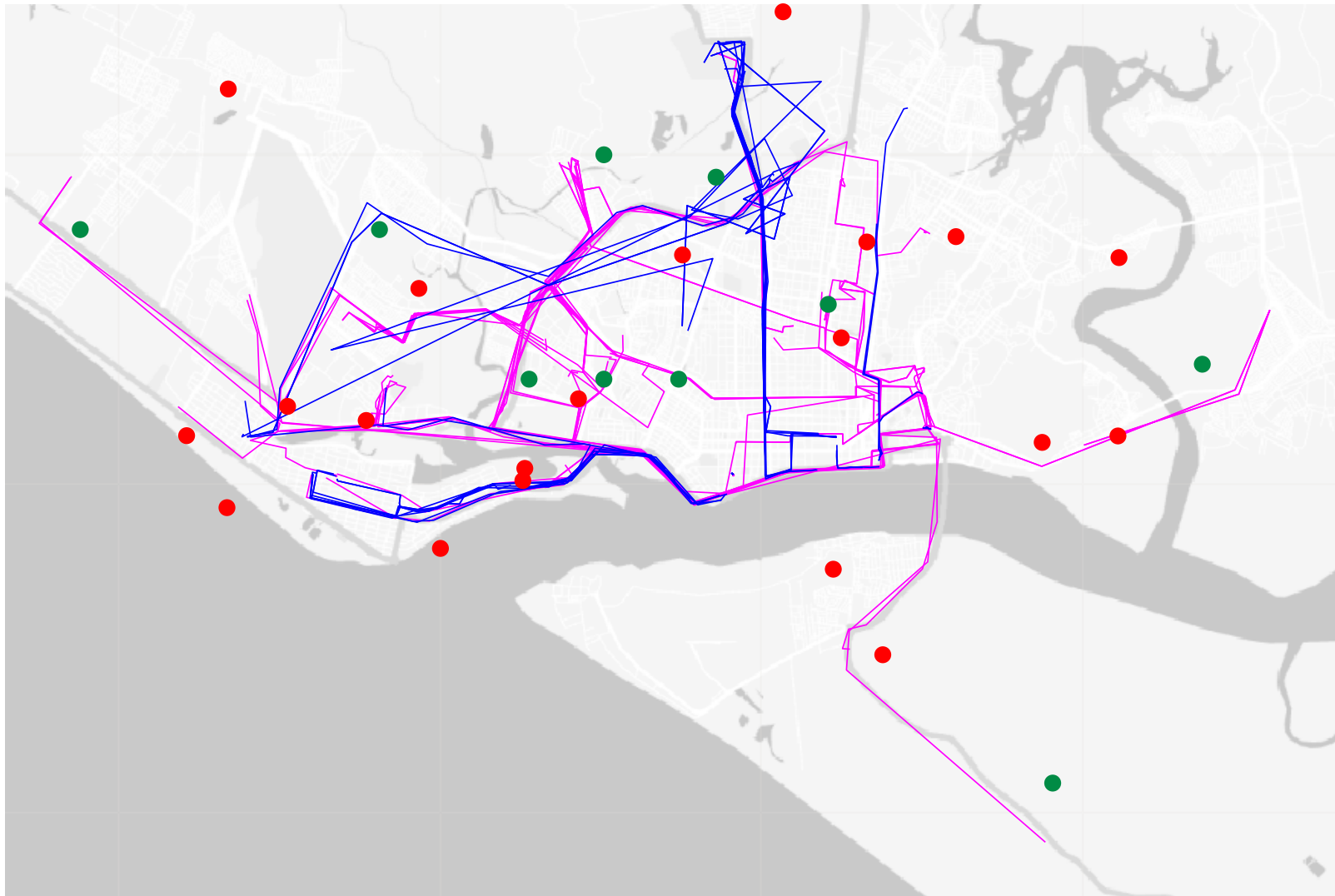# Sketching and Classifying Spatial Trajectories

Trajectories made easy! with this one simple trick
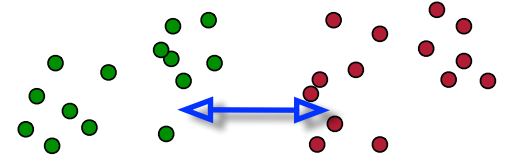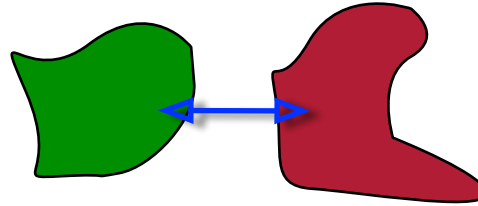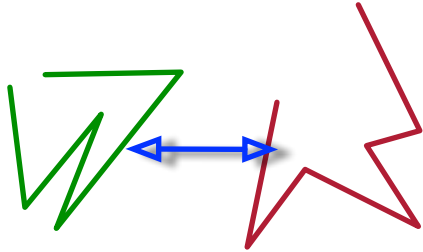
Jeff M. Phillips: University of Utah
Hasan Pourmahmood: University of Utah
Pingfan Tang: University of Utah -> Google

Classifying Trajectories?

Two sets of trajectories: **buses** and **cars**
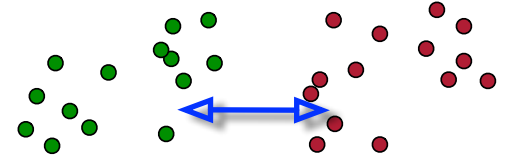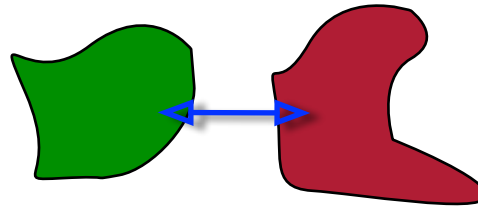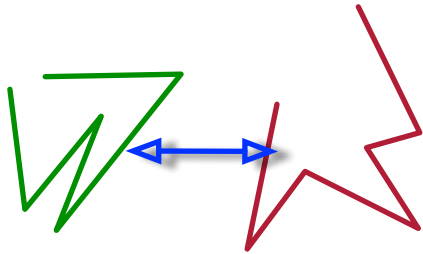Just using location, how to build classifier?

Distances between Shapes

Many distances: Hausdorff, Frechet, Dynamic Time Warping, Wasserstein, Kernel Distance, Turning Curve, …

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- understand "shape" / model real-world properties

- search queries in shape database

- *learn* something about shapes
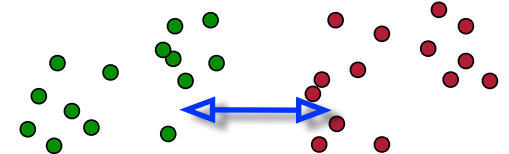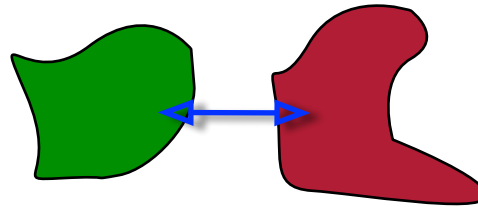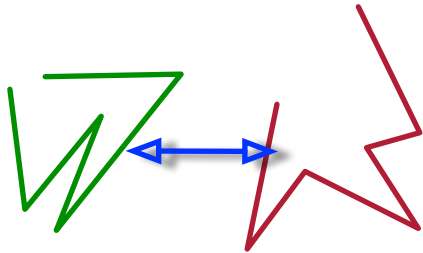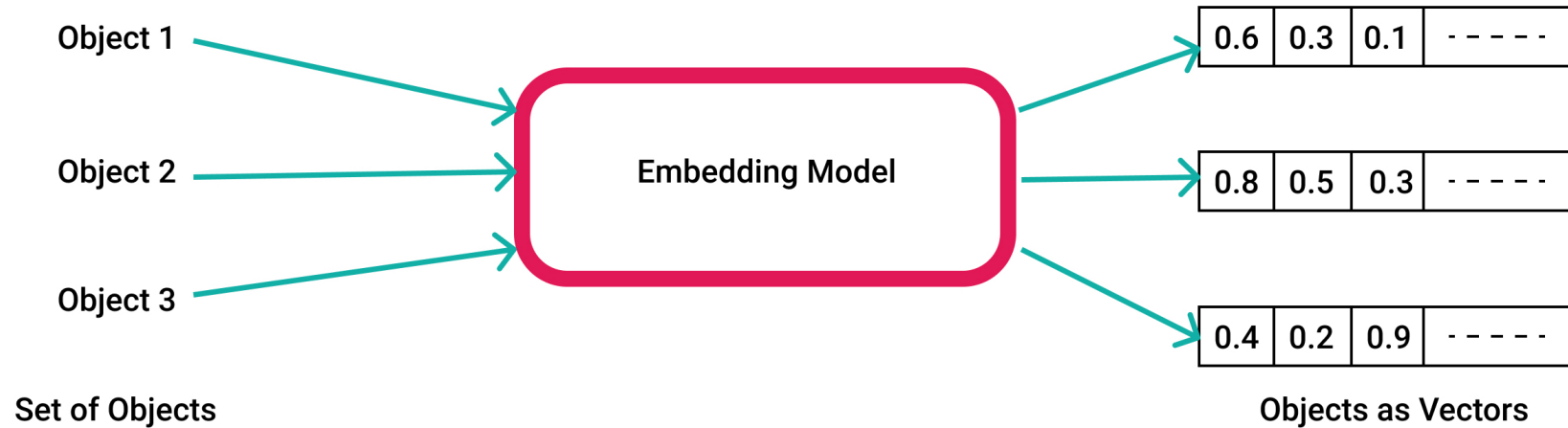
Distances between Shapes

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metric <   inner product in RKHS   <   Euclidean representation

Feature Vectors

Object 1 → Embedding Model → | 0.6 | 0.3 | 0.1 | - - - - - |

Object 2 → Embedding Model → | 0.8 | 0.5 | 0.3 | - - - - - |

Object 3 → Embedding Model → | 0.4 | 0.2 | 0.9 | - - - - - |

Set of Objects

Objects as Vectors

Object	Vector	Task

Pinecone

database for vectors
$100M in VC funding

IMAGE → IMAGE TRANSFORMER → [1.3, 0.6, 1.2, -1.3, ...] → Object recognition, deduplication, scene detection, product search, ...

Happy Birthday — TEXT → NLP TRANSFORMER → [0.3, -0.4, 1.2, 0.3, ...] → Translation, understanding, Sentiment, Question Answering, Semantic Search, ...
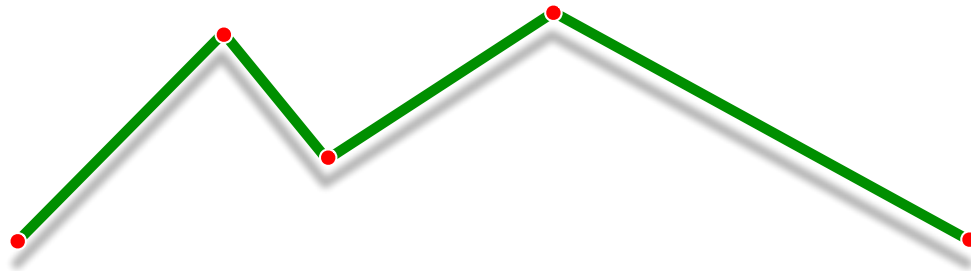
AUDIO → AUDIO TRANSFORMER → [1.2, -0.3, 0.7, -1.8, ...] → Anomaly detection, speech-to-text, music transcription, machinery malfunction, ...
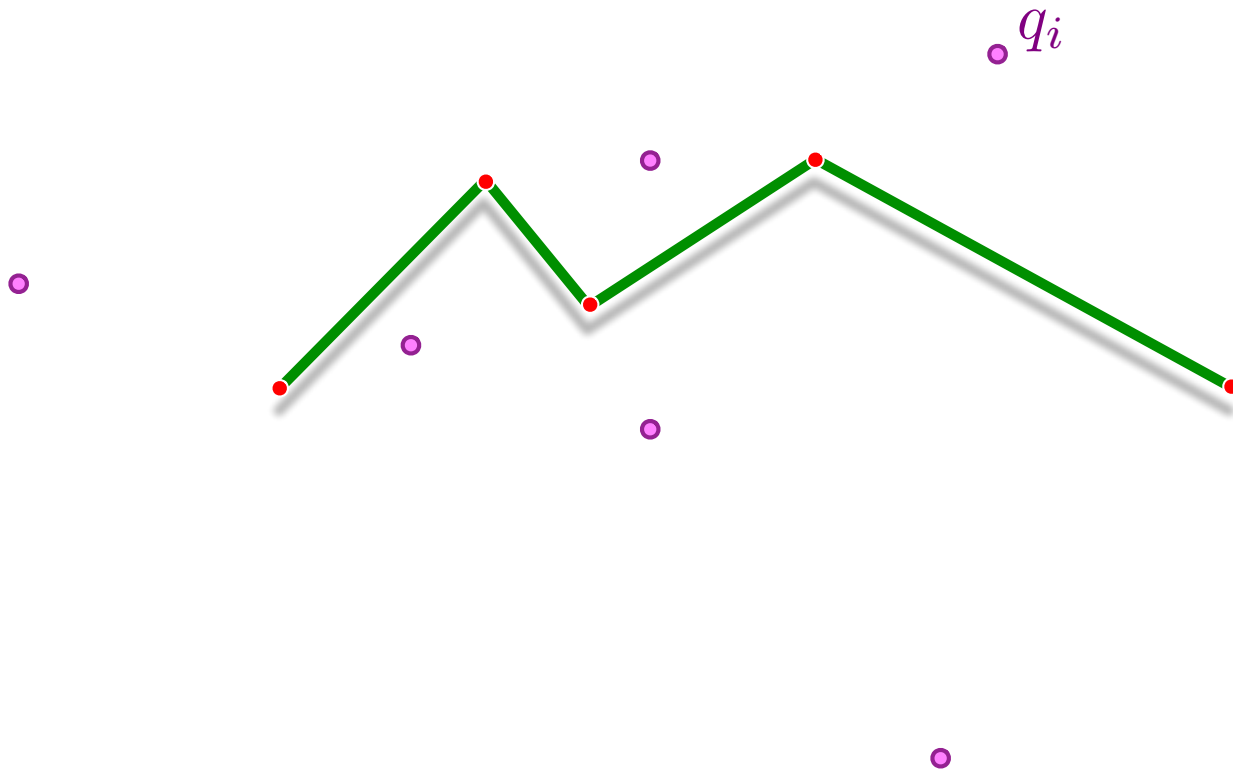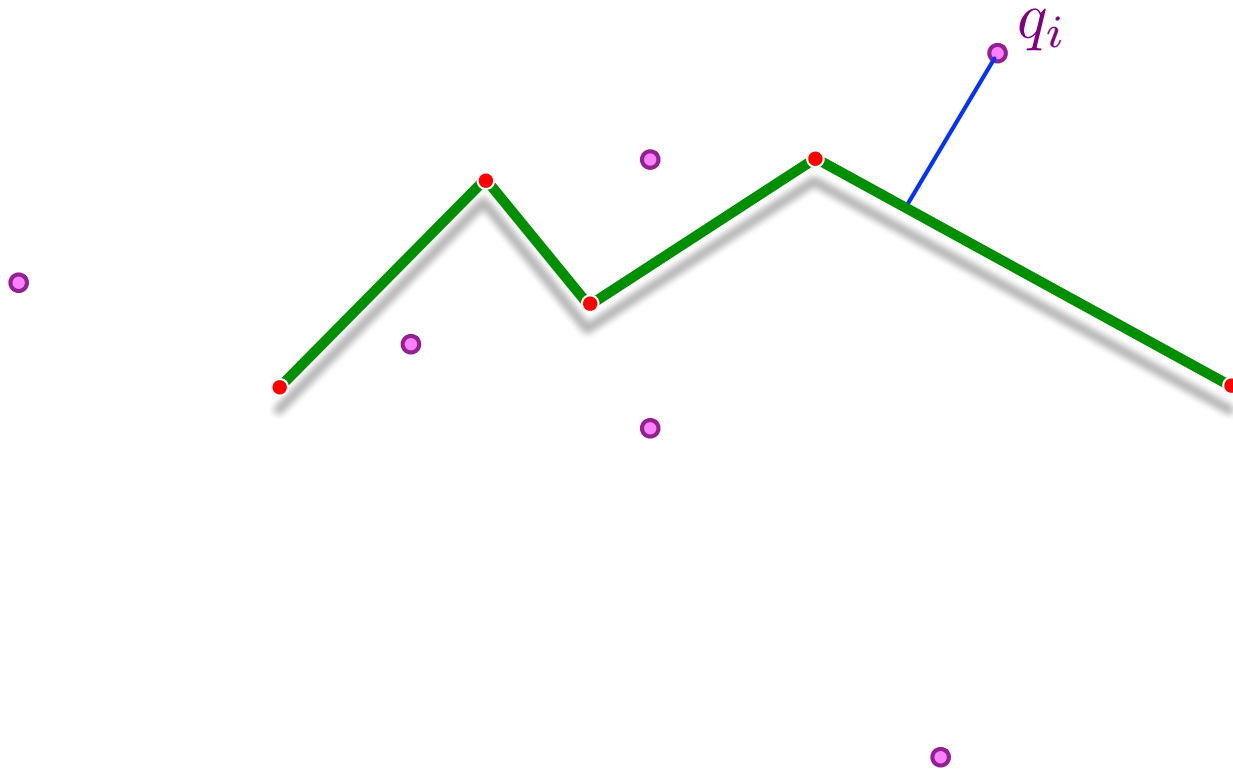
MinDist Sketch

shape J

MinDist Sketch

shape J

landmarks Q

q_i

MinDist Sketch

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minDist

$$v_i(J) = \inf_{p \in J} \|q_i - p\|$$

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$$v(J) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \dots \\ v_n \end{bmatrix}$$

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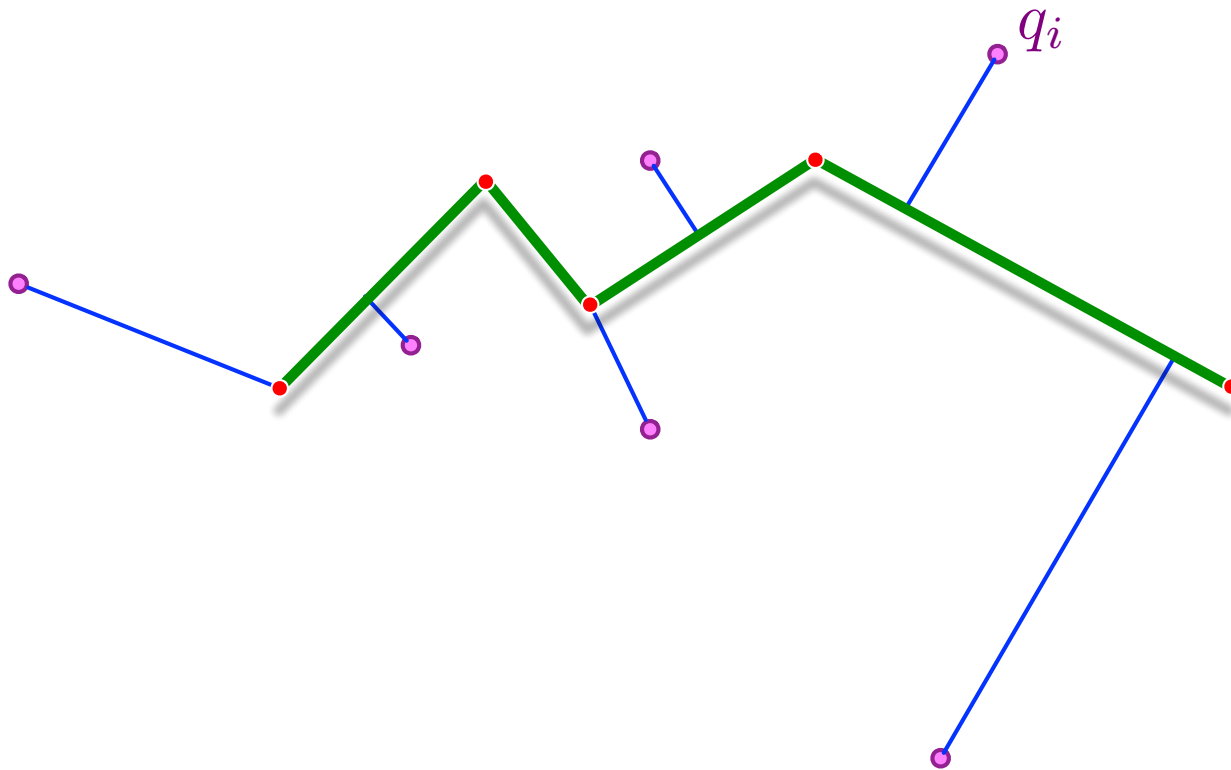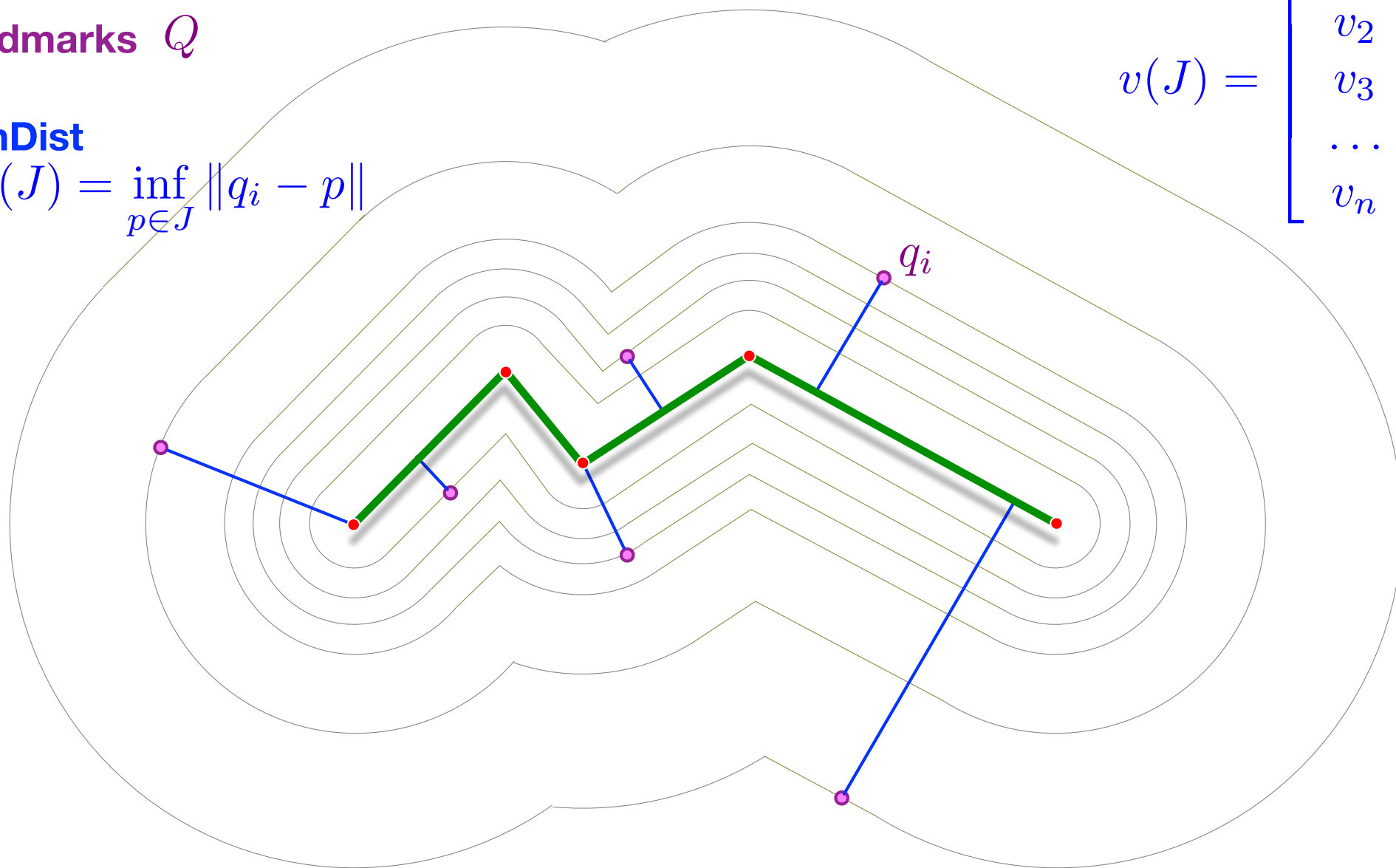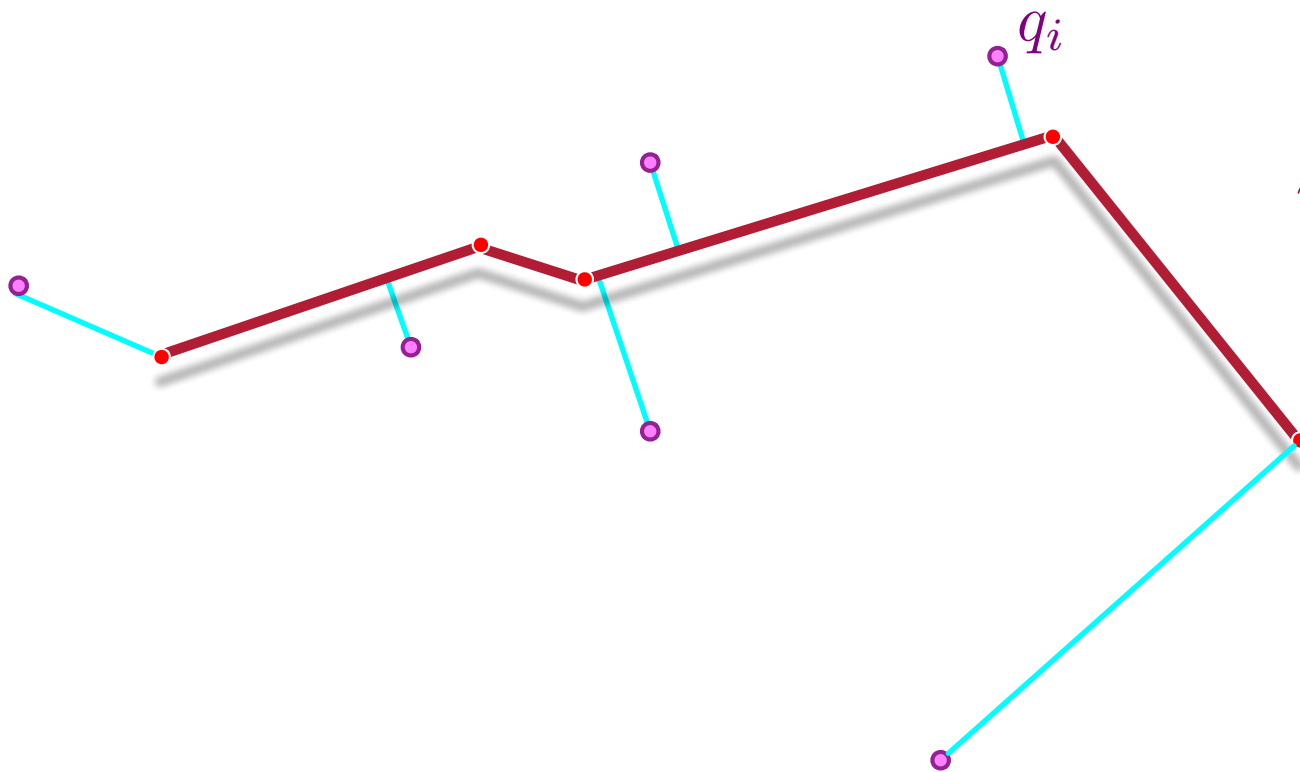
q_i

MinDist Sketch

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$$v_i(J) = \inf_{p \in J} \|q_i - p\|$$

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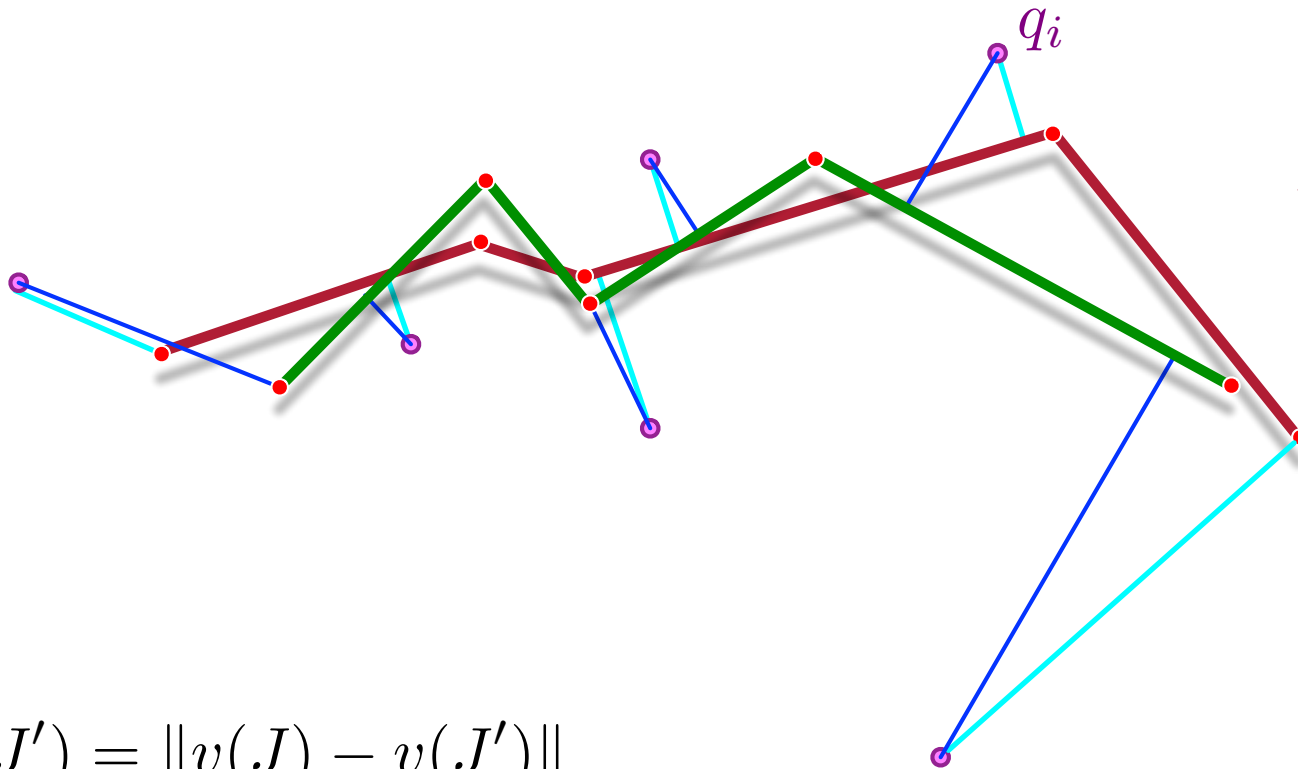
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q_i

$$d_Q(J, J') = \|v(J) - v(J')\|$$
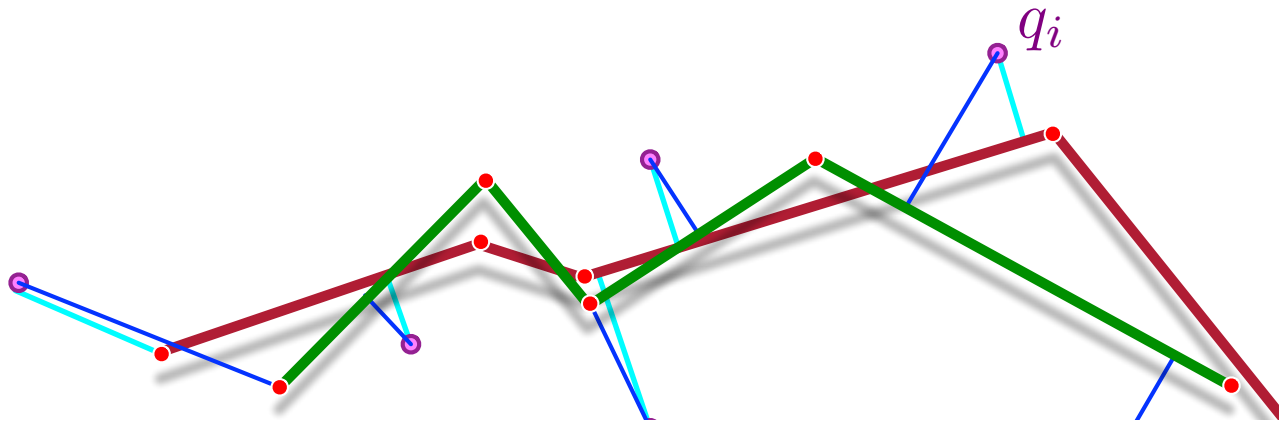
MinDist Sketch

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$$v(J') = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \\ \dots \\ v_n' \end{bmatrix}$$

q_i

OK, so is this a good distance?
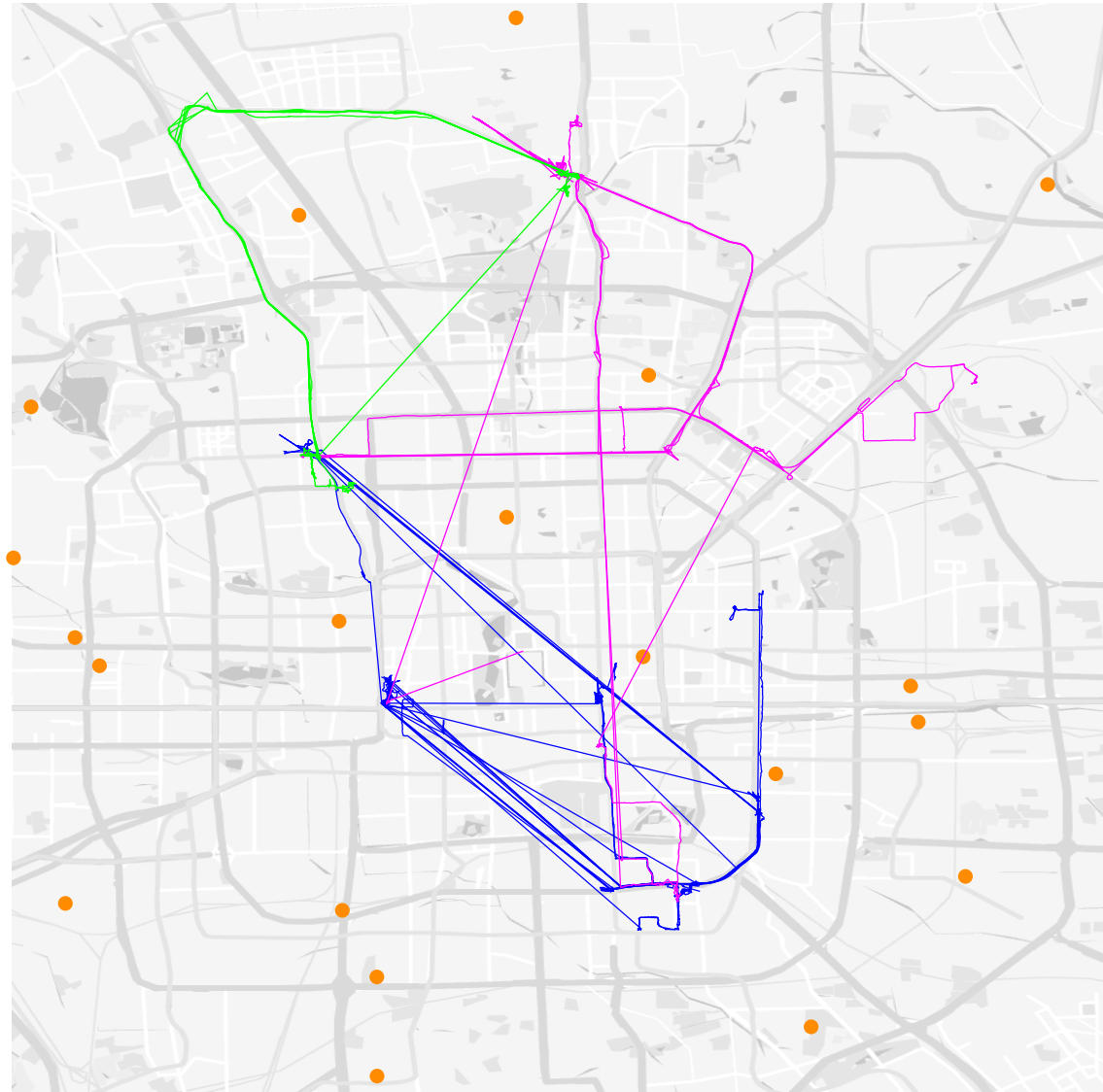
$$d_Q(J, J') = \|v(J) - v(J')\|$$

Clustering

Consider 42 (Geolife) GPS traces of car routes in Beijing

Set Q as 20 POIs

map trajectories to
$$v(\gamma_1), \ldots, v(\gamma_{42}), \in \mathbb{R}^{20}$$
run k-means!

Nearest Neighbor Queries

3 million trajectory
36 GB storage

Set Q as 20 POIs

K-Graph https://github.com/aaalgo/kgraph
an optimized Euclidean distance NN search
preprocess time 62 (s)
sketch time   109 (s)
query time 0.00037 (s)
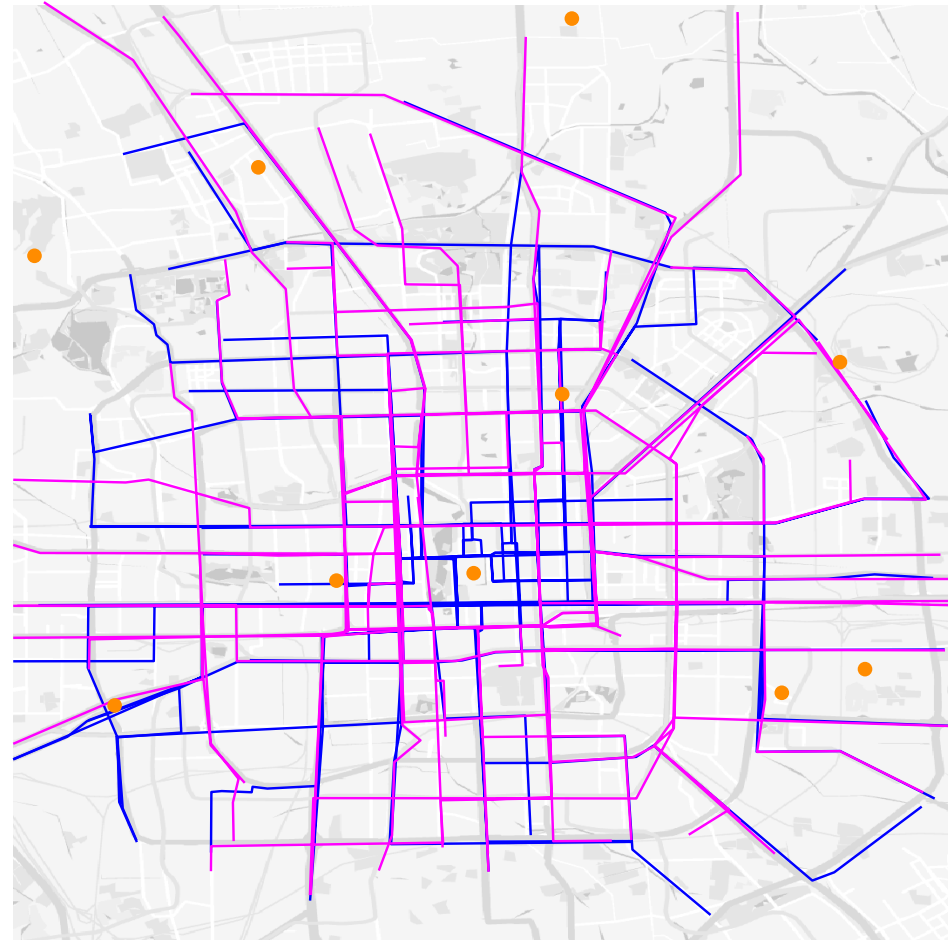
State of the art
 (*on 10-40 GB*)

Xie etal (VLDB 17)
 - Hausdorff : 50 (s)
Shang etal (SIGMOD 18)
 - DTW : 0.01 (s)
  on 256 cores

Is d$_Q$ a Metric?

$$d_Q(J, J') = \|v(J) - v(J')\|$$

For set of objects $\mathcal{T}$
a distance $d : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^+$ is a **metric** if

- symmetry: $d(a, b) = d(b, a)$

- identity: $d(a, b) = 0$ if and only if $a = b$.

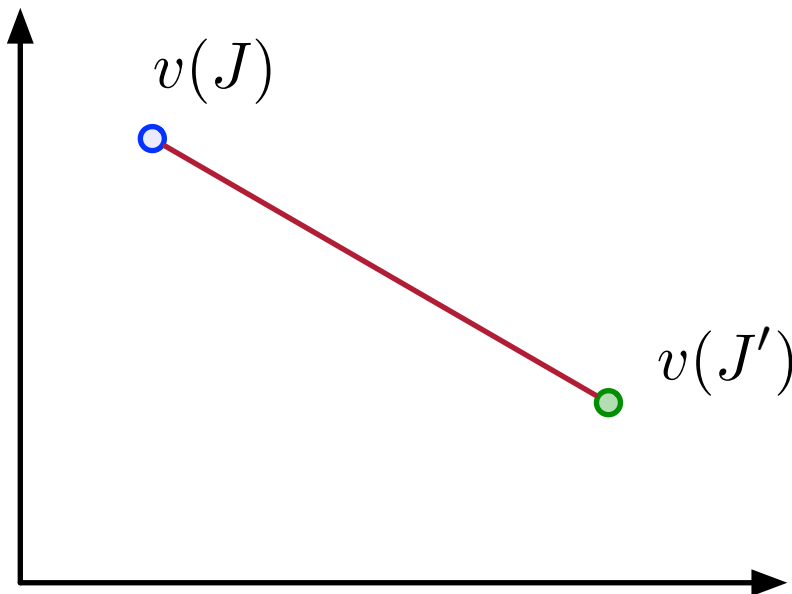- triangle inequality: $d(a, c) + d(c, b) \geq d(a, b)$

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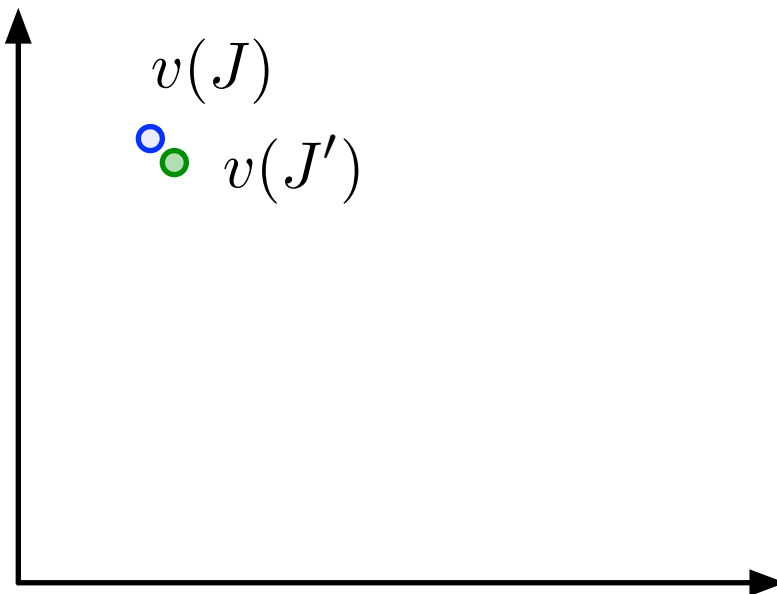- triangle inequality: $d(a, c) + d(c, b) \geq d(a, b)$

$v(J)$

$v(J')$

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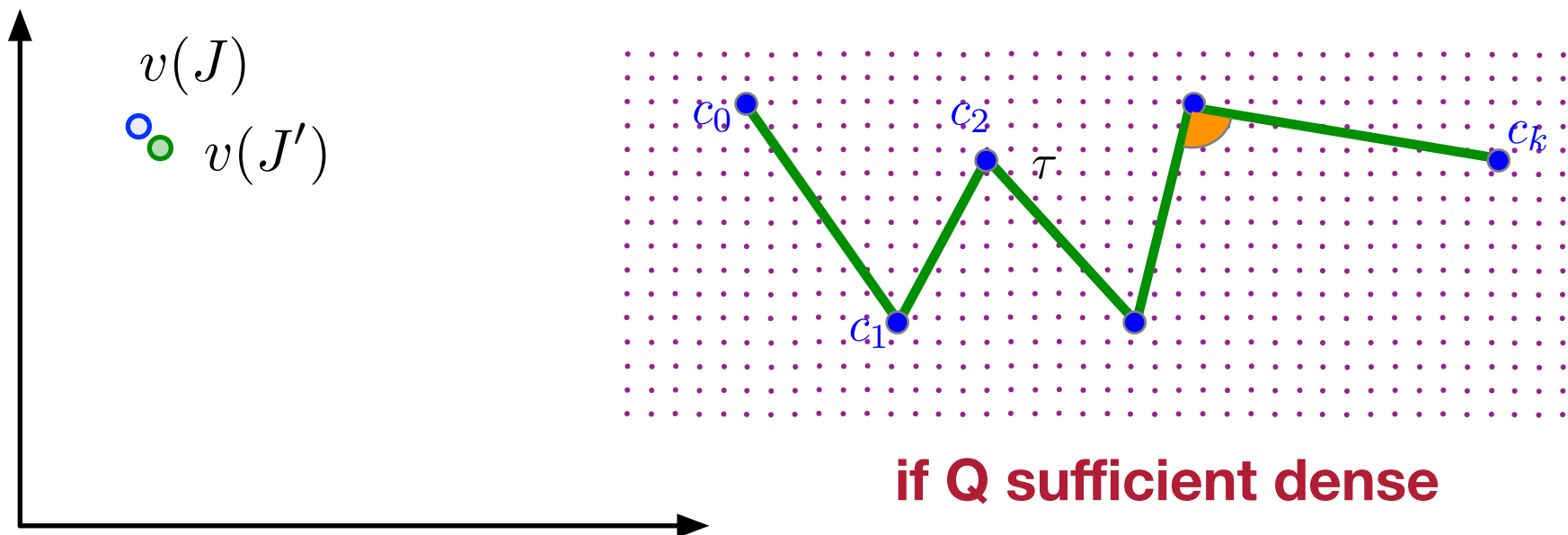- triangle inequality: $d(a, c) + d(c, b) \geq d(a, b)$

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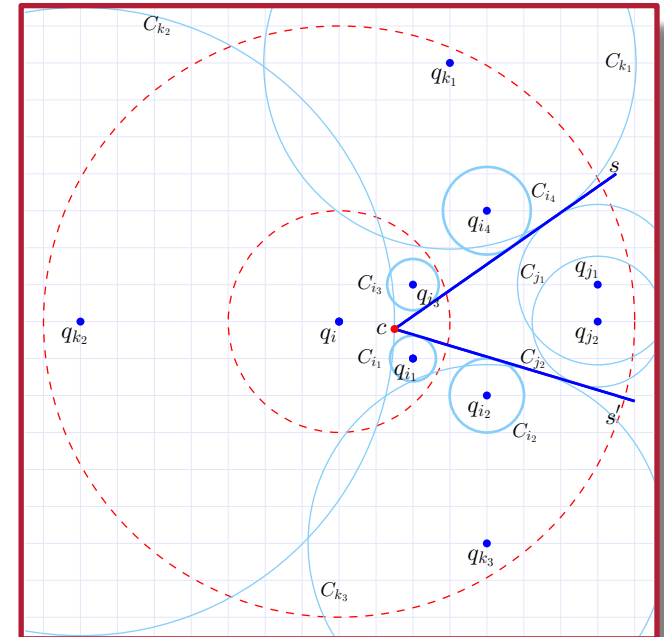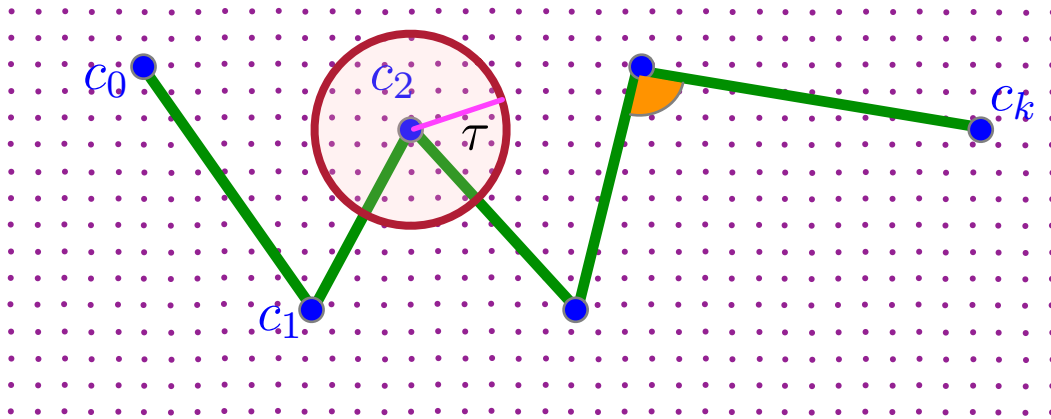a distance $d : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^+$ is a **metric** if

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- triangle inequality: $d(a, c) + d(c, b) \geq d(a, b)$

$v(J)$
$v(J')$

c_0 c_2 τ c_k
c_1

if Q sufficient dense

Curve Recontruction

Shapes are k-pl curves (with τ-separated critical points c_i).
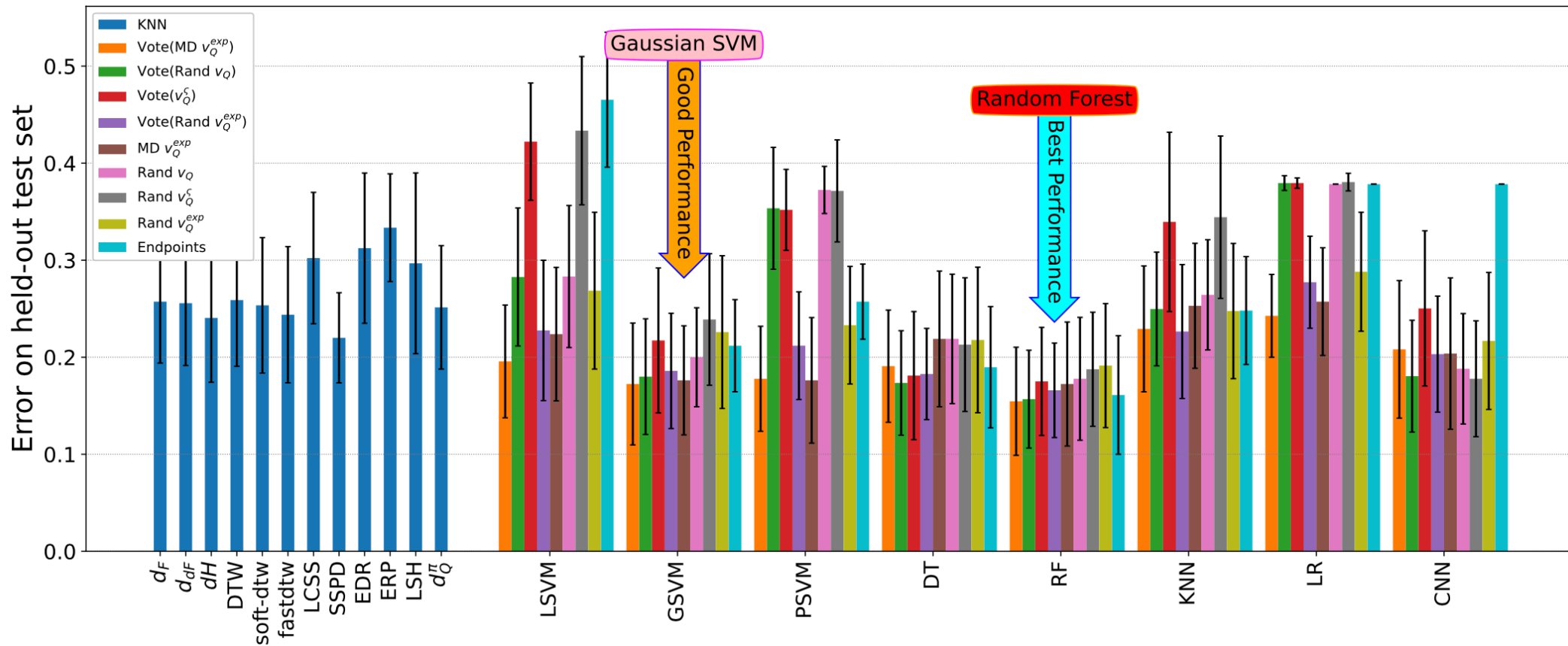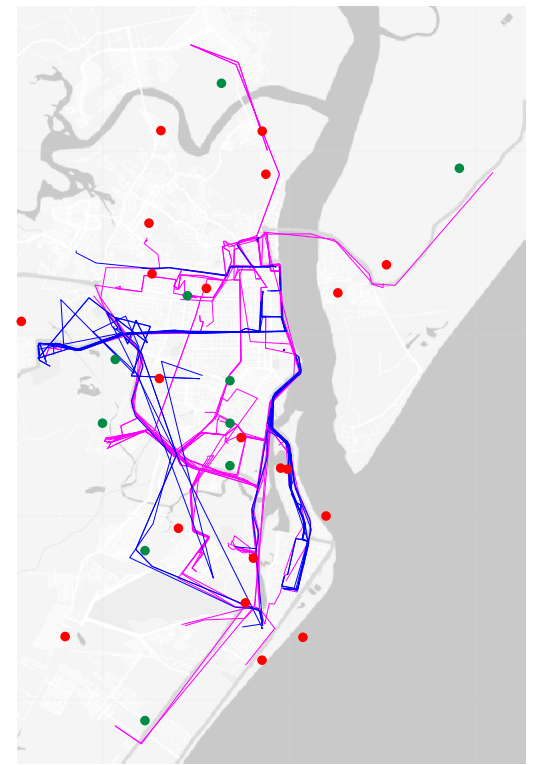If Q is dense enough, we can reconstruct J from sketch vector $v(J)$.

Classification

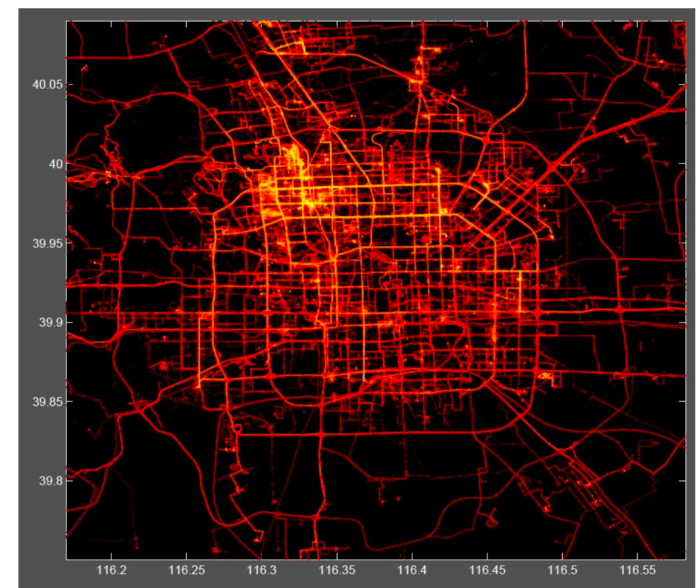Compare KNN, SVM, Random Forest, CNN, etc

to KNN on discrete Frechet, DTW, Hausdorff, LCSS, Edit distance for real sequence, …
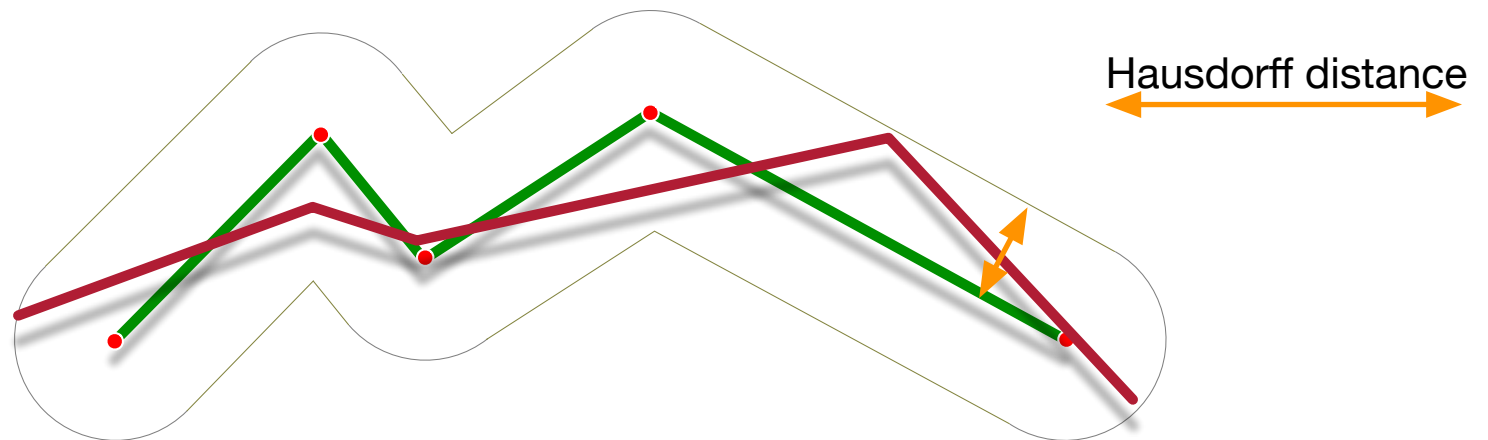
Bus vs. Car in Aracaju

Classification

Geolife Trajectory Data Set
Predict **mode** of transportation

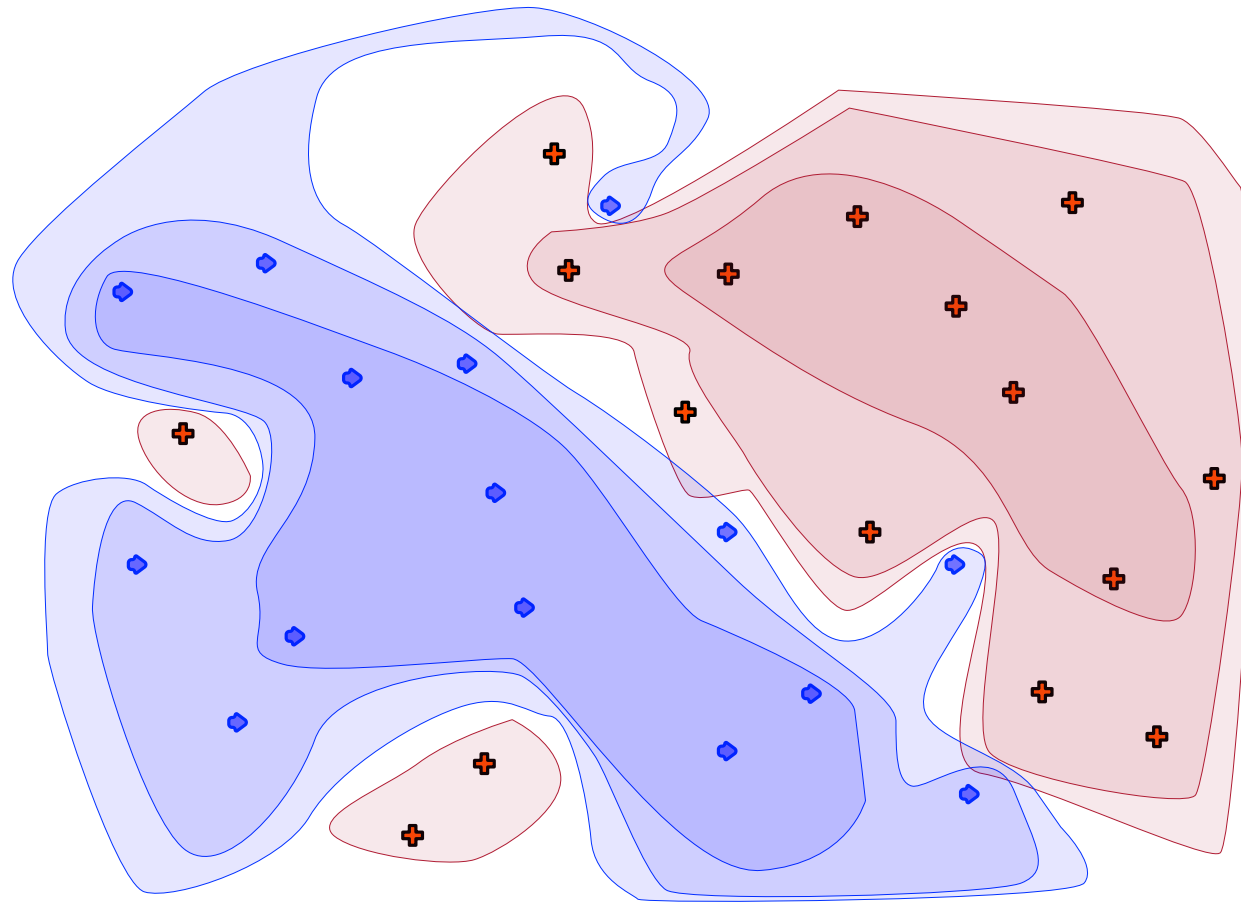| Study	Misclassifn Rate
Using CNN [ETNK2016]	32.1%
Using CNN [WLJL2017]	25.9%
Inference plus Decision Tree [ZLCXM2008]	23.8%
Using CNN [DCHR2020]	23.2%
Our Model with v_Q vectorization	18.1%
Our Model with v_Q^{S+} vectorization	16.4%
Our Model with MDv_Q^+ vectorization	15.4%
Using CNN [DH2018]	15.2%
Our Model with v_Q^+ vectorization	11.9%

Are trajectories "learnable"?

Hausdorff distance

Are trajectories "learnable"?

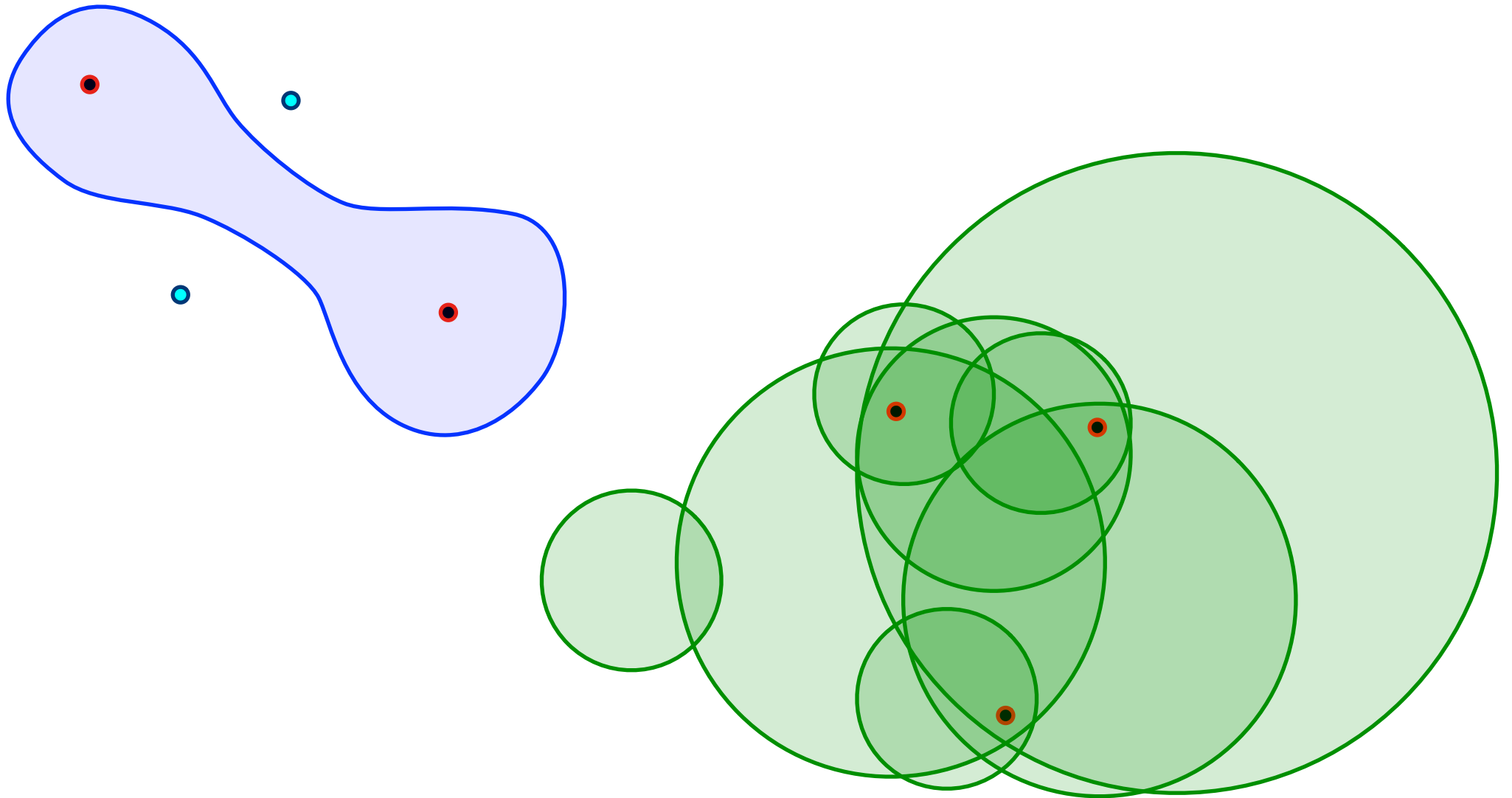Consider data (X, y) with $X \sim \mu$ labels $y_i \in \{-1, +1\}$.
Range space $(X, \mathcal{B})$ with VC-dimension ν
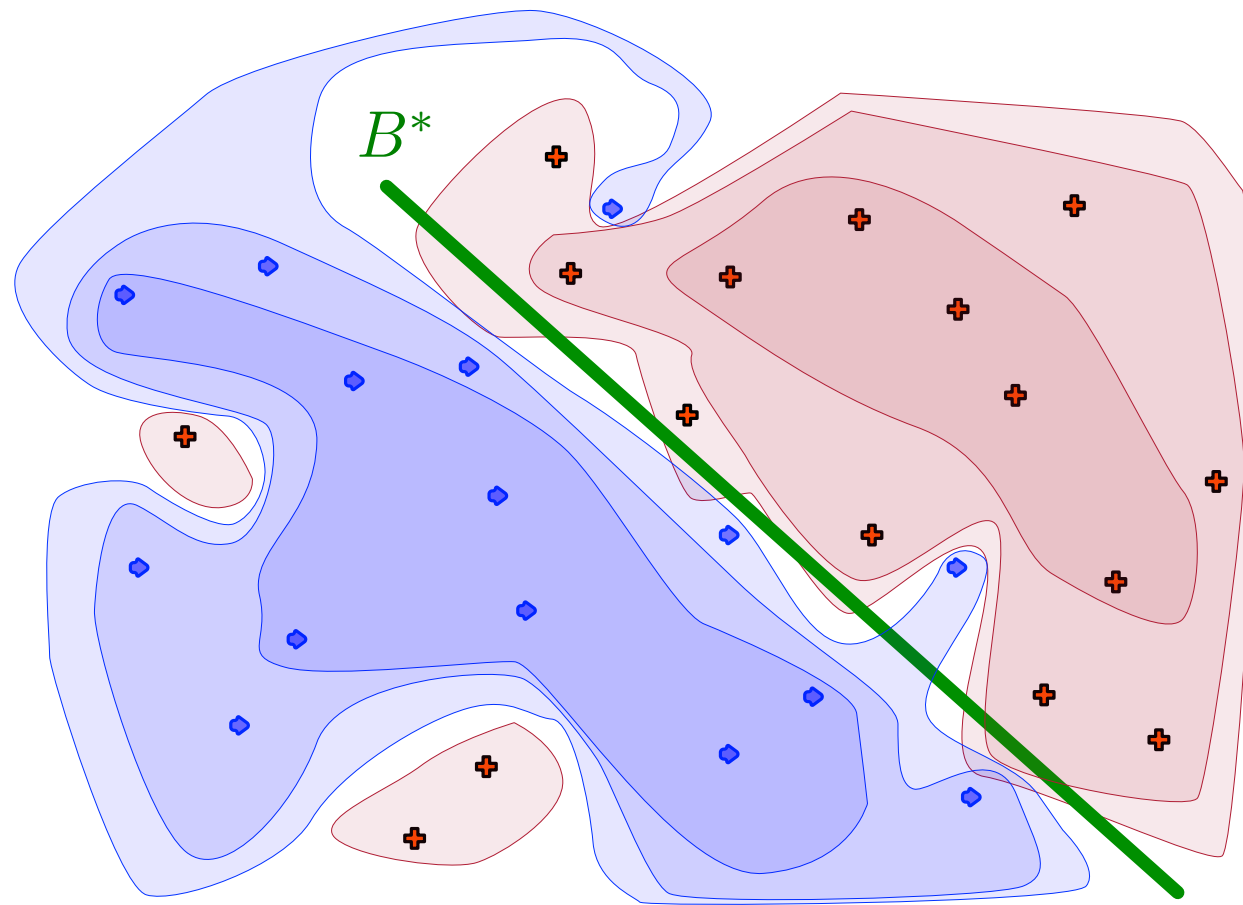
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Consider $B^* \in \mathcal{B}$, minimizing label disagreement Δ_μ on μ
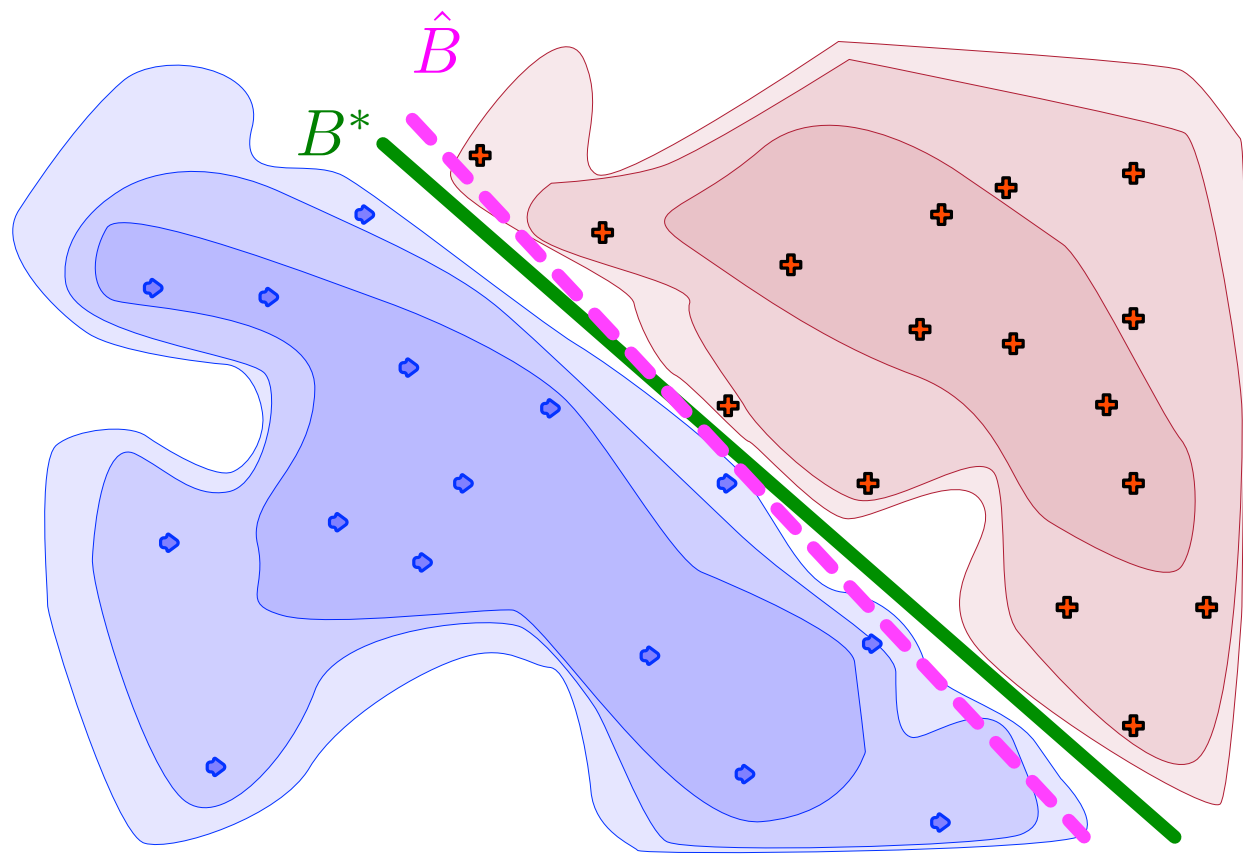
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Consider $B^* \in \mathcal{B}$, minimizing label disagreement Δ_μ on μ

To learn $\hat{B} \in \mathcal{B}$ on X so $|\Delta_\mu(\hat{B}) - \Delta_\mu(B^*)| \leq \varepsilon$

- if $\Delta_\mu(B^*) = 0$, then need $X = O((\nu/\varepsilon) \log \frac{\nu}{\varepsilon})$
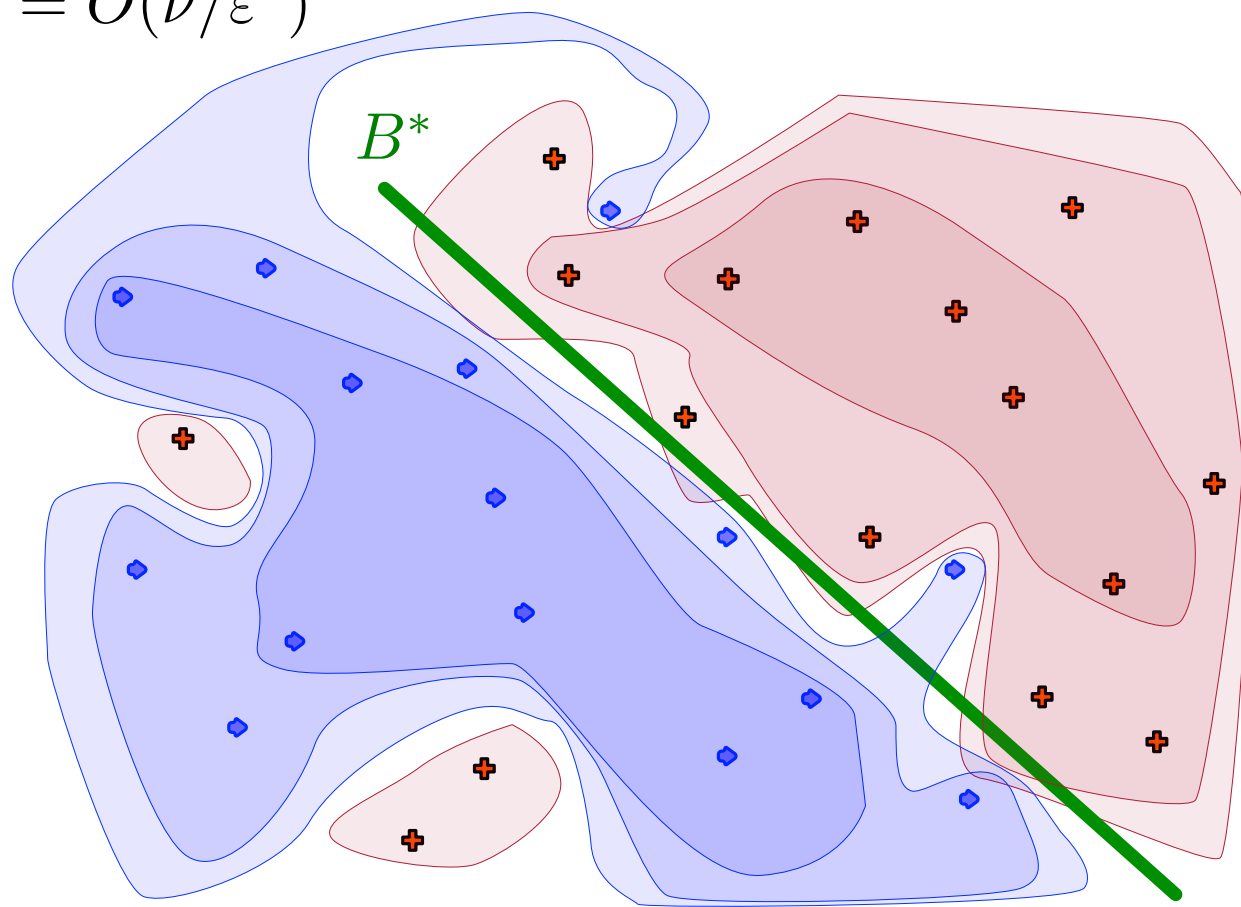
Are trajectories "learnable"?

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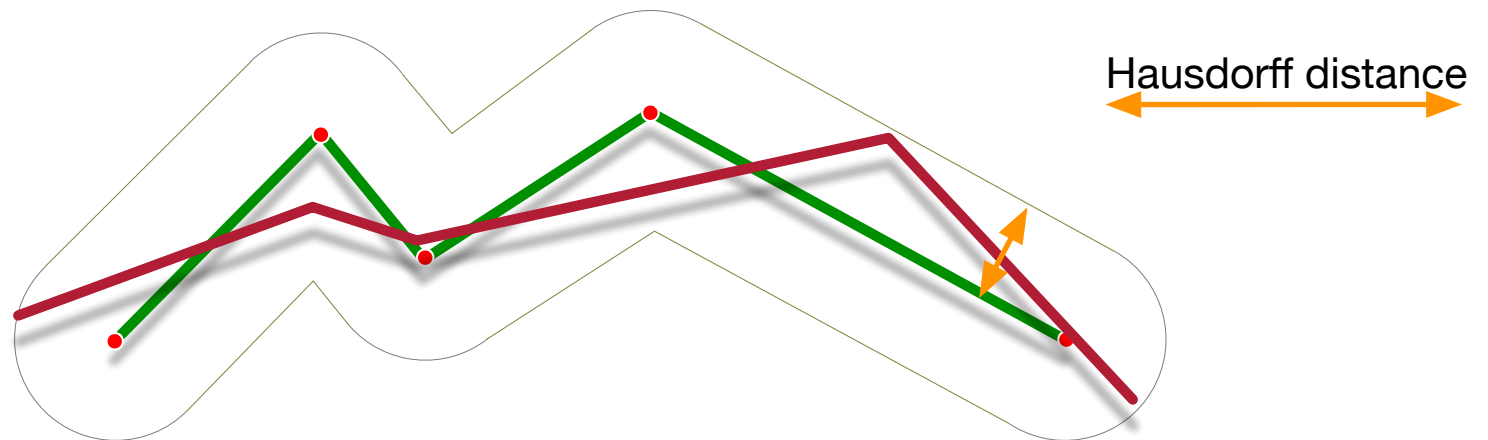**Range space** $(X, \mathcal{B})$ with VC-dimension ν

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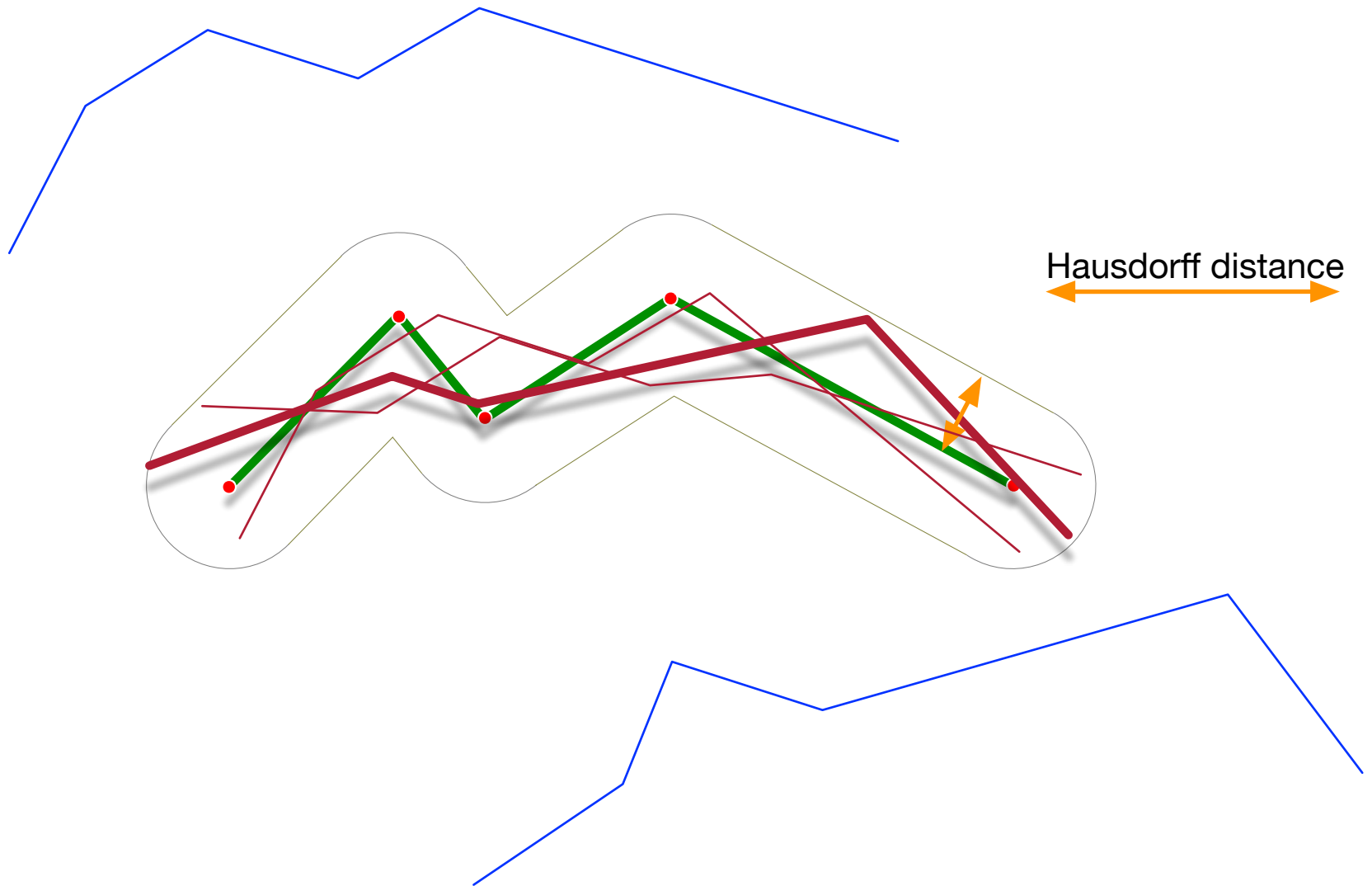- if $\Delta_\mu(B^*) = 0$, then need $X = O((\nu/\varepsilon) \log \frac{\nu}{\varepsilon})$
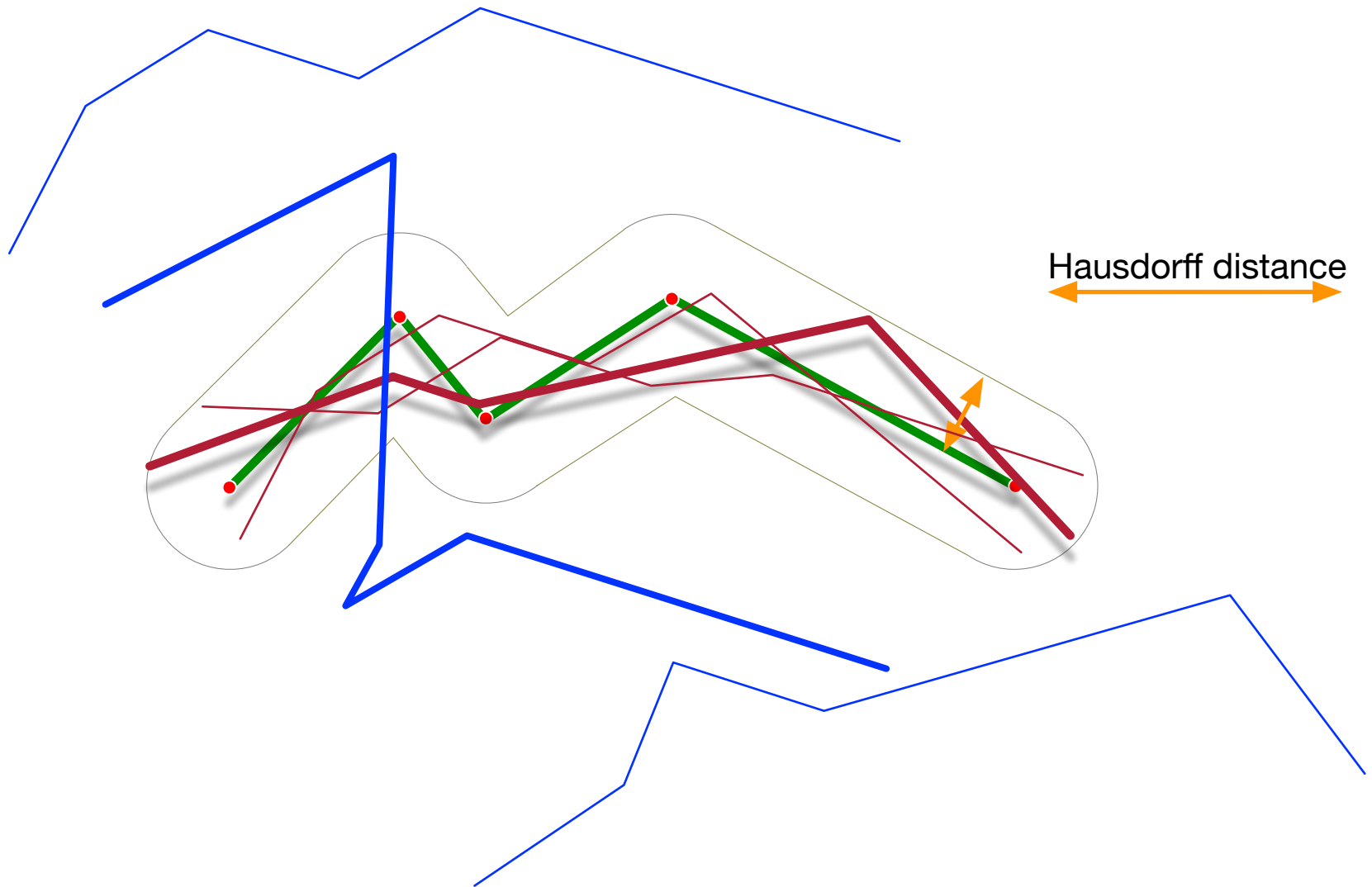- if $\Delta_\mu(B^*) > 0$, then need $X = O(\nu/\varepsilon^2)$
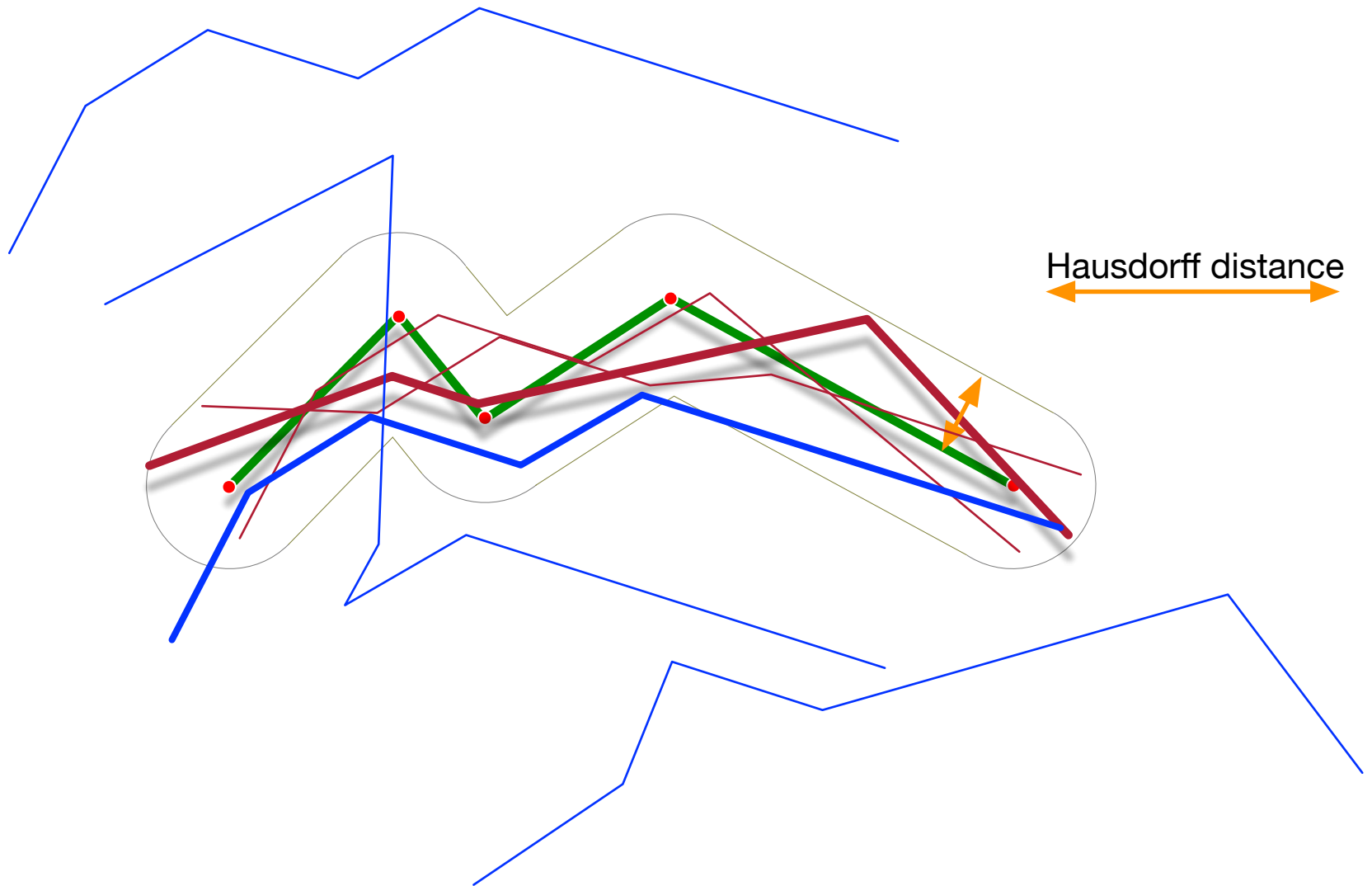
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Hausdorff distance

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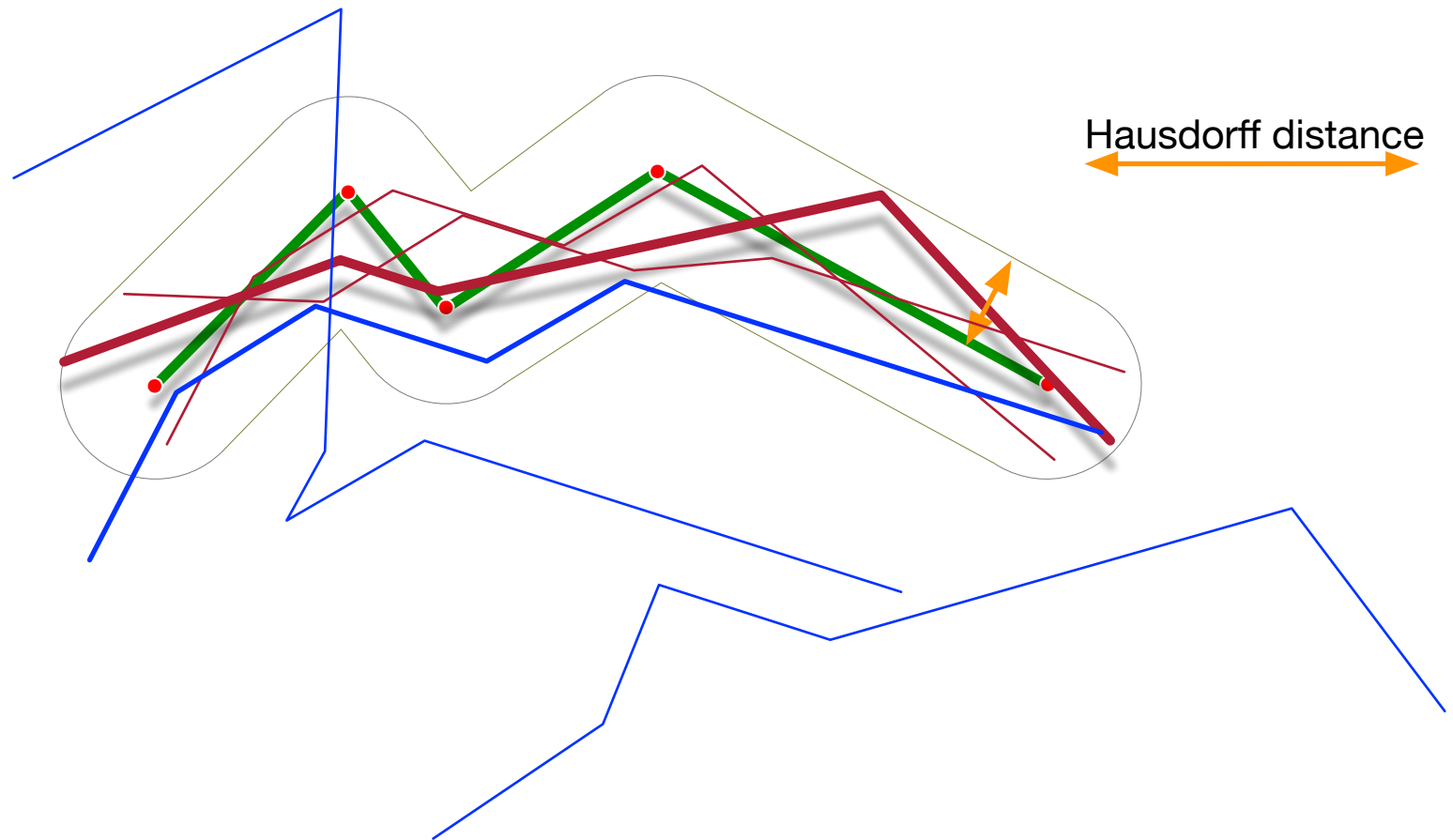**VC dimension** ν : how complex is learning

Hausdorff : $v = O(d^2 k^2 \log(dkm))$

Frechet : $v = O(d^2 k^2 \log(dkm))$

[Driemel, Nusser, Phillips, Psarros 19]

$d =$ dimension, k length of query curve, m length of data curves

Hausdorff distance

Are trajectories "learnable"?

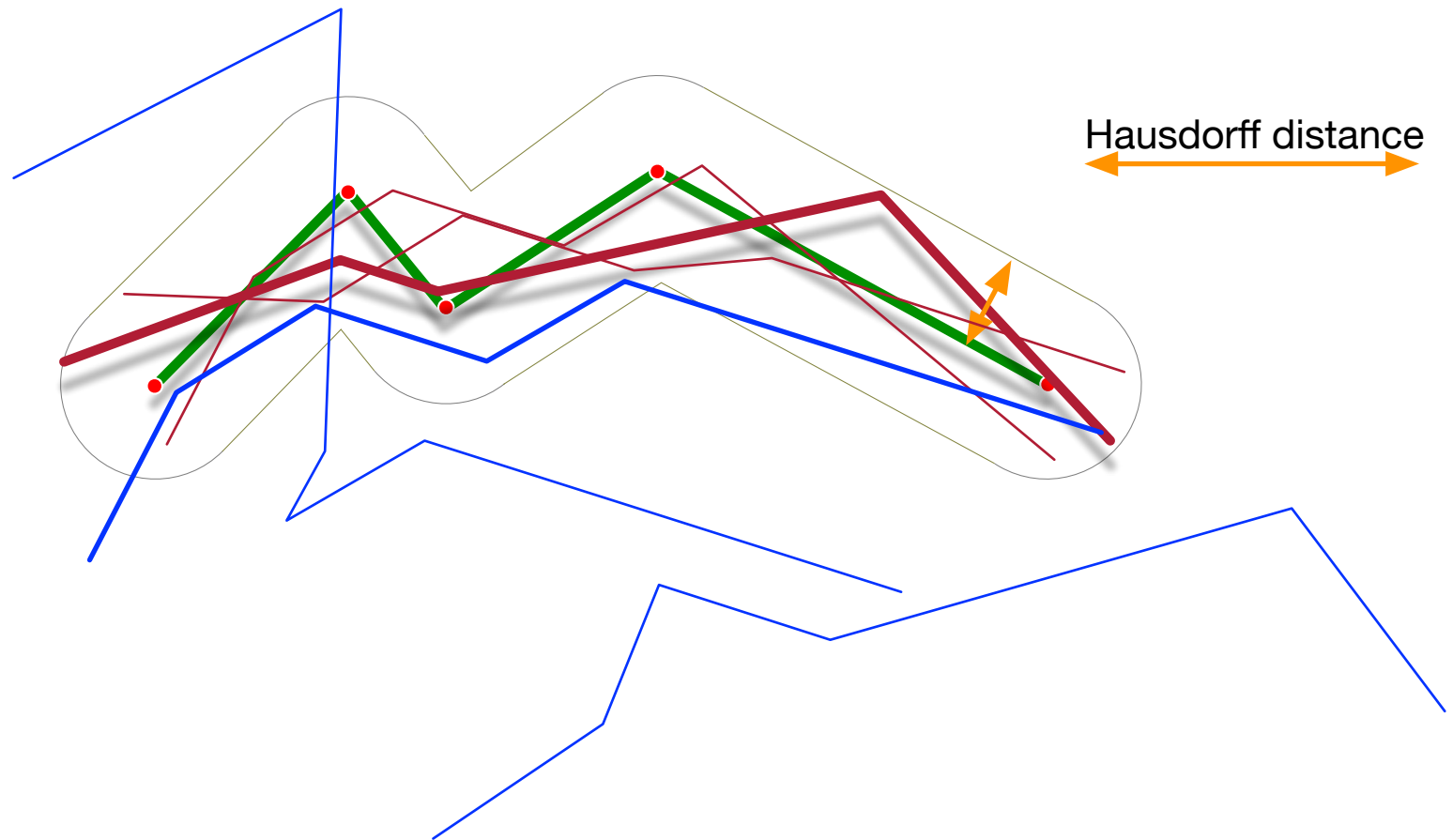**VC dimension** ν : how complex is learning

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[Driemel, Nusser, Phillips, Psarros 19]
[Brunig, Driemel: SODA 2024]

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Hausdorff distance

$v(J)$

$v(J')$
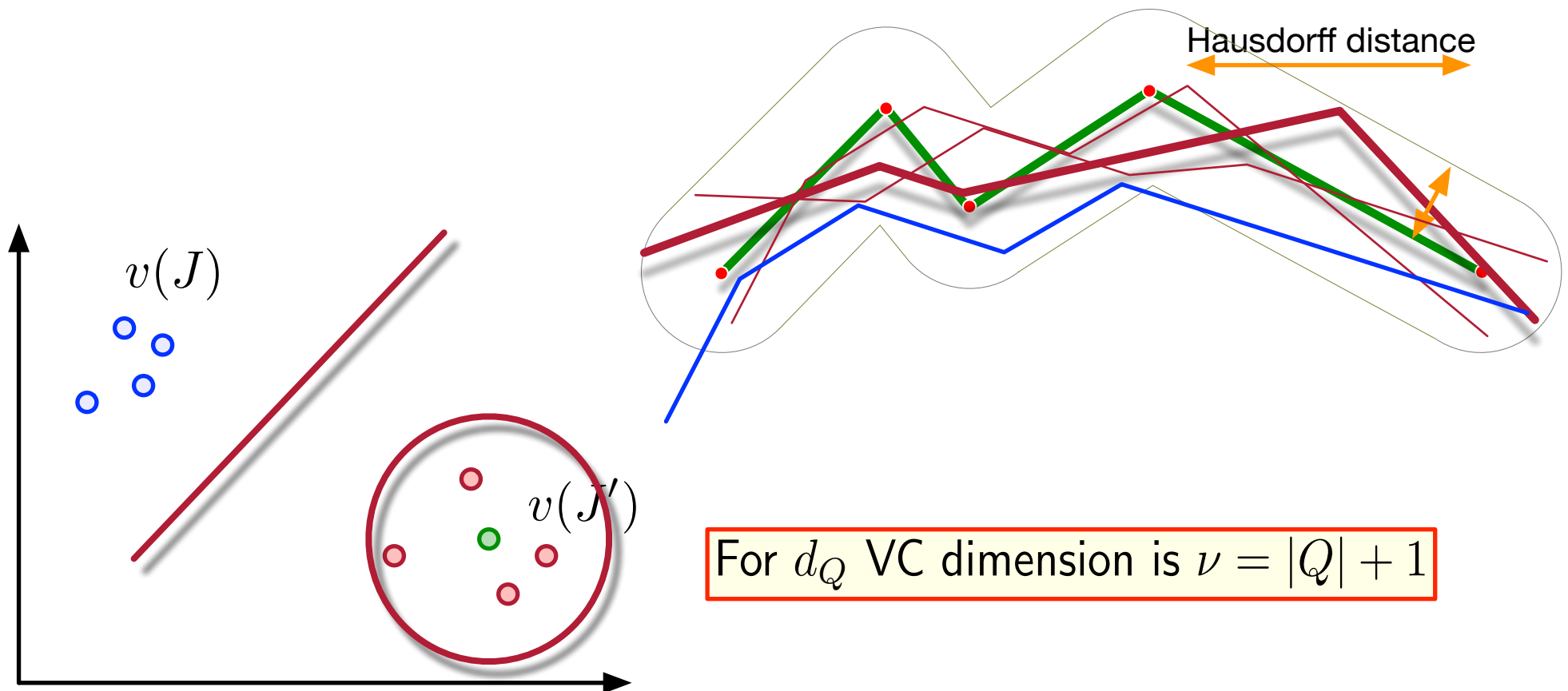
For d_Q VC dimension is $\nu = |Q| + 1$
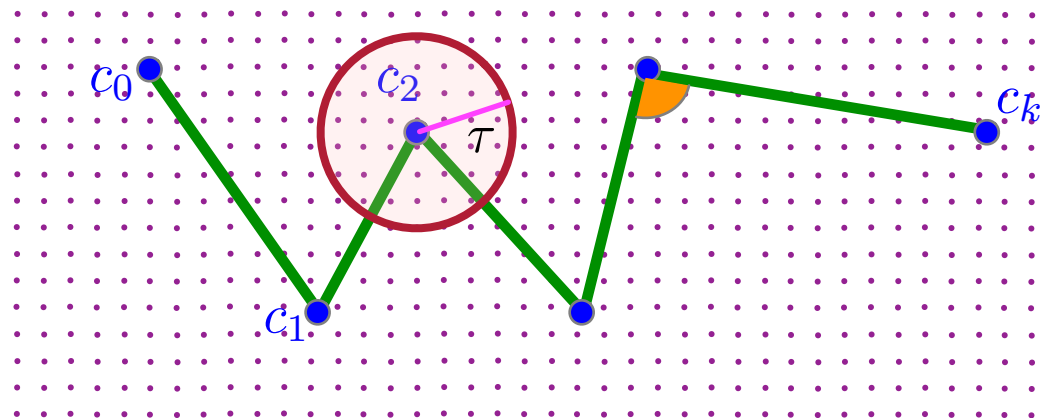
How many landmarks?

For metric properties,
 or curve reconstruction
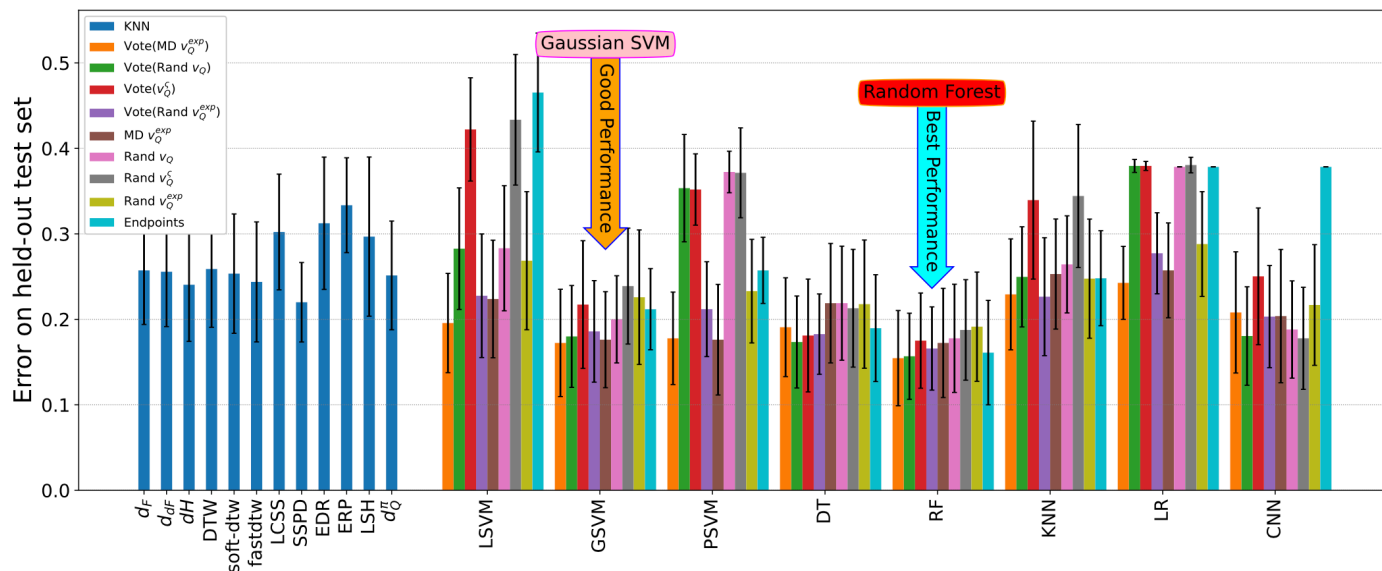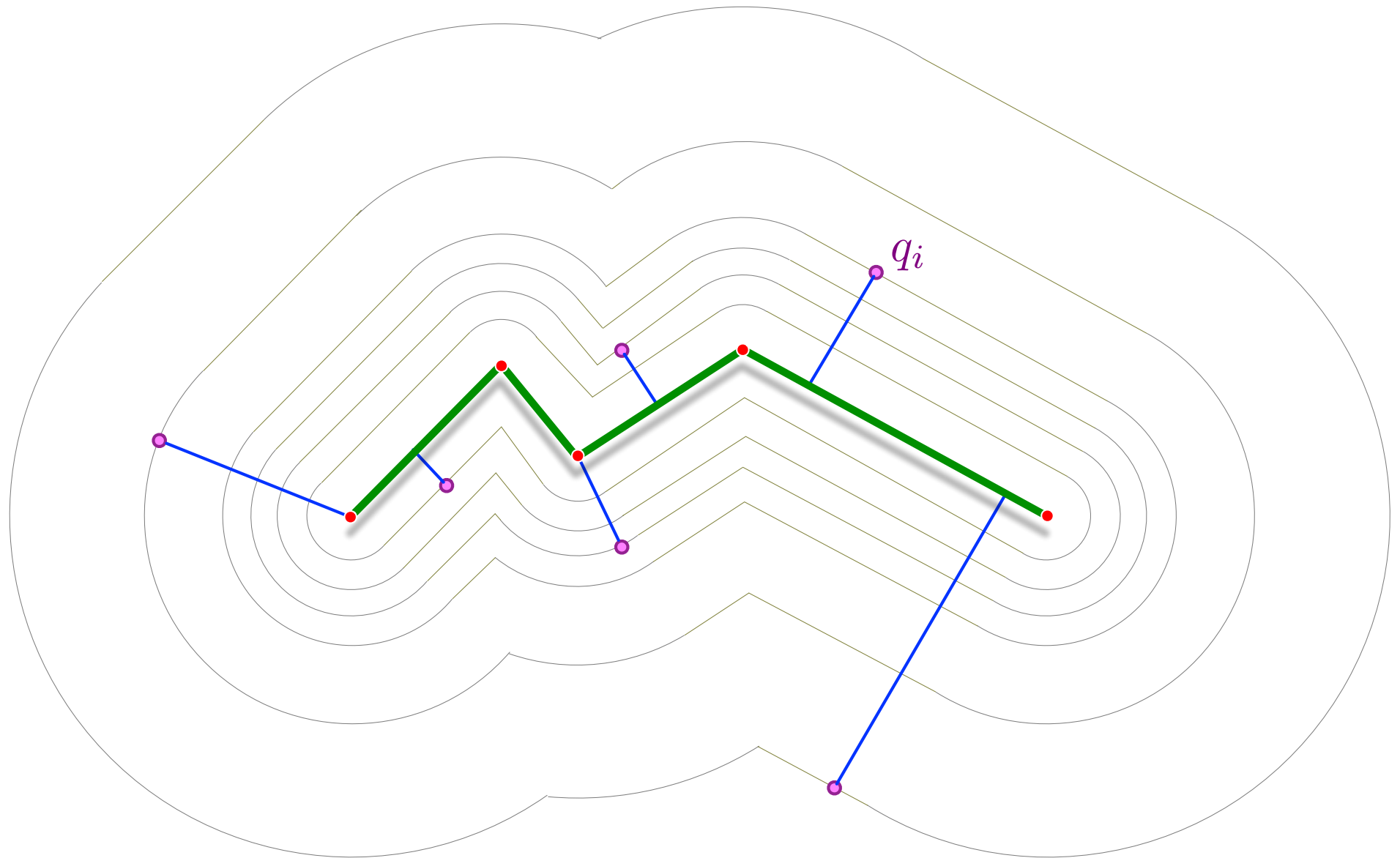 —> fine grid
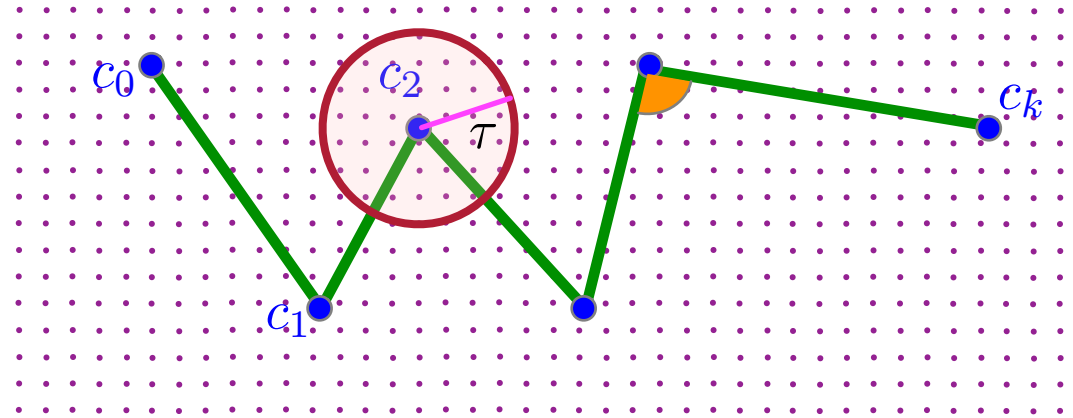
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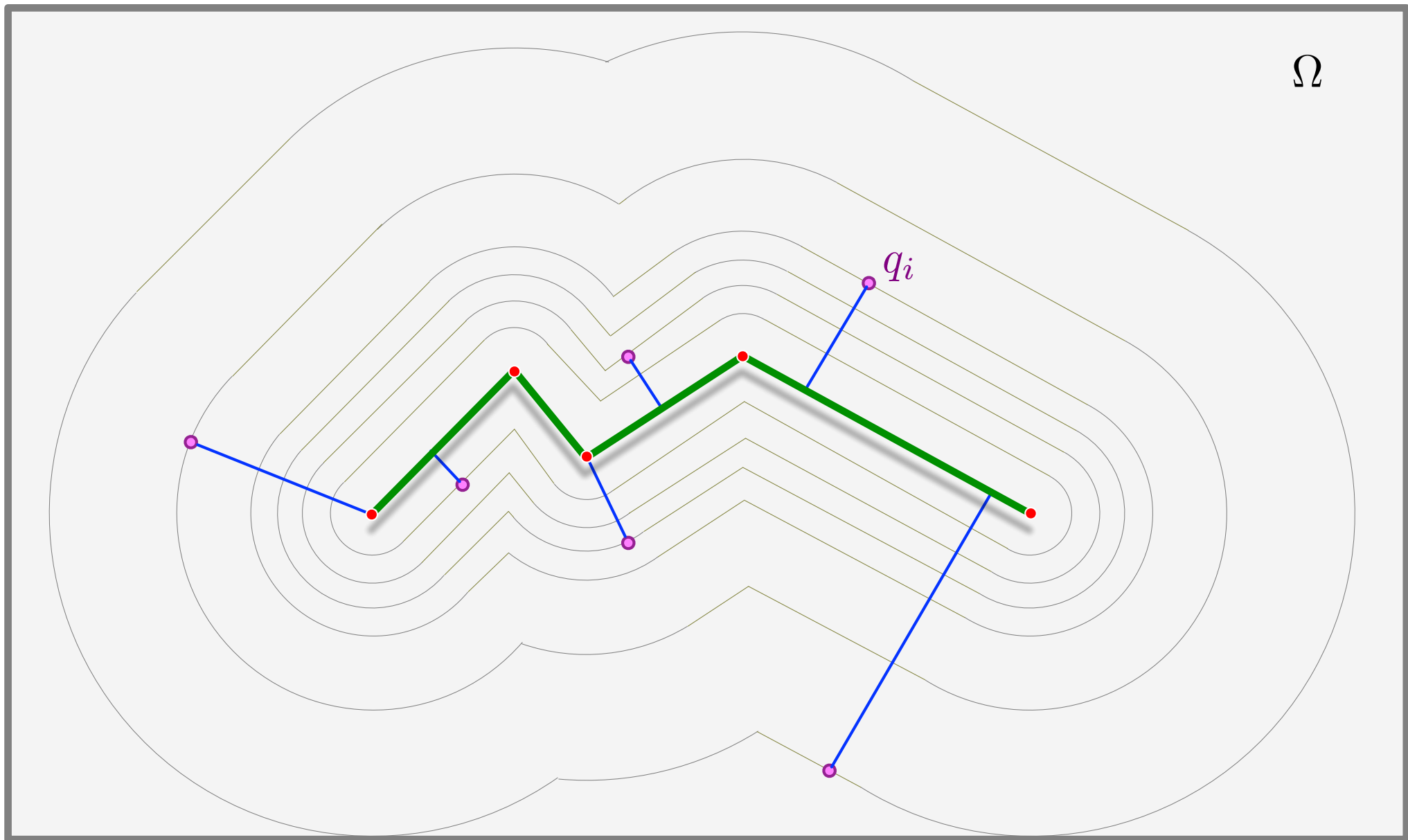

For classification
 —> 20

How many landmarks?

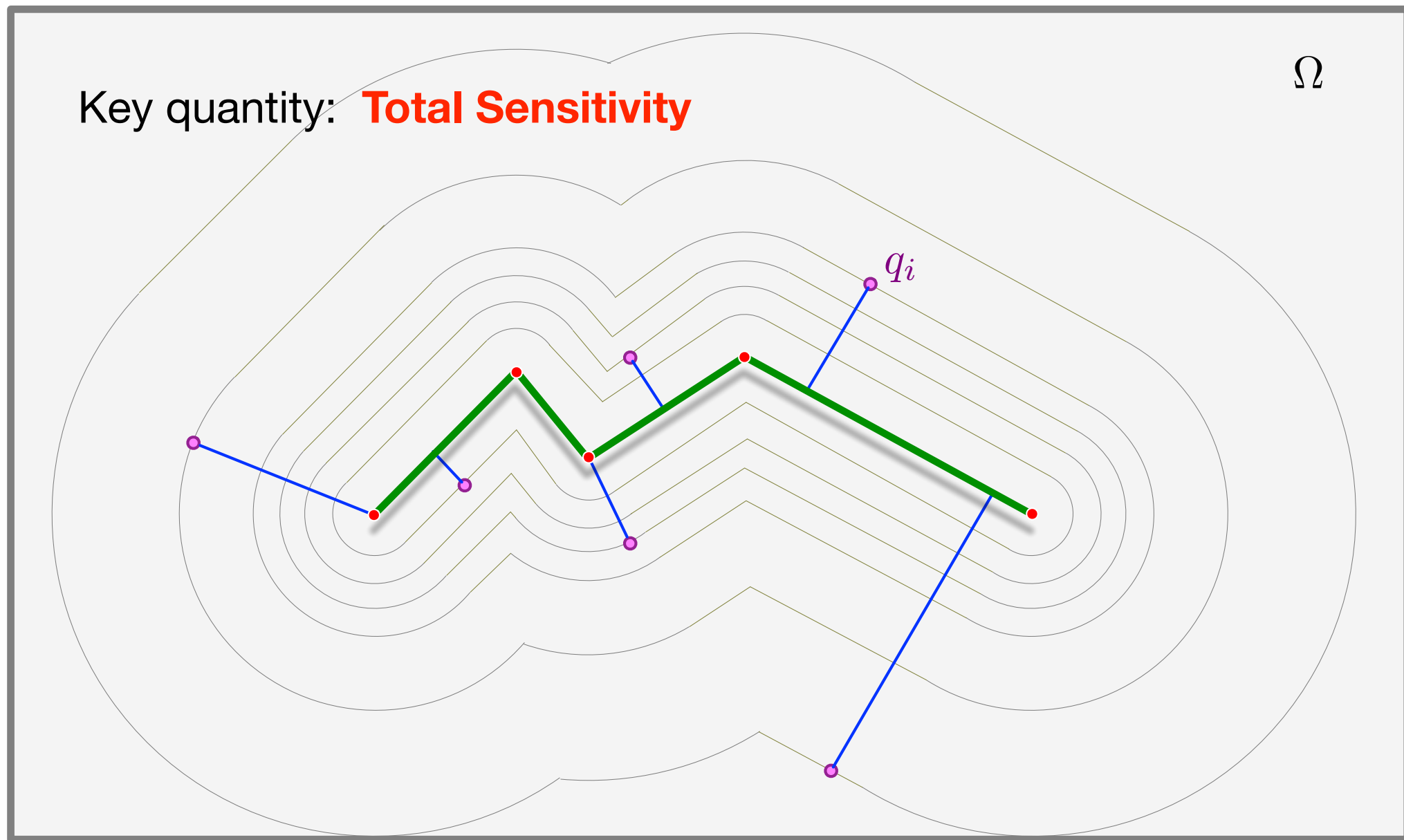

q_i

How many landmarks?

Choose Q, for $J, J' \in \Omega$ so: $(1 - \varepsilon)d_Q(J, J') \leq d_\Omega(J, J') \leq (1 + \varepsilon)d_Q(J, J')$
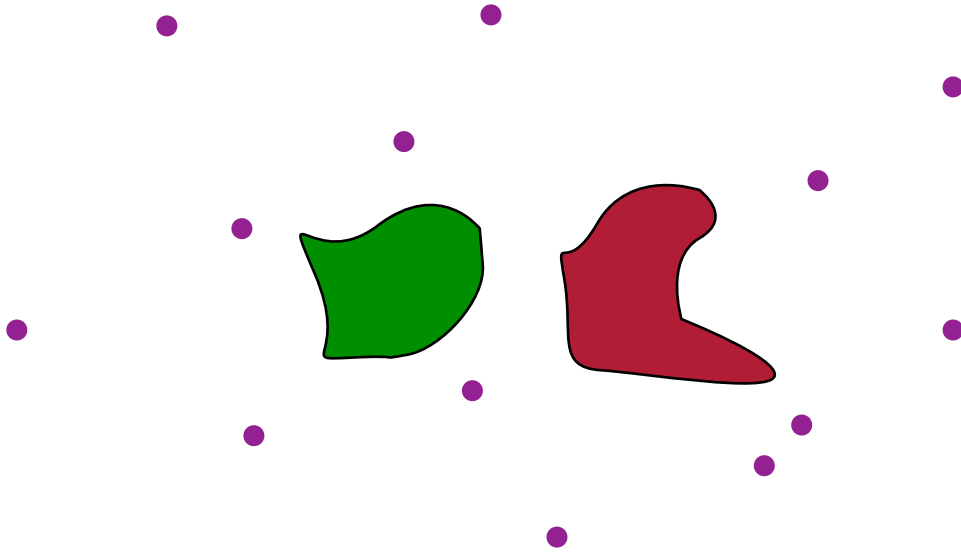
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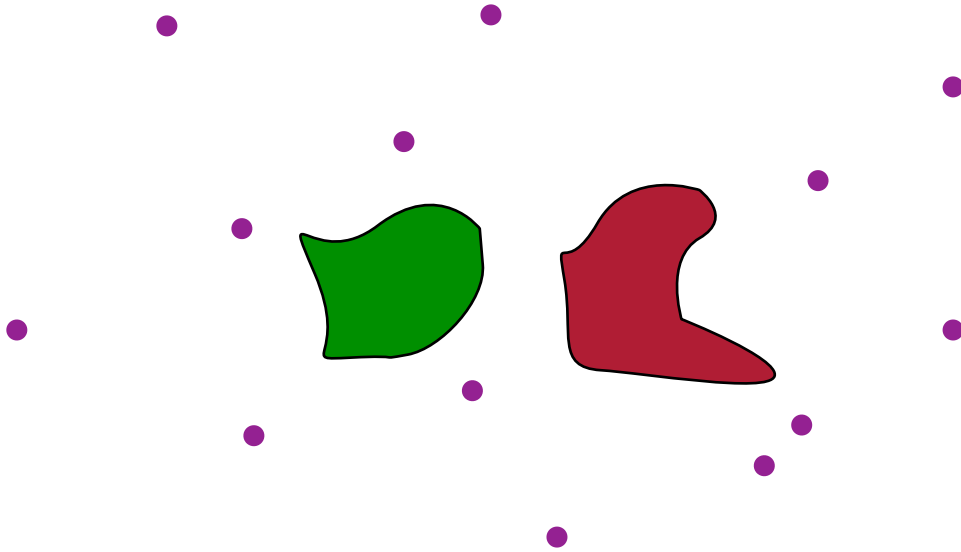*Sensitivity sampling* Feldman-Schulman-Langberg (2010,2011)

Key quantity: **Total Sensitivity**

Ω

q_i

Bounded Closed Sets

When shapes J, J' are more general, total sensitivity of Q may be unbounded.
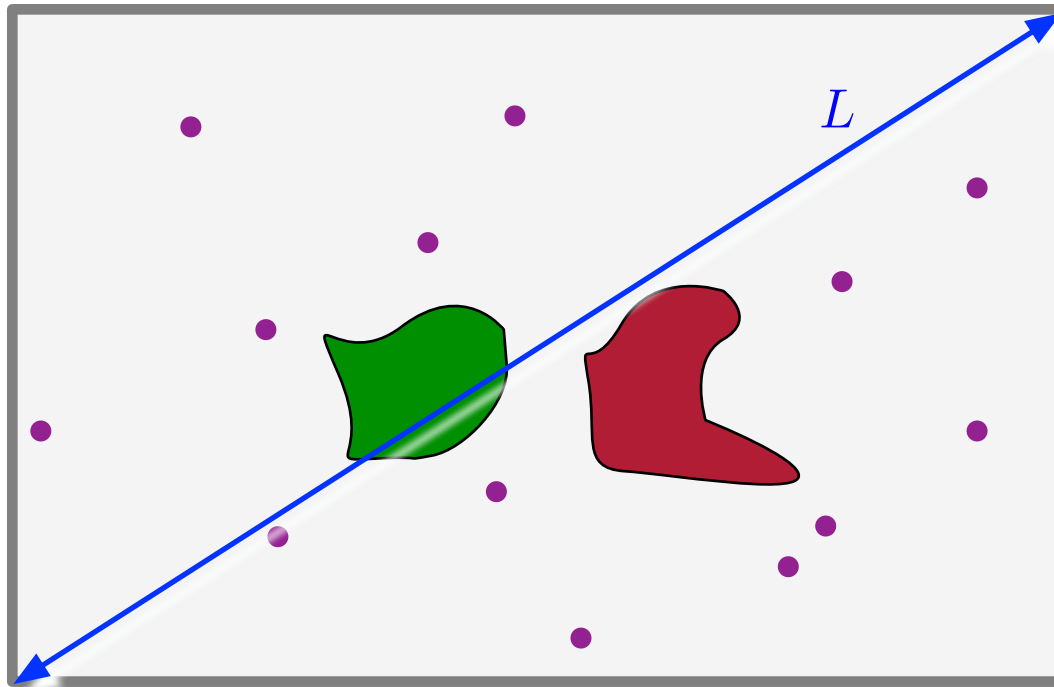
Bounded Closed Sets

When shapes J, J' are more general, total sensitivity of Q may be unbounded. Need $\frac{\text{Total Sensitivity}}{\varepsilon^2}$ samples.
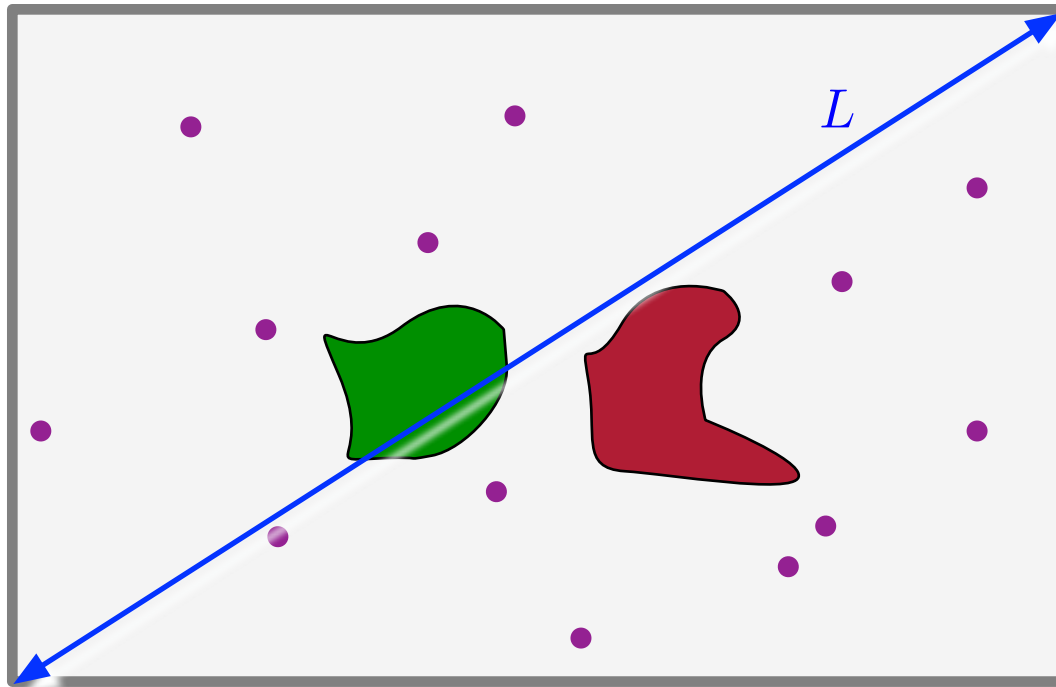
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$$d_Q(J, J') > \rho$$

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$L/\rho \leq$ Total sensitivity $\leq (L/\rho)^2$

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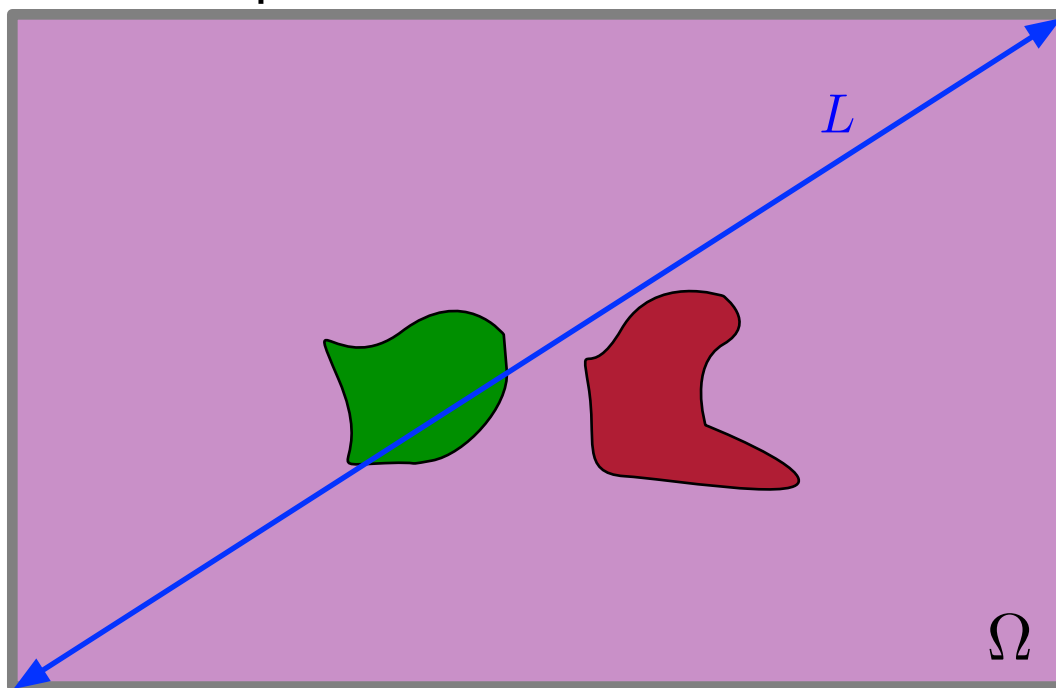

$d_Q(J, J') > \rho$

$L/\rho \leq$ Total sensitivity $\leq (L/\rho)^2$
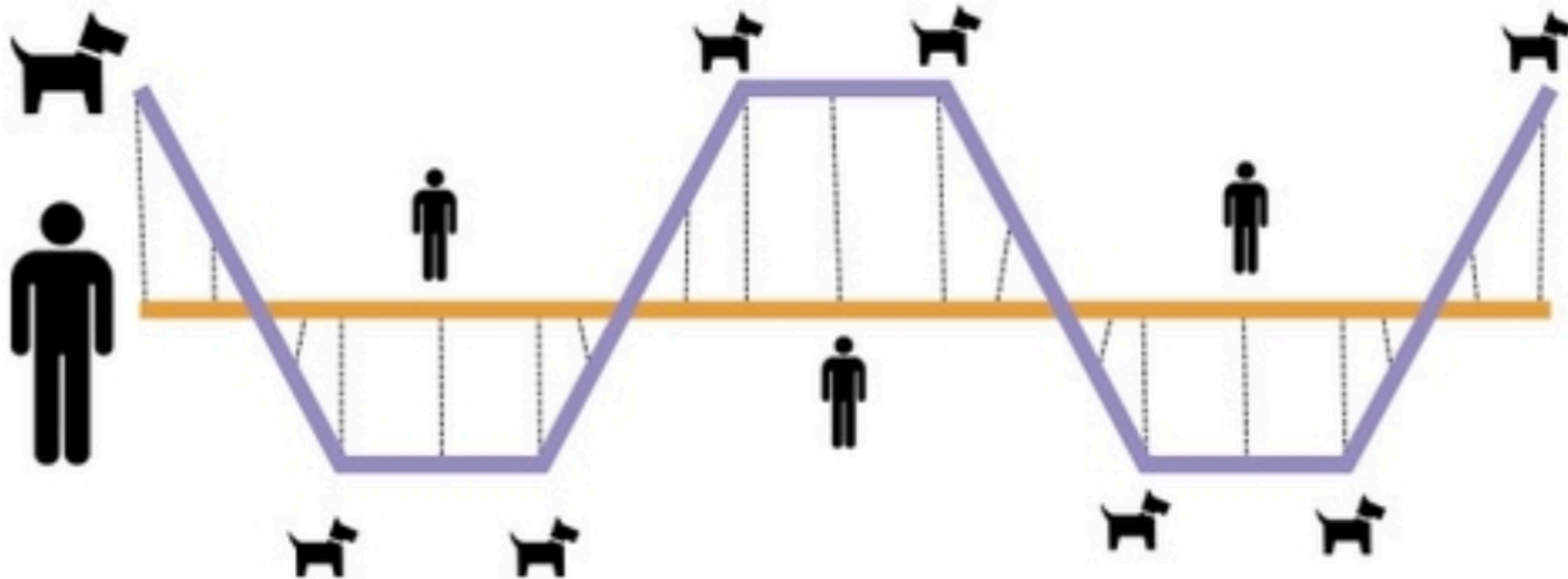
Total Sensitivity $= O(L/\rho)$ (in $d = 2$, optimal)

(general d: $(L/\rho)^{\frac{2d}{2+d}}$))
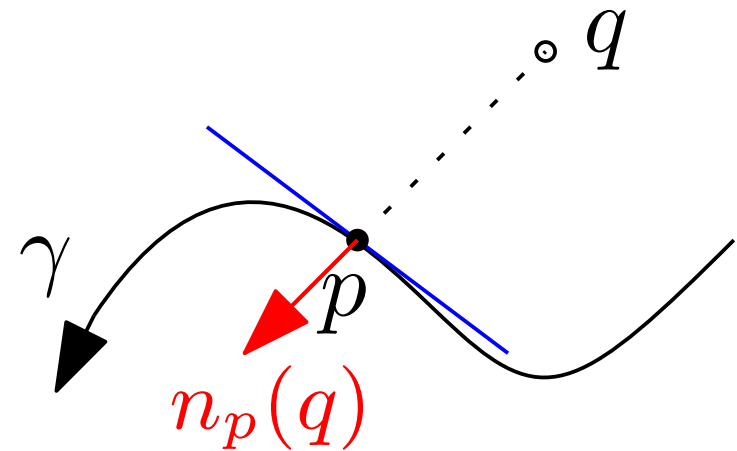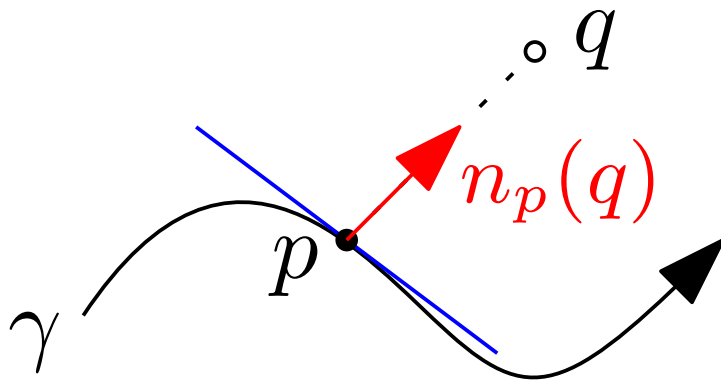
Orientation Preserving

Is a bus going to or from main station?
Is a bird flying to or from a lake?

Main common distances: Frechet, DTW, …
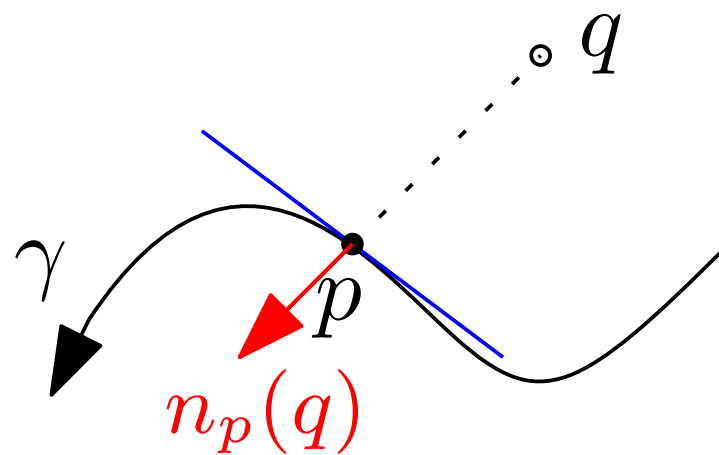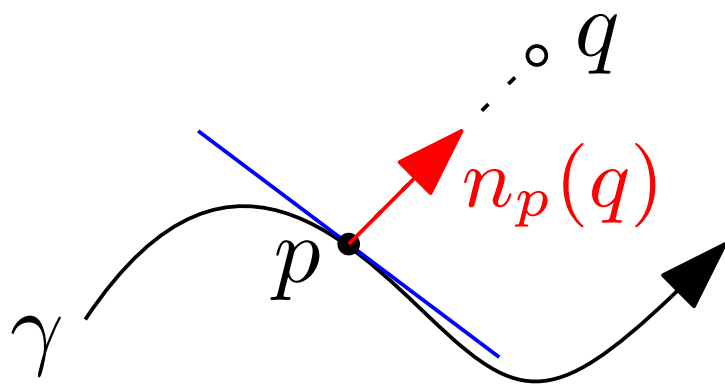
Orientation Preserving

$$p = \operatorname{argmin}_{p' \in \gamma} \|q - p'\|$$

Our MinDist distance dQ does not capture orientation!

Orientation Preserving

Let $q \in \mathbb{R}^2$ and $\sigma > 0$. For curve γ set $p = \arg\min_{p' \in \gamma} \|q - p'\|$

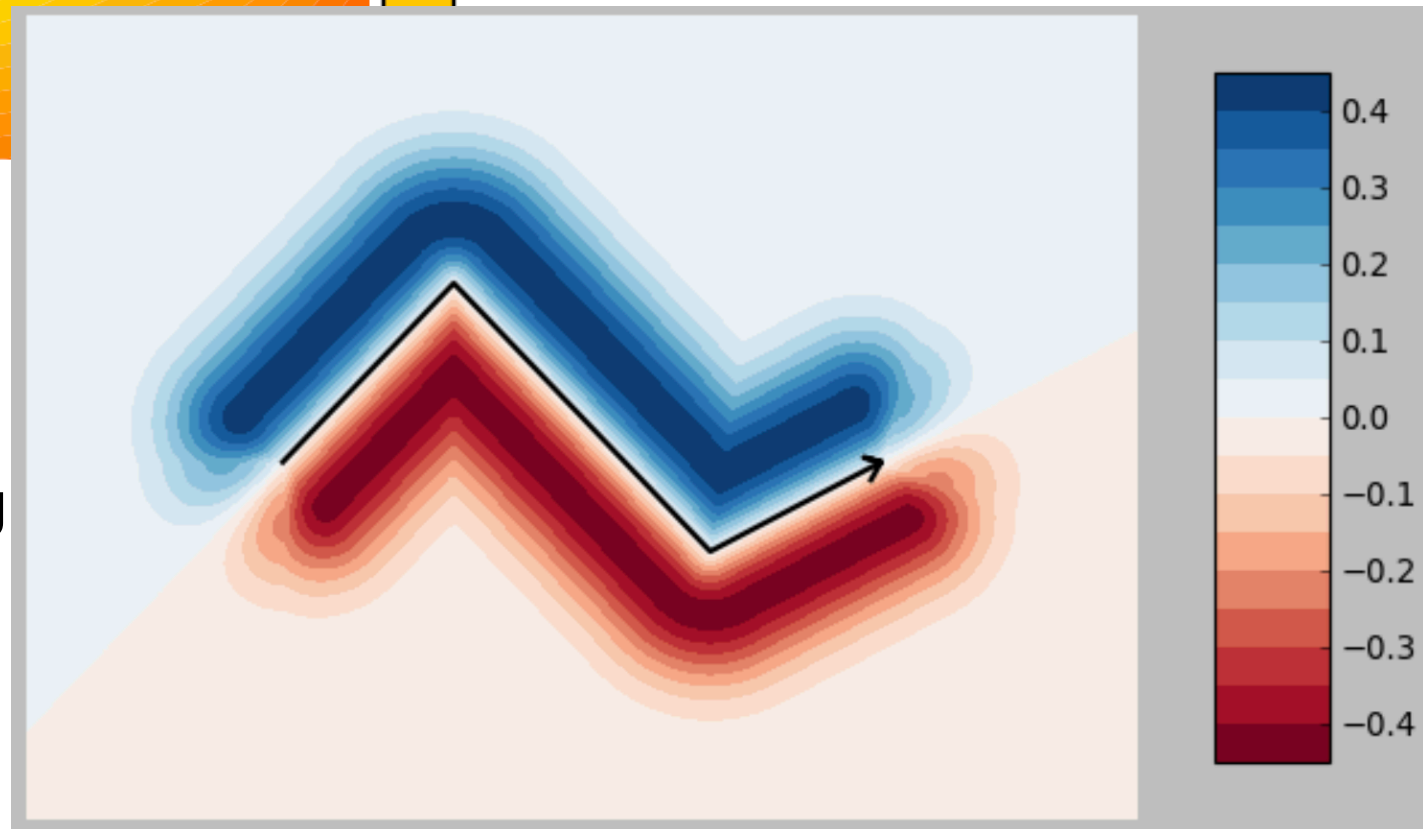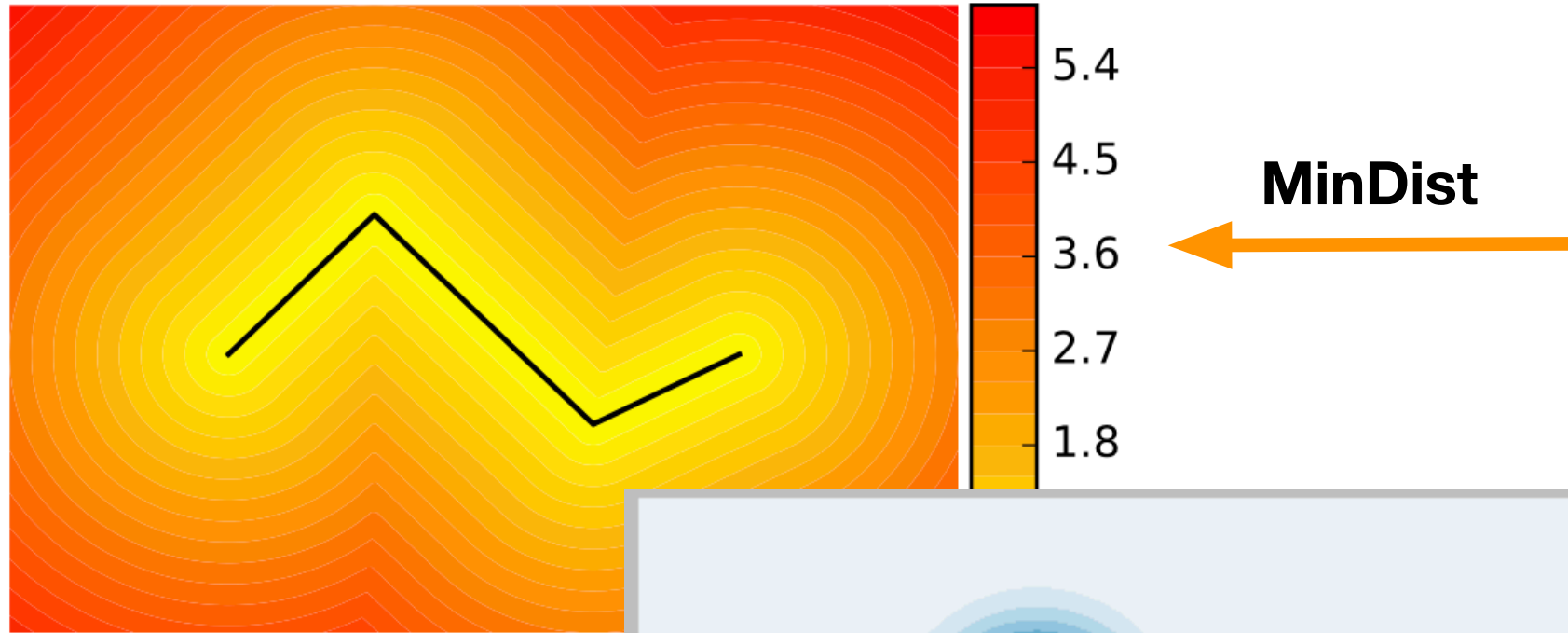$$v_q^\sigma = \langle n_p(q), q - p \rangle \frac{1}{\sigma} e^{-\frac{\|p-q\|^2}{\sigma^2}}$$

$$p = \operatorname{argmin}_{p' \in \gamma} \|q - p'\|$$

Vectorization. $v_Q^\sigma(\gamma) = (v_{q_1}^\sigma(\gamma), \ldots, v_{q_1}^\sigma(\gamma))$

Distance. $d_Q^\sigma(\gamma, \gamma') = \frac{1}{\sqrt{n}} \|v_Q^\sigma(\gamma) - v_Q^\gamma(\gamma')\|$
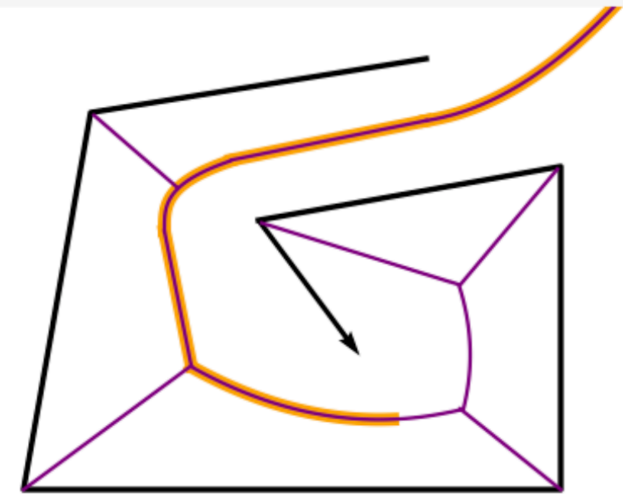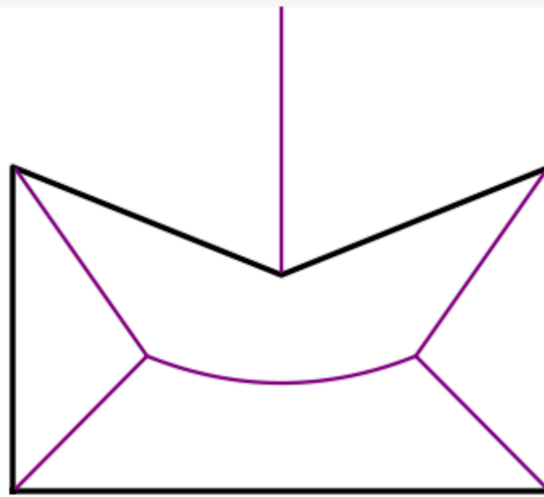
Orientation Preserving

MinDist
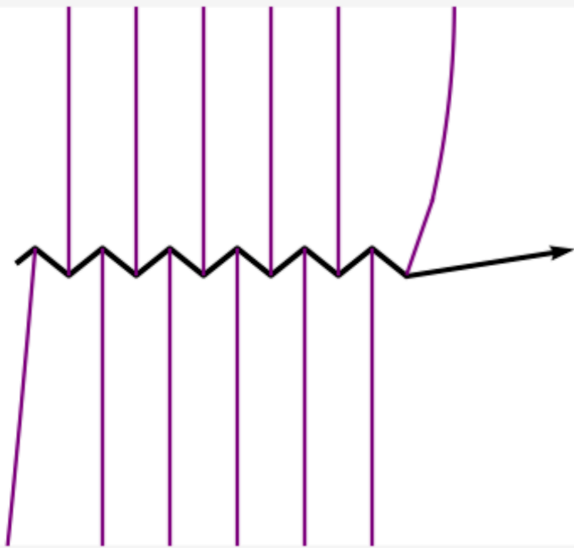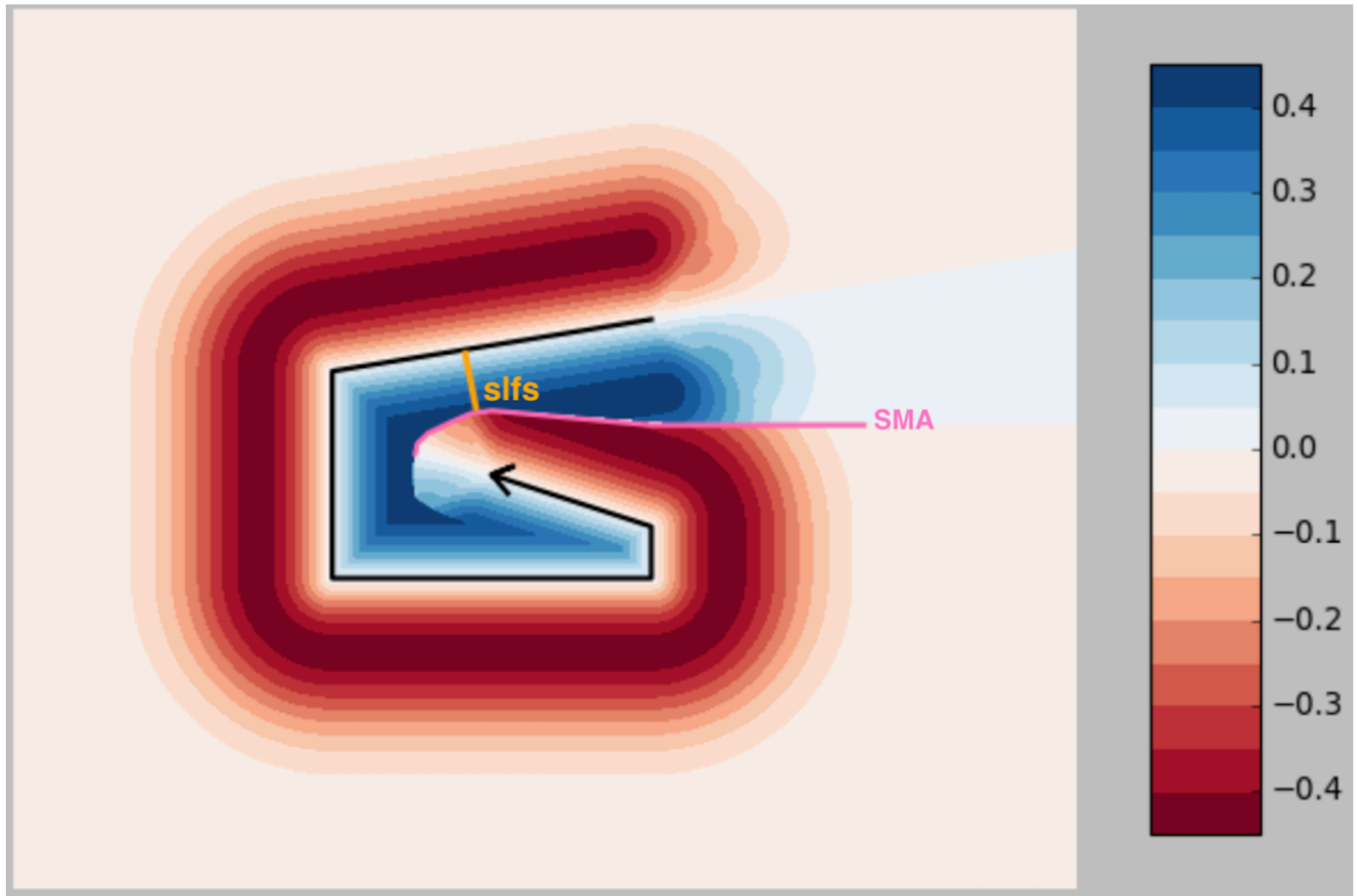
Orientation Preserving

Signed Medial Axis

The **medial axis** is the set of points where the nearest point on the curve is not unique.

The **signed medial axis** (SMA) is the subset of the medial axis where the nearest curve points differ in orientation.

Signed Local Feature Size

The **signed local feature size** (slfs) is the point on the SMA with minimum distance to the curve.
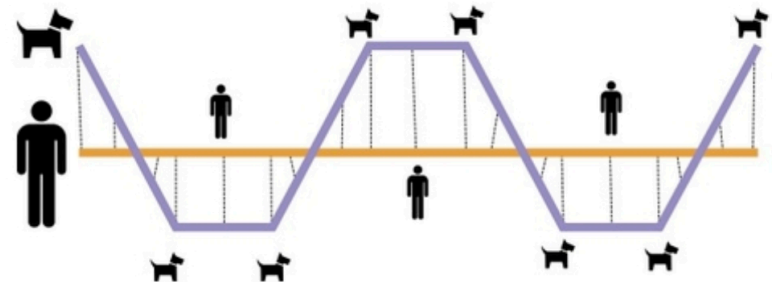
Stability

Landmark Stability

Under some conditions on q and q' (do not cross SMA)

$$|v_q^\sigma(\gamma) - v_{q'}^\sigma(\gamma)| \le \tfrac{1}{\sigma}\|q - q'\|$$

Curve Stability

Under some conditions place of q_is

$$d_Q^\sigma(\gamma, \gamma') \le \tfrac{1}{\sigma} d_{\mathsf{Frechet}}(\gamma, \gamma')$$

If Q is dense on $\Omega \subset \mathbb{R}^2$

$$d_{Q,\infty}(\gamma, \gamma') = d_{\mathsf{Hausdorff}}(\gamma, \gamma')$$
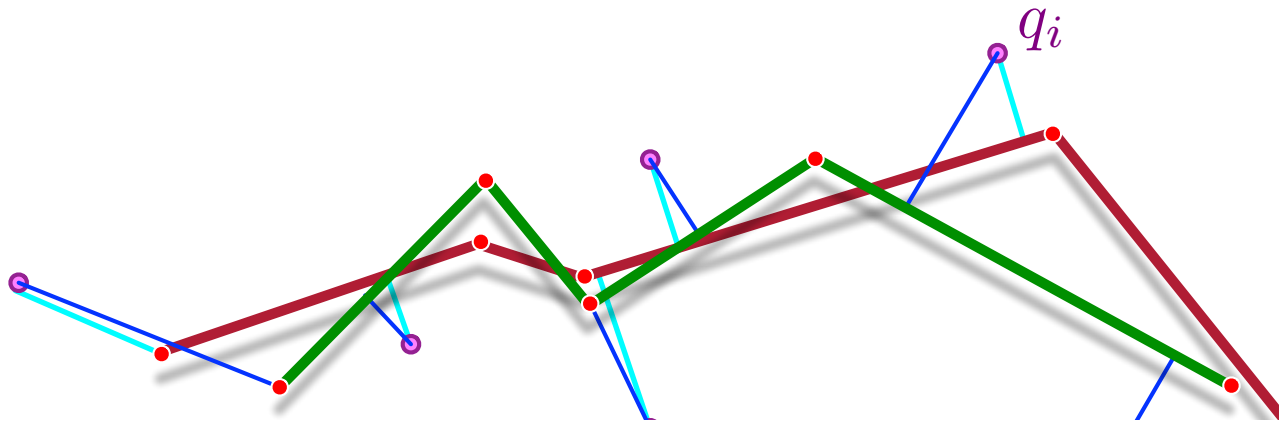
MinDist Sketch

shape J

landmarks Q

minDist
$$v_i(J) = \inf_{p \in J} \|q_i - p\|$$

$$v(J) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \dots \\ v_n \end{bmatrix}$$

$$v(J') = \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \\ \dots \\ v'_n \end{bmatrix}$$

q_i

OK, so is this a good distance?

$$d_Q(J, J') = \|v(J) - v(J')\|$$

MinDist Sketch

shape J

landmarks Q

minDist
$v_i(J) = \inf_{p \in J} \|$
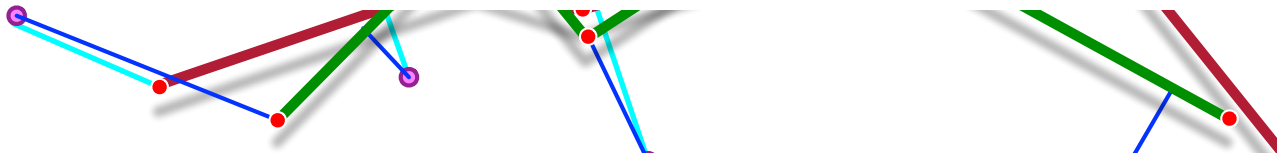
1. **Easy to use**

2. **Fast to compute (NN search)**

3. **Classifies well (good modeling)**

4. **Sketches well (small Q)**

5. **Stable wrt Q and curve perturbations**

$$(J) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \dots \\ v_n \end{bmatrix}$$

$$(J') = \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \\ \dots \\ v'_n \end{bmatrix}$$

OK, so is this a good distance?

$$d_Q(J, J') = \|v(J) - v(J')\|$$

Thanks & Next Steps

minDist

$$v_i(J) = \inf_{p \in J} \|q_i - p\|$$

$$v(J) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \dots \\ v_n \end{bmatrix}$$

1. **Better classifiers?**

2. **Rotation / shift invariant (shape)**

q_i

3. **Apply to higher-dimensional objects**