TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

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Real Root Counting Algorithms

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Initial Example

Given a matrix $M \in Sym_n(\mathbb{R})$, how can we decide if M is psd?

$$M \to \chi_M(\lambda)$$

Question about the positivity of roots of a (monic) univariate polynomial.

Typical Questions

Given a monic univariate polynomial $p \in \mathbb{R}[x]$ ...

- ... how many of the roots of p are real?
- ... how many of the roots of p are positive (negative)?

Given two monic univariate polynomials $p, q \in \mathbb{R}[x]$ ...

- ... how many $a \in \mathbb{R}$ are there such that both $p(a) = 0$ and $q(a) > 0$?

Some History - Descartes' Rule of Signs

René Descartes (1596–1650):
Look at the coefficient sequence $(c_i)_i$ of p.

- Count number $V(c)$ of "true" changes of sign in $(c_i)_i$.
- Upper bound on number of positive real roots of p.
- Exact $\mod 2$.

Examples:

- $p = x^2 - 3x + 2$ $\quad(+ - +)$ has at most (even exactly) 2 positive roots.
- $p = x^2 - x + 2$ $\quad(+ - +)$ has at most (but not exactly) 2 positive roots.

Proof idea:

- Induction on the non-zero entries of c, pass to formal derivative p'.

Some History - Descartes' Rule of Signs

Useful application:
Assume that p has only real roots (e.g. p is the characteristic polynomial of a symmetric real matrix).

Then Descartes' Rule of Signs is exact!

Example:

$$M = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

is a PSD matrix since $\chi_M(\lambda) = \lambda^3 - 6\lambda^2 + 8\lambda - 2 \qquad (+-+-)$.

Some History - Descartes' Rule of Signs

Useful application:
If p has only real roots, then Descartes' Rule of Signs is exact (ignoring 0 as a root).

Proof: Let $q := p(-x)$

- $d := \deg(p)$
- $N := V(p)$ (max nr. of pos. roots)
- $n := V(q)$ (max nr. of neg. roots)

We always have $N + n \leq d$: The coefficients of p change sign iff. those of q do not (with some tweaks for 0 coefficients).
Now by assumption $N + n = d$ and N, n are both exact. $\square$

Some More Recent History - Sturm Sequences

Jacques Charles François Sturm (1803–1855):
Consider $p \in \mathbb{R}[x]$ squarefree, $(a, b] \subseteq \overline{\mathbb{R}}$ and

- $p_0 := p$; $p_1 := p'$
- $p_{k+1} := -rem(p_k, p_{k-1})$
- $\alpha := (p_k(a))_k$; $\beta := (p_k(b))_k$

Then the number of (distinct) roots of p in $(a, b]$ equals $V(\alpha) - V(\beta)$.

Example:

- $p_0 = p = x^3 - x^2 + x - 1$; $p_1 = p' = 3x^2 - 2x + 1$
- $p_2 = -rem(p_0, p_1) = -\frac{4}{9}x + \frac{8}{9} \to -x + 2$
- $p_3 = -rem(p_1, p_2) = -9 \to -1$
- $V(0) - V(5) = 2 - 1 = 1$; $V(-\infty) - V(\infty) = 2 - 1 = 1$

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- $p_{k+1} := -rem(p_k, p_{k-1})$
- $\alpha := (p_k(a))_k; \qquad \beta := (p_k(b))_k$

Then the number of (distinct) roots of p in $(a, b]$ equals $V(\alpha) - V(\beta)$.

Proof: Move x along $a \to b$ and write $\xi = (p_k(x))_k$. If $p_k(x) = 0$, then ...

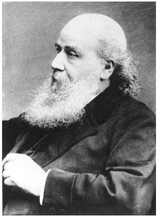
- Case 1 ($k > 0$): ... $V(\xi) = V(\xi \pm \varepsilon)$

$$p_{k-1} = qp_k - p_{k+1} \implies sign(p_{k-1}(x \pm \varepsilon)) = -sign(p_{k+1}(x \pm \varepsilon))$$

- Case 2 ($k = 0$): ... $V(\xi + \varepsilon) = V(\xi) = V(\xi - \varepsilon) - 1$

□

Generalizations of Sturm's Method



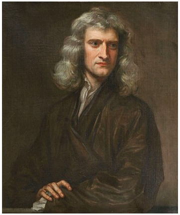
James Joseph Sylvester (1814–1897):
By replacing p' by qp' Sturm's method can be generalized to find $|\{p = 0, q > 0\}| - |\{p = 0, q < 0\}|$.

One can also add arbitrarily (finitely) many conditions of the form $q\square 0$.

There are methods that extend these results to get rid of constraints on the polynomials.

Many of the results carry over to real closed fields without change.

An Alternative Approach



Sir Isaac Newton (1642–1726):

$$p = \prod_{i=1}^{n}(x - \alpha_i) \qquad \longrightarrow \qquad \nu_k = \sum_{i=1}^{n} \alpha_i^k$$

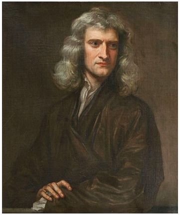
k-th Newton sum

Charles Hermite (1822–1901):

$$\mathcal{H}(p) = (\nu_{i+j}(p))_{i,j=0}^{d-1}$$

Hermite Matrix

An Alternative Approach

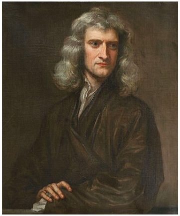


Sir Isaac Newton (1642–1726):
The k-th Newton sum can be computed inductively from its predecessors.

Charles Hermite (1822–1901):
$rk(\mathcal{H}(p)) =$ number of distinct roots of p in $\mathbb{C}$.

$sign(\mathcal{H}(p)) =$ number of distinct roots of p in $\mathbb{R}$.

Some generalizations



Sir Isaac Newton (1642–1726):
The k-th weighted Newton sum $\nu_k(p,q) = \sum q(\alpha_i)\alpha_i^k$ can be calculated inductively.

Charles Hermite (1822–1901):
The generalized Hermite Matrix $\mathcal{H}(p,q) = (\nu_{i+j}(p,q))_{i,j=0}^{d-1}$ satisfies

$rk(\mathcal{H}(p,q)) =$ number of distinct roots of p in $\mathbb{C}$ with $q \neq 0$.

$sign(\mathcal{H}(p,q)) = \sum sign(q(\alpha_i))$;
$\{p=0, q>0\} = 1/2 \sum_{e=0,1} sign(\mathcal{H}(p,q^e))$.

The End

Thank you for your interest!