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by

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Abstract

A renormalized perturbative expansion of interacting quantum fields on a globally hyperbolic spacetime is performed by adapting of the Bogoliubov Epstein Glaser method to a curved background. The results heavily rely on techniques from microlocal analysis, in particular on Radzikowski's characterization of Hadamard states by wave front sets of Wightman functions.

Quantum field theory on a curved background is supposed to describe correctly the influence of a gravitational field on quantum fields as long as the relevant length scales are much larger than the Planck length. The modifications compared to quantum field theory on Minkowski space are mainly due to some nonlocal features in the standard formulation of quantum field theory. The most important ones are the spectrum condition and the existence of a vacuum, properties expressing stability of the system and, on a more technical side, making possible the transition to a euclidean formulation. Even for free fields the changes are by no means trivial, leading to new phenomena as e.g. the Hawking radiation of black holes. It seems now to be clear that the notion of a vacuum has to be replaced by the choice of a class of admissable states [1] which was identified to be the class of Hadamard states [2, 3, 4]. Accordingly, also the Feynman propagator is no longer unique, but its singular part is fixed and has essentially the same form as on Minkowski space. Therefore the ultraviolet divergences of perturbation theory should be treatable and should be similar to those in Minkowski space. This expectation fits with the results

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on renormalization of euclidean theories on Riemannian spaces [5, 6]. It is not obvious, however, how these results can be used for renormalization on lorentzian manifolds.

For a rigorous treatment one needs an effective mathematical framework. It was first observed by Radzikowski [7] that microlocal analysis, in particular the notions of wave front sets and Fourier Integral Operators [8, 9] are ideally suited to deal with the singularities of propagators on curved spacetimes.

We choose the Bogoliubov-Epstein-Glaser method [10] for the construction of interacting fields. This method can be formulated locally and amounts to the extension of time ordered n-point distributions to coinciding points, a problem which can nicely be treated by the techniques of microlocal analysis. But an important technical ingredient in the Epstein-Glaser method is translation invariance, therefore one of the main problem which had to be solved was a generalization of this method to a situation where invariance is replaced by a suitable smoothness condition. A similar problem occurs in the presence of external fields [11].

Let us start from a free neutral scalar field φ which satisfies the Klein Gordon equation on a globally hyperbolic spacetime $\mathcal{M}$ with a metric tensor g. The commutation relations are

$$[\varphi(x), \varphi(y)] = 3Di\Delta(x, y)$$

where Δ is the difference between the (unique) retarded and advanced Green function of the Klein Gordon operator. According to Radzikowski [7] a Hadamard state ω on the algebra generated by φ is a quasifree state whose 2-point function is a bisolution of the Klein-Gordon equation with antisymmetric part $i\Delta$ and whose wave front set is the positive frequency part of the wave front set of Δ. It was shown in [12] that as an immediate consequence Wick polynomials can be defined as operator valued distributions on a dense invariant subspace of the GNS Hilbert space of ω.

The time ordered products of Wick polynomials are well defined as symmetric operator valued distributions on testfunctions f with support on noncoinciding points,

$$\mathrm{supp} f \subset \{(x_1, \ldots, x_n), x_i \in \mathcal{M}, x_i \neq x_j, i \neq j\}.$$

They have an expansion into sums of pointwise products of numerical distributions and multiple Wick products. The numerical distributions are time ordered functions, i.e. expectation values of time ordered products of sub-Wick polynomials. The problem amounts now to the extension of these time ordered functions to coinciding points such that the pointwise products remain well defined.

The first step is to guess the wave front set of these extensions. We found a condition on these wave front sets which allows to build the mentioned pointwise products. One then makes an inductive construction of time ordered functions. Finally, one defines interacting fields as formal power series of operator valued distributions.

The wave front set of a distribution on a manifold is the subset of the cotangent bundle which characterizes for every point of the manifold the directions of nonrapid decrease of a local Fourier transform,

$$\mathrm{WF}(f) = 3D\{(x, k) \in \mathrm{T}^*\mathcal{M}, k \neq 0,$$
$$\widehat{\psi f} \text{ not fast decreasing near } k \ \forall \psi \in \mathcal{D}(\mathcal{M}) \text{ with } \psi(x) \neq 0\}.$$

The Fourier transform is defined within a suitable chart at x, but the wave front set is independent of the choice of the chart.

For the Feynman propagator

$$\Delta_F^\omega(x,y) = 3D\omega(T(\varphi(x)\varphi(y))$$

associated to a Hadamard state ω it is

$$\mathrm{WF}\Delta_F^\omega = 3D\{(x,x';k,k') \in \mathrm{T}^*\mathcal{M}^2 \setminus \{0\}, (x,k) \sim (x',-k')\},$$

where $(x,k) \sim (x',k')$ means that either $x = 3Dx'$ and $k = 3Dk'$ or there is a null geodesic from x to x', k is coparallel to the tangent vector of the geodesic at x, k' is the parallel transport of k along the geodesic and $k \in V_\pm$ if the geodesic is future (past) directed.

The first problem is to find a suitable condition on the wave front set of time ordered functions. We are looking for an expansion of the form

$$T :\varphi(x_1)^{\alpha_1} : \cdots :\varphi(x_n)^{\alpha_n} := 3D \sum_{\beta \leq \alpha} t^\alpha_\beta(x_1,\ldots,x_n) :\varphi(x_1)^{\beta_1} \cdots \varphi(x_n)^{\beta_n} :$$

with

$$t^\alpha_\beta(x_1,\ldots,x_n) = 3D\binom{\alpha}{\beta}\omega(T :\varphi(x_1)^{\alpha_1-\beta_1} : \cdots :\varphi(x_n)^{\alpha_n-\beta_n} :),$$

α, β multiindices, where the time ordered products are symmetric under permutations of $\{1,\ldots,n\}$ and coincide with the operator product if the points $x_1,\ldots,x_n$ can be time ordered,

$$x_i \notin \mathcal{J}_-(x_{i+1}), i = 3D1,\ldots,n-1,$$

$\mathcal{J}_-(x)$ denoting the causal past of x. Since the Wick products are local and relatively local, this condition is compatible with the symmetry requirement. The two conditions mentioned above fix the time ordered products on a globally hyperbolic manifold uniquely on the set of noncoinciding points. The wave front set of the time ordered functions can be determined from the wave front set of the Feynman propagator and turns out to be contained in the set

$$\begin{array}{rl} \Gamma_n^{\mathrm{to}} = 3D & \{(x_1,\ldots,x_n;k_1,\ldots,k_n) \in \mathrm{T}^*\mathcal{M}^n \setminus \{0\}; \text{there is a graph } G \\ & \text{with vertices } \{1,\ldots,n\}, \text{an association of lines } l \text{ of the graph to} \\ & \text{future directed lightlike geodesics } \gamma_l \text{ from } x_{s(l)} \text{ to } x_{r(l)} \\ & \text{with covariantly constant coparallel covector fields } k_l \text{on } \gamma_l \\ & \text{with values in the closed forward lightcone such that} \\ & k_i = 3D\sum_{s(l)=3Di} k_l(x_i) - \sum_{r(l)=3Di} k_l(x_i) \quad \} \end{array}$$

We now start an inductive construction of the time ordered products by postulating that also the wave front set of the extended time ordered functions is contained in Γ_n^{to}. Note that since coinciding points can trivially be connected by future oriented lightlike geodesics, one gets a restriction only on the sum of covectors at coinciding points. This may be considered as a local version of translation invariance.

The above condition on the wave front set leads to the following version of Epstein-Glaser's Theorem 0:

Theorem 1 *Let $t \in \mathcal{D}'(\mathcal{M}^n)$ with $WF(t) \subset \Gamma_n^{to}$. Then the pointwise products*

$$t(x_1, \ldots, x_n) :\varphi(x_1)^{\beta_1} \cdots \varphi(x_n)^{\beta_n}:$$

are well defined operator valued distributions with an invariant domain of definition.

We now assume that all time ordered products of $n' < n$ factors have been constructed and satisfy the requirements of symmetry and unitarity, admit the expansion above with $\mathrm{WF}(t) \subset \Gamma_{n'}^{\mathrm{to}}$ and fulfil the factorization property

$$T \prod_{i=3D1}^{n'} :\varphi(x_i)^{k_i}:= 3DT \prod_{i=3D1}^{l} :\varphi(x_i)^{k_i}: T \prod_{i=3Dl+1}^{n'} :\varphi(x_i)^{k_i}:$$

if $x_i \notin \mathcal{J}_-(x_j)$ for $i = 3D1, \ldots, l, \quad j = 3Dl+1, \ldots, n'$.

Then on $\mathcal{M}^n \setminus \mathrm{D}_n$, $\mathrm{D_n} = 3D\{(x, \ldots, x) \in \mathcal{M}^n, x \in \mathcal{M}\}$ denoting the total diagonal, the time ordered products are uniquely defined and satisfy the previous conditions.

It remains to extend the time ordered functions to the diagonal. On Minkowski space one uses translation invariance to eliminate one coordinate so that one needs to extend the distribution only to a single point. The singularity at this point (taken to be the origin) can conveniently be desribed in terms of Steinmann's scaling degree [13]

$$\mathrm{sd}(t) 3D \mathrm{inf}\{\delta, \lambda^\delta t(\lambda\cdot) \to 0, \lambda \to 0\},$$

and one obtains the possible extensions with the same scaling degree by decomposing the space of test functions into a direct sum of the space of functions which vanish at the origin with order $\mathrm{sd}(t) - 4(n-1)$ and a complementary (finite dimensional) subspace: on the first summand there is a unique extension with the same scaling degree whereas on the second summand one may choose an arbitrary linear functional. The singularity degree $\mathrm{sd}(t) - 4(n-1)$ coincides with the degree of divergence obtained by the usual power counting rules.

On a curved spacetime one may introduce near the diagonal center of mass and relative coordinates in terms of the exponential function,

$$(x_1, \ldots, x_n) = 3D(\exp_x \xi_1, \ldots, \exp_x \xi_n), \quad \sum xi_i = 3D0, \xi_i \in \mathrm{T}_x\mathcal{M},$$

and perform the extension only with respect to the relative coordinates. But the nontrivial dependence on the center of mass coordinate requires a careful treatment for which techniques from microlocal analysis turn out to be useful; one can introduce the concept of a scaling degree relative to a submanifold (here the diagonal) whose tangent bundle is orthogonal to the wave front set, and one finds that the inductive computation of this scaling degree gives the same results as for the scaling degree in the translationally invariant situation. For details see [15].

After the construction of time ordered products of Wick polynomials one can define interacting fields in the sense of formal power series by Bogoliubov's formula

$$\varphi_g^k(x) = 3D\frac{\delta}{\delta h_k(x)} S(g)^{-1} S(g+h)_{|h=3D0},$$

where the “S-matrix” $S(g)$ is defined by

$$S(g) = 3D \sum_{n=3D0}^{\infty} \frac{i^n}{n!} \sum_{k_1,\ldots,k_n} \int dx_1 \cdots dx_n T \prod_{j=3D1}^{n} :\varphi(x_j)^{k_j}: g_{k_j}(x_j)$$

with test functions g_j. These fields are local and satisfy field equations. They depend only on the values of g in the past. Moreover, within a fixed causally closed region $\mathcal{O}$ their dependence on the values of g outside of $\mathcal{O}$ is described by a unitary transformation. Hence the structure of the local algebras of observables generated by the interacting fields is independent of the values of the couplings g outside of the region one is interested in; one therefore obtains a purely local construction (in the sense of formal power series) of the Haag-Kastler net of the interacting theory. On a curved space time, this enables us to ignore space time singularities outside the region we are interested in, but also on Minkowski space this local construction may be useful for separating the infrared problems from the ultraviolet problems.

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