Lecture 3

6. Positive polynomials

Problem: given a set $X \subseteq \mathbb{R}^n$ and a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$
find $f_{min} = \inf_{x \in X} f(x)$

connected to many optimization problems
e.g., MaxCut:

$X = \{0,1\}^n$, G weighted graph with edge weights w_{ij}

$$f(x) = -\sum_{(i,j) \in E} w_{ij} (x_i - x_j)^2$$

then $f_{min} = -\text{MaxCut}(G)$

Reformulation: $f_{min} = \sup\{\lambda \in \mathbb{R} \mid f - \lambda \text{ is nonnegative on } X\}$

$\Rightarrow$ being able to check algorithmically whether a polynomial is nonnegative would allow us to solve many opt. problems

Two main players:
1. $\{f \in \mathbb{R}[x_1, \dots, x_n] \mid f(x) \geq 0 \ \forall x\} =: P$
   the convex cone of nonnegative poly
2. $\{f \in \mathbb{R}[x_1, \dots, x_n] \mid f(x) = \sum g_i^2(x)$ for some $g_i \in \mathbb{R}[x_1, \dots, x_n]\} =: \Sigma$

The univariate case:

Theorem: a univariate polynomial $f \in \mathbb{R}[x]$ is nonnegative if and only if it is a sum of squares.

$\hookrightarrow$ a nonnegative polynomial must have even degree

There are similar results for nonnegative polynomials on $[a,b]$

The general case: Hilbert

Minkowski conjectured that nonnegative $\neq$ sum of square

Let's work now with homogeneous polynomials

$\rightsquigarrow$ homogenize and dehomogenize does not change ≥ 0 or s.o.s.

We will denote by $P_{n,d} = \{ f \in \mathbb{R}[x_1, \dots, x_n]_d \mid f \geq 0 \}$

$$\Sigma_{n,d} = \{ f \in \mathbb{R}[x_1, \dots, x_n]_d \mid f \text{ is sos} \}$$

Theorem (Hilbert 1888): Let $n \geq 2$ and d even. Then $\Sigma_{n,d} = P_{n,d}$ iff

(i) $n = 2$
(ii) $d = 2$
(iii) $n = 3$, $d = 4$

"the Hilbert cases"

Proof: (partial)

(i) dehomogenization gives univariate polynomials

(ii) $f(x) = x^T A x$ with A symmetric. $f \geq 0 \iff A \succcurlyeq 0$ (positive semidefinite)

$\iff A = B^T B$ (Cholesky factorization)

$\iff f(x) = x^T B^T B x = \|Bx\|^2$

$\iff f$ sos

(iii) technical proof, based on the fact that for any nonnegative ternary quartic f there exists g s.t. $\deg g = 2$ and $f - g^2 \geq 0$.

The remaining cases?

- $n \geq 3$, $d \geq 6$ the counterexample is based on the Motzkin polynomial

$$f(x,y,z) = z^6 - 3x^2y^2z^2 + x^2y^4 + x^4y^2$$

- $n \geq 4$ the counterexample is based on the polynomial

$$f(w,x,y,z) = w^4 + x^2y^2 + x^2z^2 + y^2z^2 - 4wxyz$$

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However, Hilbert's 17^{th} problem ask if we can write every nonnegative polynomial as a sum of rational functions.

Theorem: Any nonnegative polynomial can be written as a sum of squares of rational functions.

Putinar's in L4

We can prove it using a version of the Positivstellensatz

Theorem: $g_1, \dots, g_k \in \mathbb{R}[x_1, \dots, x_n]$, $S = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \ \forall i = 1, \dots, k\}$,

$P = \{ \sum_{i \in \{0,1\}^k} \sigma_i \, g_1^{i_1} \cdots g_k^{i_k} \mid \sigma_i \in \mathbb{R}[x_1, \dots, x_n] \text{ sos} \}$, $f \in \mathbb{R}[x_1, \dots, x_n]$. Then

(i) $f > 0$ on $S \iff \exists\, g, h \in P$ with $fg = 1 + h$

(ii) $f \geq 0$ on $S \iff \exists\, g, h \in P, m \geq 0$ with $fg = f^{2m} + h$

(iii) $f = 0$ on $S \iff \exists\, g \in P, m \geq 0$ with $f^{2m} + g = 0$

Using this we can prove Hilbert 17^{th} problem:

$\to$ we have $k = 0$, so $f \geq 0$ on $\mathbb{R}^n \iff \exists\, g, h$ sos, $m \geq 0$ such that

$$fg = f^{2k} + h$$

$$\Rightarrow f = \left(\frac{1}{g}\right)^2 g(f^{2k} + h) \quad \text{is a rational sos}$$

Theorem (Schmüdgen) $f \in \mathbb{R}[x_1, \dots, x_n]$ such that $f > 0$ on S (above), then $f \in P$, namely

$$f = \sum_{i \in \{0,1\}^k} \sigma_i \, g_1^{i_1} \cdots g_k^{i_k} \quad \text{with } \sigma_i \text{ sos}$$

7. SOS cone and optimization

Back to the beginning: $f_{min} = \sup\{\lambda \in \mathbb{R} \mid f - \lambda \geq 0\}$

$\Rightarrow f_{sos} = \sup\{\lambda \in \mathbb{R} \mid f - \lambda \text{ is sos}\}$

Since $\{\text{sos}\} \subsetneq \{\geq 0\}$, then $f_{sos} \leq f_{min}$

BUT we can compute f_{sos}!

A polynomial $f \in \mathbb{R}[x_1,\dots,x_n]$ is sos if and only if $f = m^T A m$, $A \succeq 0$

vector of monomials

Example: $f = y^4 - xy^2 + x^2 + 1$

then $m = \{1, x, y, x^2, xy, y^2\}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{bmatrix}$$

→ I can also just take a subset of monomials, if that's enough: $\tilde{m} = \{1, x, y^2\}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$A \succeq 0$ and in fact $f = \frac{3}{4}(x - y^2)^2 + \frac{1}{4}(x+y)^2 + 1$

Using this remark we can compute f_{sos} via a semidefinite program

We look for a decomposition

$$f - \lambda = m^T A m \quad , \quad \lambda \in \mathbb{R} \, , \; A \succeq 0.$$

depends on λ

Then

$$\boxed{\begin{aligned} f_{sos} &= \sup \lambda \\ &\text{subject to } A \succeq 0 \\ &\qquad\qquad f - \lambda = m^T A m \end{aligned}}$$

semidefinite condition

linear conditions

Example: $f - \lambda = m^T A m$ with $\{A\}_{ij} = a_{ij}$

$$x^4 - xy^2 + x^2 + (1 - \lambda) = a_{11} + (a_{22} + a_{14} + a_{41}) x^2 + \dots + a_{66} y^4$$

$$\Rightarrow f_{sos} = \sup \; 1 - a_{11}$$

$$\text{subject to } \begin{cases} A \succeq 0 \\ a_{22} + a_{14} + a_{41} = 1, \dots, a_{66} = 1 \end{cases}$$

What is in general an SDP?

Let A be a matrix with unknown entries, such as

$$A = \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & z \\ x & 0 & 2 & y \\ 0 & z & y & y \end{bmatrix}$$

Then the associated SDP is

$$\begin{aligned} &\text{maximize} && \ell(x,y,z) \\ &\text{subject to} && A \succcurlyeq 0 \end{aligned}$$

the points satisfying this constraint form the "feasible set"

What is the feasible set?

Def: a spectrahedron is the intersection of the cone of positive semidefinite matrices with a linear space.

Equivalently, it is a set with the following shape

$$\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i A_i + A_0 \succcurlyeq 0 \right\}.$$

Example: the spectrahedron associated to $A = \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & z \\ x & 0 & 2 & y \\ 0 & z & y & y \end{bmatrix}$ is:

① the cone Σ of PSD 4×4 symmetric matrices $(a_{ij})_{i,j}$ intersected with the linear space

$$\{ a_{11} = a_{22} = 1,\ a_{33} = 2,\ a_{12} = a_{14} = a_{23} = 0,\ a_{34} = a_{44} \}$$

② $A_1 = \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$, $A_2 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$, $A_0 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$

then the spectrahedron is $\{ xA_1 + yA_2 + zA_3 + A_0 \succcurlyeq 0 \}$

↓

Semialgebraic, convex sets → NICE!

↳ many properties: see exercises