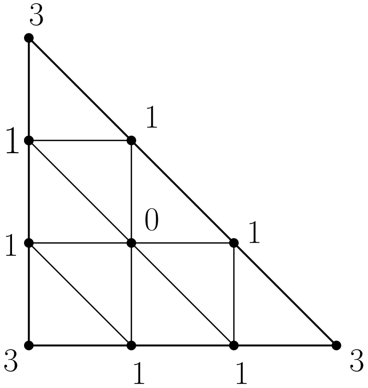
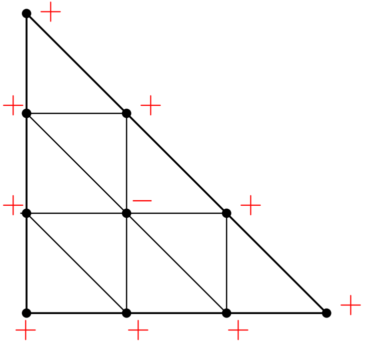
Then, for small enough t, the curve $C = V(f)$ topologically looks like the contour L, i.e. there is a homeomorphism of $\mathbb{RP}^2$ and $\diamondsuit$ which sends C to L.

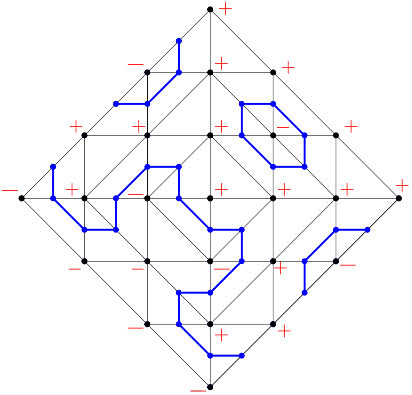
Viro's Patchworking

In our running example, we have

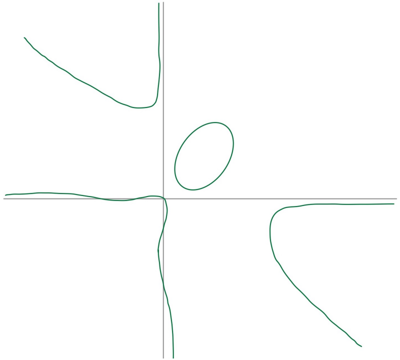
$$f = \sum_{(i,j)\in V(T)} \text{sgn}(i,j) \cdot t^{w(i,j)} \cdot x^i y^j z^{d-i-j}$$
$$= -xyz + t(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) + t^3(x^3 + y^3 + z^3)$$







(a) Our model



(b) The curve $V(f)$

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- If X is a topological space with an involution σ, and X^σ is the fixed locus of the involution, then

$$b_*(X^\sigma; \mathbb{F}_2) \leq b_*(X; \mathbb{F}_2).$$

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- $(*)$ implies that $\#\text{connected components} = k \leq g + 1.$

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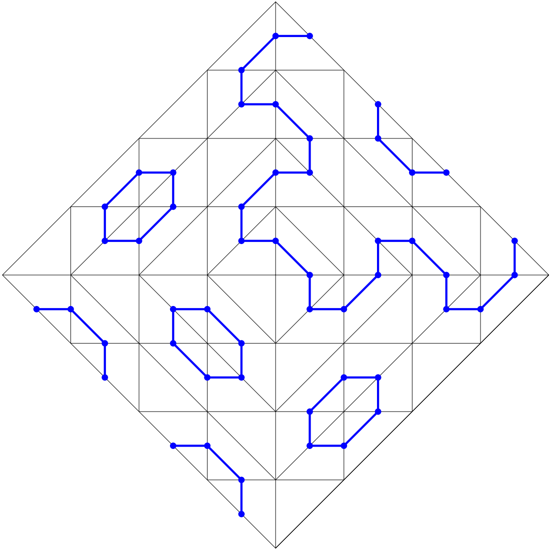
- Consider the triangle T with the standard unimodular triangulation and the Harnack sign function $\operatorname{sgn} : V(T) \to \{\pm\}$ given by

$$\operatorname{sgn}(i,j) = \begin{cases} - & \text{if } i,j \text{ even} \\ + & \text{otherwise} \end{cases}.$$

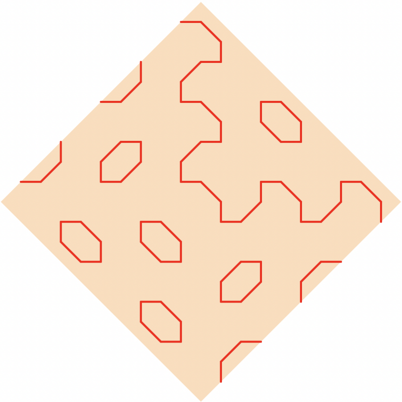
Maximal Curves

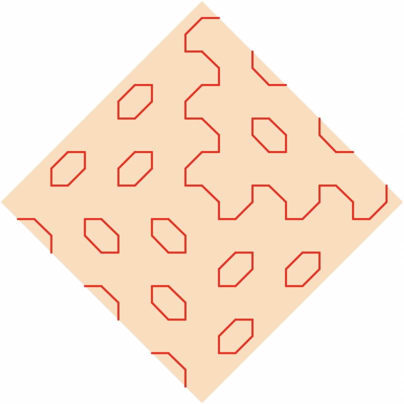
- In degree 4, we get



Maximal Curves

- In degrees 5 and 6, we get:





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Thank you!

References

(1) J. A. De Loera, J. Rambau, F. Santos. *Triangulations. Structures for Algorithms and Applications.*

(2) J. Hinssen. *The topology of real loci of* $\mathbb{R}$*-varieties.*

(3) I. Itenberg. *Viro's method and T-curves.*

Also, cool online tool for Viro's patchworking:
`https://math.uniandes.edu.co/~j.rau/patchworking_english/patchworking.html`