

A brief intro to

Chow rings and Kähler package

I. Definitions

II. Properties/examples

I. Poincaré Duality

Hard Lefschetz
Hodge-Riemann

II. Examples

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10 2023

Chow ring of a variety

X is an algebraic variety

A k -cycle on X is a finite form sum

of k -dimensional varieties of X with integer coefficients

$$\sum n_i V_i$$

$$\mathbb{Z}_k(X) := \langle \text{ dim-}k \text{ cycles of } X \rangle.$$

$[Y]$ is rationally equivalent to $[Z]$ if

\exists a subvariety $V \subseteq X \times \mathbb{P}^1$ s.t

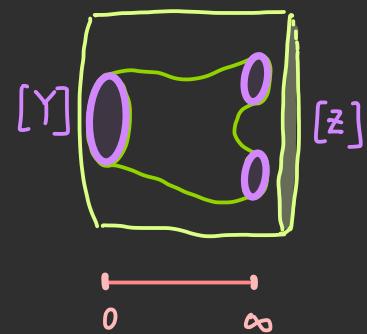
(1) $X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ induces a dominant map $f: V \rightarrow \mathbb{P}^1$

(2) $[f^{-1}(0)] = [Y]$ and $[f^{-1}(\infty)] = [Z]$

$\text{Rat}_k(x) := \dim - k$ cycles rationally equivalent to zero

The Chow group $A_\bullet(X)$ of X is

$$A_\bullet(X) = \bigoplus_{k=0}^{\dim X} A_k(X) = \frac{\mathcal{Z}_k(X)}{\text{Rat}_k(X)}$$



Functorial properties I: Pushforward

If $f: X \rightarrow Y$ is a proper morphism,



then $f_* : \mathcal{Z}_k(X) \rightarrow \mathcal{Z}_k(Y)$

$$V \mapsto [R(V) : R(W)] W \quad \begin{array}{ll} \text{if } \dim V = \dim W, \\ 0 & \text{otherwise.} \end{array}$$

$f_*(\text{Rat}(X)) \subseteq \text{Rat}(Y) \Rightarrow f_* : A_k(X) \rightarrow A_k(Y)$ complete

Example If the structure morphism $f: X \rightarrow \text{Spec } k$ is proper

$$\text{then } \deg(\alpha) = \int_X \alpha := f_*(\alpha) \quad \begin{array}{ll} \text{if } \alpha \in A_0(X) \\ 0 & \text{otherwise} \end{array}$$

complete
Example If the structure morphism $f: X \rightarrow \text{Spec } k$ is proper

$$\text{then } \deg(\alpha) = \int_X \alpha := f_*(\alpha) \quad \begin{cases} \text{if } \alpha \in A_0(X) \\ 0 \quad \text{otherwise} \end{cases}$$

\oplus

$$A_0(\text{Spec } k) = \mathbb{Z}$$

Functorial properties I: Pullbacks

If $f: X \rightarrow Y$ is a flat morphism,

Ex open embedding, projection of vector bundle,
projection of cartesian products of pure dim

Non-Ex

$$\begin{array}{ccc} & \text{---} & \\ & | & \\ \text{IA}^2 & \xrightarrow{f} & \text{IA}^1 \end{array}$$

$f^{-1}(0)$ is not equidim

Functorial properties I: Pullbacks

If $f: X \rightarrow Y$ is a flat morphism,

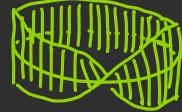
Ex open embedding, projection of vector bundle,
projection of cartesian products of pure dim

then $f^*: \mathcal{Z}_k(Y) \rightarrow \mathcal{Z}_k(Y)$

$$[V] \mapsto [f^{-1}V]$$

is well-defined and $f^*(\text{Rat}(Y)) \subseteq \text{Rat}(X)$.

Affine Bundles



E is an affine bundle of rank n over X



with $p: E \rightarrow X$ if X has an open covering $\{U_\alpha\}$

with the property that

$$p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{A}^n.$$

Prop The pullback $p^*: A_k(X) \rightarrow A_{k+n}(E)$ is surjective.

Example $A_k(\mathbb{A}^n) = \begin{cases} 0 & k < n, \\ \mathbb{Z} & k = n. \end{cases}$

A useful sequence

If $i : Y \hookrightarrow X$ is closed subscheme of X ,

and let $j : X - Y \hookrightarrow X$ be the inclusion,

then $A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \rightarrow 0$ is exact.

Example $\mathbb{P}^n = /A^n \sqcup \mathbb{P}^{n-1}$

$$\Rightarrow A_k(\mathbb{P}^{n-1}) \rightarrow A_k(\mathbb{P}^n) \rightarrow A_k(/A^n) \rightarrow 0$$

is exact.

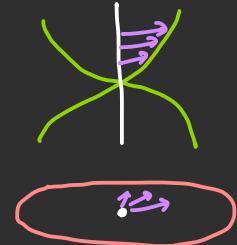
- If X is equidimensional, $\text{codim } Y := \dim X - \dim Y$.
- If X is smooth, $A^k(X) := A_{\dim X - k}(X)$.
- Setting $[x] \in A^0(X)$ as 1, $[Y][Z] = [Y \cap Z]$
for smooth Y, Z intersecting transversely gives $A^\bullet(X)$
a ring structure.
- If X smooth, $Y = \mathbb{P}^n$,
then $A^\bullet(X) \otimes A^\bullet(Y) \equiv A^\bullet(X \times Y)$

Chow ring of a blow-up.

If $Y \subseteq X$ is a closed smooth subscheme of X ,

then the smooth projective $B_Y X \xrightarrow{\pi} X$ is

$$\text{Proj } \bigoplus_{n=0}^{\infty} I_Y^n$$



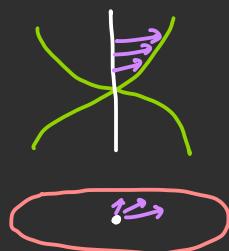
(1) π is proper and birational

(2) π is an isomorphism on $B_Y X - \pi^{-1}(Y)$.

$$\begin{array}{ccc} \pi^{-1}(Y) & \hookrightarrow & B_Y X \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

exceptional divisor

- (1) π is proper and birational (if Y is not dense)
- (2) $\pi|_E$ is an isomorphism on $B^{\ell_Y} X - \pi^{-1}(Y)$.
- (3) $E = \pi^{-1}(Y) \hookrightarrow B^{\ell_Y} X$
- $$\begin{array}{ccc} \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$
- exceptional divisor*



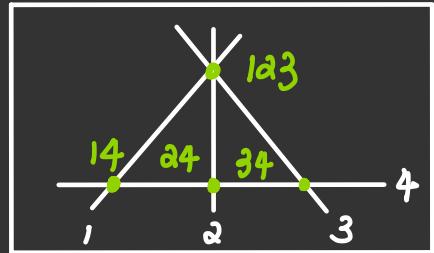
Proposition There is a split exact sequence of groups

$$0 \rightarrow A_k(Y) \rightarrow A_k(E) \oplus A_k(X) \rightarrow A_k(B^{\ell_Y} X) \rightarrow 0$$

Wonderful compactification of
the complement of a hyperplane
arrangement De Concini - Procesi 1995

EX

PL



L = a vector space over k

A = a finite set of hyperplanes
intersecting only at the origin

$$\{H_e : e \in E\}$$

For each $S \subseteq E$, $L_S = \bigcap_{e \in S} H_e$, $\text{rk}(S) = \dim L / L_S$

A flat is a subset $F \subseteq E$ maximal among its rank.

$$\Phi_A : \mathbb{P}L \dashrightarrow \overline{\mathbb{P}}(L/L_F)$$

nonempty flats F

Def The wonderful variety of A is
 the closure of the image of $\mathbb{P}L$ under Φ_A :

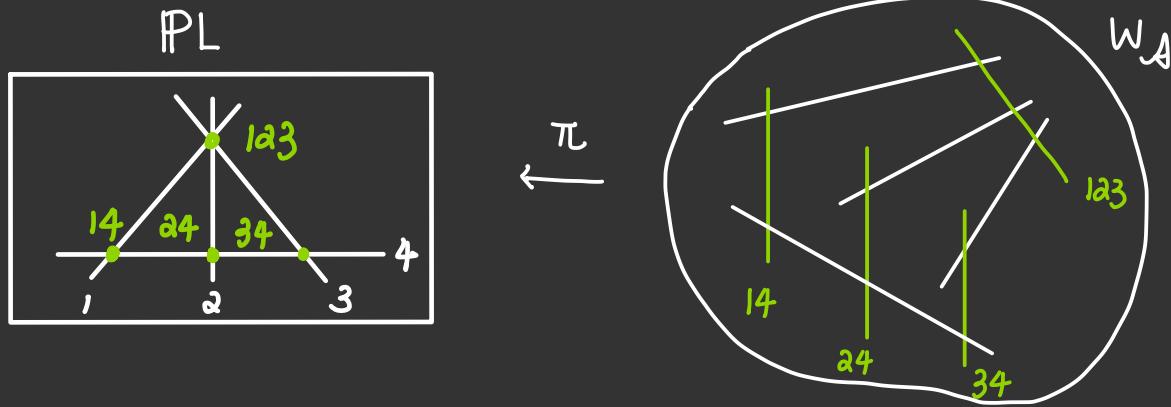
$$W_A := \overline{\Phi_A(\mathbb{P}L)} \subseteq \overline{\mathbb{P}(L/L_F)}.$$

nonempty
flats F

Def The wonderful variety of \mathcal{A} is
 the closure of the image of PL under $\Phi_{\mathcal{A}}$:

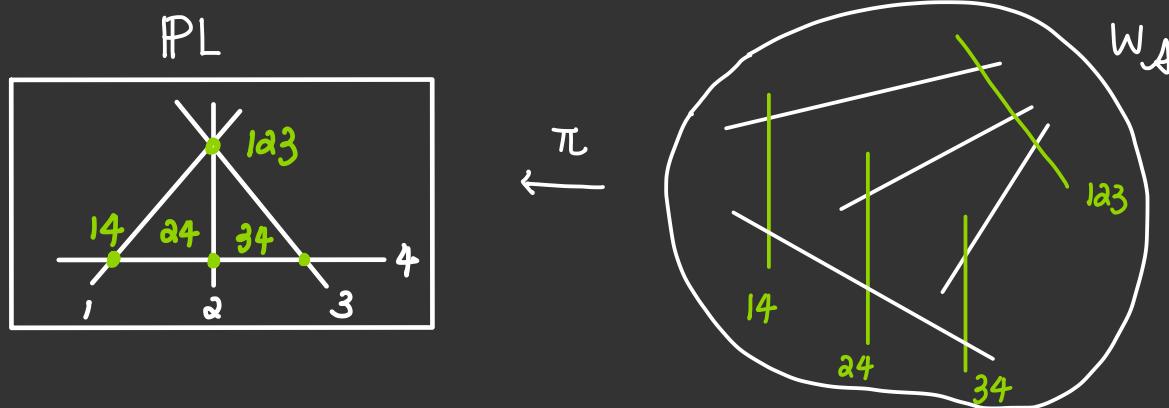
$$W_{\mathcal{A}} := \overline{\Phi_{\mathcal{A}}(\text{PL})} \subseteq \prod \mathbb{P}(L/L_F).$$

nonempty
flats F



Feichtner - Yuzvinsky 2004

$$A(W_A) = \frac{\mathbb{E}[x_F : \text{nonempty flats}]}{\langle x_F x_G : F, G \text{ incomparable} \rangle} + \left\langle \sum_{F \leq G} x_G : \text{rk } F = 1 \right\rangle$$



Feichtner - Yuzvinsky 2004

$$A(w_A) = \frac{\mathbb{E}[x_F : \text{nonempty flats}]}{\langle x_F x_G : F, G \text{ incomparable} \rangle} + \left\langle \sum_{F \leq G} x_G : \text{rk } F = 1 \right\rangle$$

Backman - Eur - Simpson 2021

$$A(w_A) = \frac{\mathbb{E}[h_F : \text{nonempty flats}]}{\langle (h_F - h_{F \vee G})(h_G - h_{F \vee G}) : F, G \text{ incomparable} \rangle} + \left\langle h_F : \text{rk } F = 1 \right\rangle$$

Key : h_F 's have elegant intersection properties.

Chow ring of a matroid

- N a free abelian group \mathbb{Z}^m
- Given a matroid M , $S \subseteq E$,

$$\bar{e}_S = \text{the image of } e_S = \sum_{i \in S} e_i \text{ under } \mathbb{R}^E \rightarrow \mathbb{R}^E / \mathbb{R} \mathbf{1}.$$

- $\text{rk}: 2^E \rightarrow \mathbb{N}_{\geq 0}$ $S \mapsto \max_{I \subseteq S} |I|$
- A set F is a flat if it is maximal among its rank
 \Leftrightarrow adding non-member preserves rank.

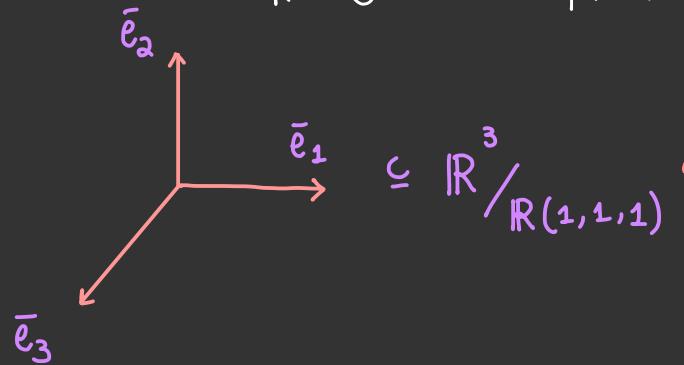
Def The Bergman fan Σ_M is the fan consisting of cones of the form

$$\sigma_F = \mathbb{R}_{\geq 0} \left\{ \bar{e}_{F_1}, \dots, \bar{e}_{F_m} \right\}$$

for each chain of nonempty proper flats

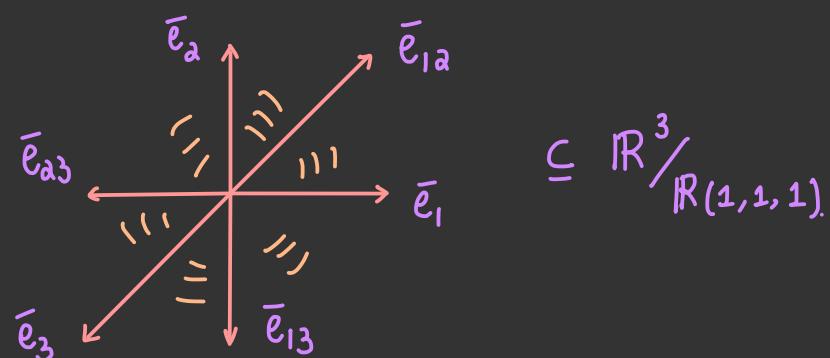
Example $U_{2,3}$

$$rk : S \mapsto \min \{|S|, 2\}$$



B_3

$$rk : S \mapsto |S|$$



$$\subseteq \mathbb{R}^3 / \mathbb{R}(1,1,1)$$

Def A fan Σ is

- (1) pure-dim if every maximal cone has the same dim.
- (2) simplicial if for every cone σ , $\dim \sigma = |\text{generators of } \sigma|$.
- (3) complete if the support $|\Sigma| = N_{\mathbb{R}}$.
- (4) balanced if at each σ ,

$$\sum_{\substack{\tau \supseteq \sigma \\ \dim \tau = \dim \sigma + 1}} u_{\tau/\sigma} \in \mathbb{R}^{\sigma}$$

Example Σ_M is pure of dim $r-1$, simplicial, balanced, but often not complete

Chow ring of a simplicial Σ

$$A^*(\Sigma) = \frac{\mathbb{R}[x_p : p \text{ a ray}]}{I_\Sigma + J_\Sigma}$$

with

$$I_\Sigma = \left\langle x_{p_1} \cdots x_{p_n} \mid \text{if } p_1, \dots, p_n \text{ don't form a cone} \right\rangle$$

$$J_\Sigma = \left\langle \sum_{\text{ray } p} \langle u_p, v \rangle x_p, v \in \mathbb{Z}^m \right\rangle$$

AHK '18 Balanced $\Rightarrow \deg : A^*(\Sigma) \rightarrow \mathbb{R}$ $\prod_{p \in \max \Sigma} x_p \mapsto 1$

Kähler package (A^\bullet, \deg, K)

A^\bullet is a finite dimensional, graded real vector space

\deg is a symmetric bilinear form $A^\bullet \times A^{d-\bullet} \rightarrow \mathbb{R}$

K is a convex set of linear operators $L: A^\bullet \rightarrow A^{\bullet+1}$

$$\text{s.t. } \deg(Lx, y) = \deg(x, Ly)$$

(1) Poincaré duality $K \leq \frac{d}{2}$

The symmetric bilinear pairing

$$A^K \times A^{d-K} \rightarrow \mathbb{R}$$

$$\alpha \quad \beta \quad \mapsto \deg(\alpha \beta)$$

is non-degenerate.

Example $(\mathbb{P}^1)^3$

$$H^0((\mathbb{P}^1)^3) = \frac{\mathbb{R}[x_1, x_2, x_3]}{\langle x_1^2, x_2^2, x_3^2 \rangle} \xrightarrow{\deg} \mathbb{R}$$

PD

$$k=0, \deg(x_1 x_2 x_3) = 1 \quad x_1 x_2 x_3 \quad 3$$

$$k=1, \deg(x_i x_i x_j) = 0 \quad x_1 x_2 \quad x_2 x_3 \quad x_1 x_3 \quad 2$$

$$\text{but } \deg(x_i x_j x_k) = 1. \quad x_1 \quad x_2 \quad x_3 \quad 1$$

$$1 \quad 0$$

(2) Hard Lefschetz $k \leq \frac{d}{2}$

For any $\ell_1, \ell_2, \dots, \ell_{d-2k}$ in K , the symmetric bilinear form

$$A^k \rightarrow A^{d-k}$$

$$\alpha \mapsto \left(\prod_{i=1}^{d-2k} \ell_i \right) \alpha$$

is an isomorphism.

Example $(\mathbb{P}^1)^3$

$$A^{\bullet}((\mathbb{P}^1)^3) = \frac{\mathbb{R}[x_1, x_2, x_3]}{\langle x_1^2, x_2^2, x_3^2 \rangle} \xrightarrow{\deg} \mathbb{R}$$

HL

$k=0$

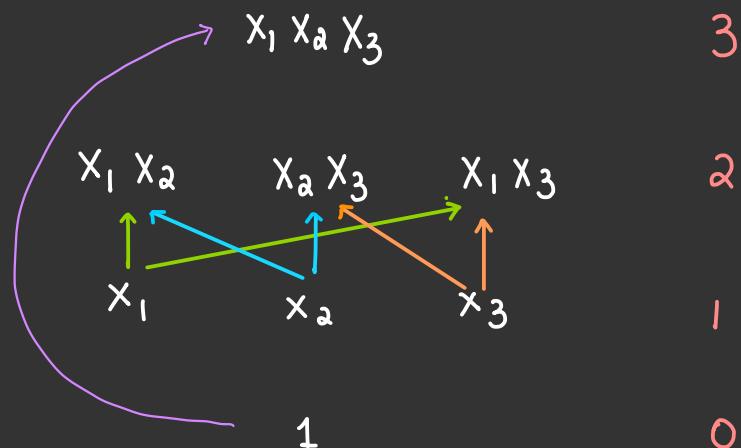
$$1 \mapsto 6x_1x_2x_3$$

$R=1$

$$x_1 \mapsto x_1x_2 + x_1x_3$$

$$x_2 \mapsto x_1x_2 + x_2x_3$$

$$x_3 \mapsto x_1x_3 + x_2x_3$$



The Hard Lefschetz property

$$\Rightarrow \ell : A^i \longrightarrow A^{i+1} \quad \ell \in K$$

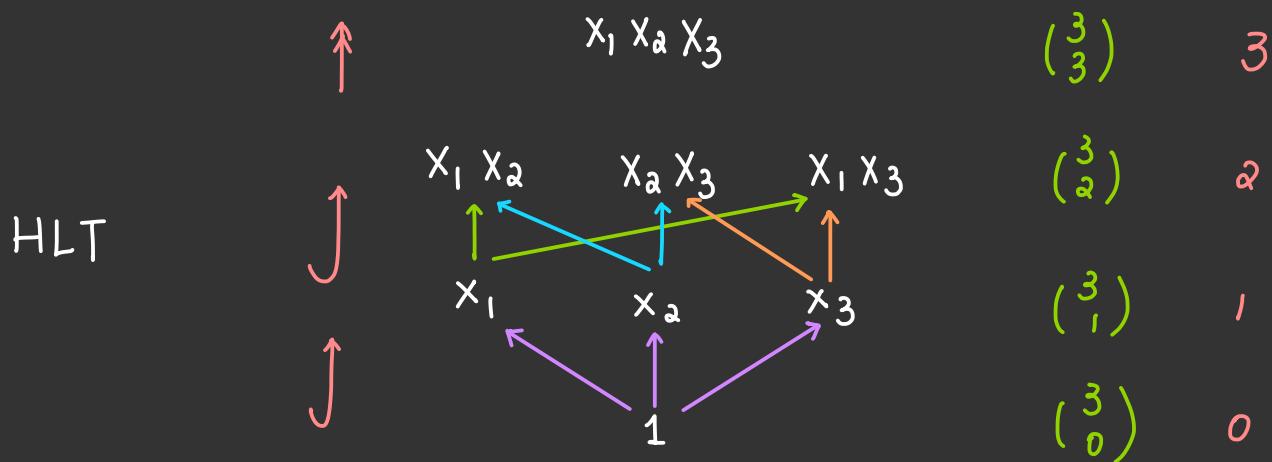
is

injective for i no greater than $\frac{d}{2}$

surjective for i no smaller than $\frac{d}{2}$.

Example $(\mathbb{P}^1)^3$

$$A^*(\mathbb{P}^1)^3 = \frac{\mathbb{R}[x_1, x_2, x_3]}{\langle x_1^3, x_2^3, x_3^3 \rangle}$$



(3) Hodge-Riemann Relations

For any $\ell_0, \dots, \ell_{d-k} \in K$, the symmetric bilinear form

$$A^k \times A^k \longrightarrow \mathbb{R}$$

$$(\alpha, \beta) \mapsto (-1)^k \deg \left(\alpha \left(\prod_{i=1}^{d-k} \ell_i \right) \beta \right)$$

is positive definite on the kernel of the linear map

$$\begin{aligned} A^k &\longrightarrow A^{d-k+1} \\ \alpha &\mapsto \left(\prod_{i=0}^{d-k} \ell_i \right) \alpha. \end{aligned}$$

Example $(\mathbb{P}^1)^3$

$$A^{\bullet}((\mathbb{P}^1)^3) = \frac{\mathbb{R}[x_1, x_2, x_3]}{\langle x_1^2, x_2^2, x_3^2 \rangle} \xrightarrow{\deg} \mathbb{R}$$

HR $k=0$ $\ker \begin{pmatrix} A^0 & \longrightarrow & A^4 = 0 \\ \alpha & \mapsto & \ell^4 \alpha \end{pmatrix} = A^0$

The form $A^0 \times A^0 \rightarrow \mathbb{R}$
 $\alpha \quad \beta \quad \mapsto \quad \deg(\alpha \ell^3 \beta) = 6$

$k=1$ $\ker \begin{pmatrix} A^1 & \rightarrow & A^3 \\ \alpha & \mapsto & \ell^2 \alpha \end{pmatrix}$ has basis $x_1 - x_2, x_1 - x_3$

The form $A^1 \times A^1 \rightarrow \mathbb{R}$
 $(x_1 - x_i, x_1 - x_j) \mapsto -\deg(\alpha \ell \beta) = -1 \text{ or } -2.$

Example

- X is a smooth projective variety
an special case of complex Kähler manifolds

The Kähler metric induces

- (1) the Hodge decomposition on $H^\bullet(X; \mathbb{C})$
- (2) the Lefschetz decomposition on $H^\bullet(X; \mathbb{Q})$

Example

- X is a smooth projective variety
an special case of complex Kähler manifolds

The Kähler metric induces

- (1) the Hodge decomposition on $H^{\bullet}(X; \mathbb{C})$

$$H^i(X; \mathbb{C}) = \bigoplus_{p+q=i} \sum H^{p,q}(X; \mathbb{C})$$

where $H^{p,q}(X; \mathbb{C})$ = classes of closed forms of type (p, q) .

Example

- X is a smooth projective variety of dim n
an special case of complex Kähler manifolds
- $\left(\bigoplus_{i=0}^n H^{i,i}(X, \mathbb{R}), \text{ Poincaré pairing, ample divisor classes} \right)$

- M is a matroid

If Σ_M simplicial and complete

$\Rightarrow D = \sum C_p \times p$ gives a piecewise-linear function ψ_D

- X is a smooth projective variety of dim n
an special case of complex Kähler manifolds

$\left(\bigoplus_{i=0}^n H^{i,i}(X, \mathbb{R}), \text{Poincaré pairing, ample divisor classes} \right)$

- M is a matroid

φ_D is ample if $\varphi_D(u) + \varphi_D(v) < \varphi_D(u+v)$

$\forall u, v \in N_{\mathbb{R}}$ in different cones.

- X is a smooth projective variety of dim n
an special case of complex Kähler manifolds

$\left(\bigoplus_{i=0}^n H^{i,i}(X, \mathbb{R}) , \text{ Poincaré pairing, ample divisor classes} \right)$

- M is a matroid

Since Σ_M is often not complete,

D is ample if φ_D is a restriction of φ'_D
on a complete $\Sigma' \supseteq \Sigma_M$.

- X is a smooth projective variety of dim n
an special case of complex Kähler manifolds

$\left(\bigoplus_{i=0}^n H^{i,i}(X, \mathbb{R}), \text{ Poincaré pairing, ample divisor classes} \right)$

- M is a matroid

$\left(A^*(M), \deg, \text{ ample divisor classes} \right)$