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# Asymptotically Euclidean manifolds and twistor spinors

by

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# Asymptotically Euclidean Manifolds and Twistor Spinors

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**Abstract**

Based on the Berger–Simons holonomy classification, we classify all Riemannian spin manifolds carrying a twistor spinor with at least one zero. In particular, the dimension $n$ of the manifold is either even or $n = 7$. The metric is conformal to either a flat metric or a Ricci flat and locally irreducible metric.

## 1 Introduction and Result

On a Riemannian manifold with spin structure there lives the Dirac operator and also another natural differential operator, the *twistor operator.* It is also called *Penrose operator* since it was first introduced in General Relativity by R.Penrose. The kernel of this operator is formed by the *twistor spinors* which satisfy the *twistor equation*

$$\nabla_X \phi + \frac{1}{n} X \cdot D\phi = 0 \tag{1}$$

for every vector field $X$. Here $\nabla_X \phi$ is the spinor derivative of the spinor field $\phi$ in direction of the vector field $X$. $D$ denotes the *Dirac operator*, the dot $\cdot$ denotes the Clifford multiplication and $n = \dim M$ is the dimension of the manifold. One can regard a twistor spinor on $(M, g)$ as a Killing vector field on a canonically associated supermanifold, cf. [ACDS].

Particular cases of twistor spinors are *parallel spinors* which satisfy $\nabla\phi = 0$ and *real Killing spinors* which satisfy the equation $\nabla_X \phi = \lambda X \cdot \phi$ for all $X$ and some real number $\lambda \neq 0$. A manifold with a parallel spinor is Ricci flat and a manifold carrying a real Killing spinor is Einstein with positive scalar curvature $4\lambda^2(n-1)n$. It was remarked by Hitchin [Hi] that a simply–connected and irreducible complete spin manifold of dimension $n$ with a parallel spinor has special holonomy.

In particular one concludes from the Berger–Simons theorem on holonomy: If the dimension $n > 8$ then either the manifold has holonomy $\mathbb{SU}(m)$ with $n = 2m$ and carries a two–dimensional space of parallel spinors or the manifold is hyperkähler, i.e. has holonomy $\mathrm{Sp}(m)$ where $n = 4m$ and carries an $(m+1)-$dimensional space of parallel spinors, cf. [Wa]. According to Bär [Ba] one can reduce the case of manifolds carrying real Killing spinors to the case of manifolds carrying parallel spinors as follows: If a compact $(M, g)$ carries a real Killing spinor then the cone $(\mathbb{R}^+ \times_r M, dr^2 + r^2 g)$ over $M$ is either flat or irreducible and carries a parallel spinor.

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If $u = \langle \phi, \phi \rangle$ is the *length* of the twistor spinor $\phi$ then the function $A_\phi = 4u\|D\phi\|^2 - n^2\|\nabla u\|^2$ is a non–negative constant. Moreover, the conformally equivalent metric $\overline{g} = u^{-2}g$ is an Einstein metric with non–negative scalar curvature $s = (n-1)A_\phi/n$. The last statement holds everywhere except at the zero set of $\phi$. If $A_\phi > 0$ then the metric $\overline{g}$ carries a real Killing spinor, if $A_\phi = 0$ then the metric $\overline{g}$ carries a parallel spinor, cf. [Fr, 5.3]. Hence the case of a manifold with a twistor spinor without zero can be reduced by a conformal change to the case of a manifold carrying a parallel spinor. In the case of a manifold carrying a twistor spinor with zero things are different. One first observes that the set of zeros of a twistor spinor is discrete. Although there are compact manifolds carrying parallel spinors respectively real Killing spinors which are not conformally flat we have the following result. The proof uses the solution of the Yamabe problem, i.e., the existence of a metric of constant scalar curvature in the conformal class:

**Theorem 1.1** (Lichnerowicz [Li, Thm.7]) *A compact Riemannian spin manifold carrying a non–trivial twistor spinor with zero is conformally equivalent to the standard sphere.*

In [KR1] we showed that a manifold carrying a twistor spinor $\phi$ with zero is conformally flat if the associated conformal vector field $V_\phi$ defined by $\langle V_\phi, X \rangle = \sqrt{-1}\langle \phi, X \cdot \phi \rangle$ is non–trivial, under additional curvature and completness assumptions this was shown before by K.Habermann [Ha]. The authors gave in [KR2] resp. [KR3] complete Riemannian metrics $g$ on $\mathbb{R}^4$ resp. $\mathbb{R}^{2m}$ carrying a two–dimensional space of twistor spinors with a common zero point which are not conformally flat. If a 4–dimensional manifold carries a twistor spinor then the manifold is *half–conformally flat.* The conformally equivalent metric Ricci flat metric $\overline{g}$ is in these examples *asymptotically euclidean.* In dimension 4 it is the *Eguchi–Hanson metric* and in dimension $2m$ higher dimensional analogues which can be used to define complete metrics on on a line bundle over $\mathbb{C}P^n$ with holonomy $\mathbb{SU}(m)$ due to Calabi [Ca], cf. also [FG].

Our main result is the following:

**Theorem 1.2** *Let $(M, g)$ be an $n$–dimensional Riemannian spin manifold carrying a twistor spinor with non–empty set $Z_\phi$ of zeros. Then the following hold:*

*(a) The inverted normal coordinates around every $p \in Z_\phi$ (as defined in section 2) give an asymptotically euclidean coordinate system of order 3 for the metric $\overline{g} = \|\phi\|^{-4} g$ outside $p$.*

*(b) The metric $\overline{g}$ is flat or Ricci flat and locally irreducible.*

Using recent results by M.Herzlich [He] one obtains a partial version of a converse statement: Let $(M, g)$ be a Riemannian spin manifold which carries a parallel spinor and has an end with an asymptotically euclidean coordinate system of order 2. Then by adding a point $q$ we can conformally compactify this end and obtain a $C^2$–metric on $M \cup \{q\}$ with a twistor spinor having a zero at $q$.
Using the Berger–Simons theorem on holonomy as in [Hi, p.8, footnote p.54] and [Wa, Prop.], we conclude from our main result the following classification theorem:

**Theorem 1.3** *Let $(M,g)$ be a Riemannian spin manifold carrying a twistor spinor $\phi$ with non–empty set $\mathrm{Z}_\phi$ of zeros. Then the conformally equivalent and Ricci flat metric $\overline{g} = \|\phi\|^{-4} g$ on $\overline{M} = M - \mathrm{Z}_\phi$ carries a parallel spinor and is either flat or locally irreducible.*
*If $(M,g)$ is not conformally flat the following holds: Denote by $N$ the dimension of the space of twistor spinors on $(M,g)$. Then for all twistor spinors on $(M,g)$ the set of zeros equals $\mathrm{Z}_\phi$ and for the restricted holonomy $\mathrm{Hol}^0$ of the conformally equivalent and Ricci flat metric $(\overline{M}, \overline{g})$ one of the following holds:*

*a)* $n = 2m, m \geq 2, \mathrm{Hol}^0 = \mathbb{SU}(m)$ *and* $N = 2$.

*b)* $n = 4m, m \geq 2, \mathrm{Hol}^0 = \mathrm{Sp}(m)$ *and* $N = m+1$.

*c)* $n = 8, \mathrm{Hol}^0 = \mathrm{Spin}(7)$ *and* $N = 1$.

*d)* $n = 7, \mathrm{Hol}^0 = G_2$ *and* $N = 1$.

The results of [KR3] give examples for case a) in all even dimensions.

Another consequence of our result is an alternative proof of Theorem1.1 due to Lichnerowicz in the compact case. Instead of the solution of the Yamabe problem we apply the Cheeger–Gromoll splitting theorem to conclude that there is only one zero and the rigidity part of the Bishop volume comparison theorem to conclude that the manifold with the conformally equivalent metric $\overline{g}$ is actually flat, see Corollary 3.6.

## 2 Inverted normal coordinates

In Riemannian normal coordinates around a point $p$ a Riemannian metric is euclidean up to a term of order 2. By using an *inversion* of the Riemannian normal coordinates and a conformal change which coincides with the inversion in the case of the euclidean space one obtains an asymptotic euclidean metric of order 2. Here one obtains that the metric coefficients are euclidean up to order 2, the first derivatives of the metric coefficients vanish of order 3 and the second derivative vanishes of order 4. We give a precise formulation in the particular case, where the curvature tensor vanishes at $p$, then the orders can be improved.

**Definition 2.1** *Let $U$ be an open subset of a Riemannian manifold. We call a diffeomorphism $y \in \{y \in \mathbb{R}^n \,;\, \|y\|^2 > R\} \mapsto \phi(y) \in U$ an* asymptotically euclidean coordinate system of order $\tau$ *if in the coordinates $(y_1, \ldots, y_n)$ we have*

$$g_{ij} = \delta_{ij} + O(\rho^{-\tau}); \partial_k g_{ij} = O(\rho^{-\tau-1}), \partial_k \partial_l g_{ij} = O(\rho^{-\tau-2})$$

*for $\rho = \|y\| \to \infty$. A shorthand notation for this asymptotic behaviour is*

$$g_{ij} = \delta_{ij} + O''(\rho^{-\tau}).$$

*A Riemannian manifold $N$ is called* asymptotically euclidean *of order $\tau$ if there is a decomposition $N = N_0 \cup N_\infty$ with $N_0$ compact such that $N_\infty$ carries asymptotic coordinates of order $\tau$.*

Let $x = (x_1, \ldots, x_n)$ be *Riemannian normal coordinates* around a point $p \in M$, at which the Riemann curvature tensor $R$ vanishes, i.e. $R(p) = 0$. Then

$$g_{ij}(x) = \delta_{ij} + h_{ij}\,,\ h_{ij} = O(r^3)$$

where $r^2 = \sum_k x_k^2$. Now introduce *inverted normal coordinates* $z = (z_1, \ldots, z_n)$ outside $p$ with $z_i = r^{-2} x_i$. Let $\rho^2 = \sum_k z_k^2$ then $\rho = 1/r$. Then the conformally equivalent metric $\hat{g} = \rho^4 g$ is given in the inverted normal coordinates $z_i$ as follows:

$$\begin{aligned} \hat{g}_{ij}(z) &= \hat{g}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \delta_{ij} + h_{ij} - \frac{2}{\rho^2}\left(z_i \sum_k z_k h_{kj} + z_j \sum_l z_l h_{il}\right) \\ &\quad + \frac{4}{\rho^4} z_i z_j \sum_{k,l} z_k z_l h_{kl}\,. \end{aligned} \tag{2}$$

Then one computes that

$$\hat{g}_{ij}(z) = \delta_{ij} + O''(\rho^{-2})$$

which is the shorthand notation for:

$$\hat{g}_{ij}(z) = \delta_{ij} + O(\rho^{-2}); \frac{\partial}{\partial z_k}\hat{g}_{ij}(z) = O(\rho^{-3}); \frac{\partial^2}{\partial z_k \partial z_l}\hat{g}_{ij}(z) = O(\rho^{-4})$$

Hence we proved the following

**Lemma 2.2** (cf. [LP]) *If the curvature tensor vanishes at p then the inverted normal coordinates are asymptotically euclidean coordinates of order* 3.

For the proof of Lichnerowicz's Theorem1.1 using Theorem1.2 we use the following

**Lemma 2.3** *Let $(M, g)$ be a Riemannian manifold with a decomposition $M = M_0 \cup M_\infty$ where $M_0$ is compact and with a diffeomorphism $\{y \in \mathbb{R}^n\,;\, \|y\|^2 > R_0\} \mapsto \phi(y) \in M_\infty$ for some positive $R_0 > 0$. With respect to this coordinates $y = (y_1, \ldots, y_n)$ the metric coefficients $g_{ij}$ for $y \to \infty$ are of the form*

$$g_{ij} = \delta_{ij} + \mathrm{O}(r^{-\tau})$$

*for some $\tau > 1$ and with $r^2 = |y|^2 = \sum_j y_j^2$. Then the following holds: If the Ricci curvature of $(M, g)$ is non–negative then the manifold is isometric to the euclidean space $\mathbb{R}^n$.*

Proof. Choose $p \in M_0$, denote by $B_t(p) = \{y \in M \mid d(y,p) < t\}$ the distance ball of radius $t$ around $p$. Denote $A(t) = M_0 \cup \{\phi(y) \mid R_0 < \|y\| < t\}$. Since the closure of $A(t)$ is compact there is a $t_0 > 0$ such that $A(2R_0) \subset B_{t_0}(p)$. Let $q = \phi(y) \in A(2R_0 + t)$ for some $t > 0$. Choose the path $\gamma : s \in [2R_0, 2R_0 + t] \mapsto \phi(sy/\|y\|)$. The estimate $g(\dot\gamma(s), \dot\gamma(s)) = 1 + \mathrm{O}(s^{-\tau})$ with $\tau > 1$ implies that there is a constant $\lambda > 0$, such that $L(\gamma) \le t + \lambda$ for all $t > 0$. The triangle inequality implies $d(p,q) \le d(p, \gamma(2R_0) + d(\gamma(2R_0), q) \le t_0 + t + \lambda$ for all $t > 0$. Hence we have shown that there is a positive constant $\mu$ such that for all $t > 0$ :

$$A(2R_0 + t) \subset B_{t+\mu}\,.$$

The euclidean volume element $\mathrm{vol}_{\mathbb{R}^n} = dx_1 \cdots dx_n$ can be written in polar coordinates with radial coordinate $r$ in the form $r dr \mathrm{vol}_{S^{n-1}}$, where $\mathrm{vol}_{S^{n-1}}$ is the volume element of the standard $(n-1)$–sphere. Therefore the estimate $\sqrt{\det(g_{ij})} = 1+\mathrm{O}(r^{-\tau})$ implies that $\mathrm{vol}A(s) = \omega_n s^n(1+\mathrm{O}(r^{-\tau}))$, here $\omega_n = \mathrm{vol}S^n$ is the volume of the standard $n$–sphere. Hence we conclude:

$$\lim_{t\to\infty} \frac{\mathrm{vol}B_{t+\mu}(p)}{\omega_n r^n} \geq \lim_{t\to\infty} \frac{\mathrm{vol}A(2R_0+t)}{\omega_n r^n} = 1.$$

Then the Bishop volume comparison theorem, in the formulation due to Gromov, implies that $(M,g)$ is isometric to the euclidean $n$–space $\mathbb{R}^n$. For the reader's convenience we state this comparison result for manifolds of non–negative curvature as the following Lemma.

**Lemma 2.4** (cf. [E, Thm.5.5, Rem.5.7]) *Let $(M,g)$ be a complete Riemannian manifold of non–negative Ricci curvature and let*

$$t \in \mathbb{R}^+ \mapsto f(t) = \frac{\mathrm{vol}B_t(p)}{\omega_n t^n} \in \mathbb{R}^+$$

*be the quotient of the volume of the distance ball of radius $t$ around $p$ in $(M,g)$ and the euclidean volume of a ball of radius $t$. Then $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is monotone decreasing and $\lim_{t\to 0} f(t) = 1$. If $f(t_0) = 1$ for some $t_0 > 0$ then the ball $B_{t_0}(p)$ is isometric to the euclidean ball of the same radius.*

# 3 Twistor spinors with zeros

We study the asymptotic behaviour of the length of a twistor spinor with zero in a neighborhood of the zero point. First we recall the following proposition which we already mentioned in the introduction:

**Proposition 3.1** *Let $u = \langle\phi,\phi\rangle$ be the* length *of a twistor spinor then the function*

$$A_\phi = u\|D\phi\|^2 - \frac{n^2}{4}\|\nabla u\|^2$$

*is a non–negative constant. Moreover, the conformally equivalent metric $\overline{g} = u^{-2}g$ is an Einstein metric with non–negative scalar curvature $s = 4(n-1)n^{-1}A_\phi$. The last statement holds everywhere except at the zeros of $\phi$. If $s > 0$ then the metric $\overline{g}$ carries a real Killing spinor, if $s = 0$ then the metric $\overline{g}$ carries a parallel spinor.*

Since the twistor equation is conformally invariant we can choose a convenient metric in the conformal class:

**Lemma 3.2** *Let $(M,g)$ be a Riemannian manifold of dimension $n \geq 3$. Then there is in a coordinate neighborhood of $p$ a conformally equivalent metric $\overline{g} = f^{-2}g$ such that its Ricci tensor $\overline{\mathrm{Ric}}$ vanishes at $p$.*

PROOF. The $(0,2)$ Ricci tensors $\mathrm{Ric}, \overline{\mathrm{Ric}}$ of the metrics $g, \overline{g}$ are related by the following formula:

$$\overline{\mathrm{Ric}} - \mathrm{Ric} = f^{-2}\left((n-2)f\nabla^2 f + [f\Delta f - (n-1)g(\nabla f, \nabla f)] \cdot g\right) \tag{3}$$

where the gradient, the hessian and the laplacian are defined with respect to the metric $g$. Let $x = (x_1, \ldots, x_n)$ be coordinates in a neighborhood of the point $p$. We identify $p$ with the zero point of the coordinates. In addition the coordinate vectors $\partial/\partial x_i$ for $i = 1, \ldots, n$ form an orthonormal basis of the tangent space $T_pM$ consisting of eigenvectors of the Ricci tensor at $p$. Hence there are real numbers $\lambda_1, \ldots, \lambda_n$ such that $\mathrm{Ric}(e_i, e_j) = \lambda_i \delta_{ij}$. In a neighborhood of $p$ we define $f(x) = 1 + \sum_j \mu_j x_j^2$ where the numbers $\mu_1, \ldots, \mu_n$ are uniquely determined by the linear system

$$-\lambda_i = 2(n-2)\mu_i + 2\sum_{j=1}^{n} \mu_j$$

for $n \geq 3$. Then it follows from Equation 3 that $\overline{\mathrm{Ric}}(p) = 0$.

**Lemma 3.3** *Let $\phi$ be a twistor spinor with zero $p$ and let $u = \langle \phi, \phi \rangle$ be its length function. If $\mathrm{Ric}(p) = 0$ we obtain in normal coordinates $x$ around $p$ with $r^2 = \sum_k x_k^2$ :*

$$u(x) = cr^2 + O(r^5),$$

*where $c = 2\|D\phi(p)\|^2/n^2$.*

PROOF. Since $u$ is a non–negative function $\nabla u(p) = 0$. The hessian of $u$ is given by

$$\nabla^2 u(X,Y) = \langle X, L(Y)\rangle u + \frac{2}{n^2}\langle X, Y\rangle \|D\phi\|^2$$

cf. [KR1, Eq.(18)]. Here

$$L(X) = \frac{1}{n-2}\left(\frac{s}{2(n-1)}X - \mathrm{ric}(X)\right)$$

is the *Schouten–Weyl tensor,* here ric denotes the $(1,1)$–Ricci tensor. Hence at the zero point the hessian is proportional to the metric:

$$\nabla^2 u(p) = \frac{2}{n^2}\|D\phi(p)\|^2 \, g$$

and

$$\begin{aligned}\nabla^3 u(Z,X,Y) &= \langle X, (\nabla_Z L)(Y)\rangle u + \\ &\quad \langle X, L(Y)\rangle\langle Z, \nabla u\rangle + \frac{2}{n}\langle X, Y\rangle \mathrm{Re}\langle L(Z)\phi, D\phi\rangle\,.\end{aligned}$$

Here we use that a twistor spinor satisfies

$$\nabla_Z D\phi = \frac{n}{2}L(Z)\phi\,, \tag{4}$$

cf. [BFGK, (1.34)] and here $\mathrm{Re}\langle\phi_1,\phi_2\rangle = 1/2(\langle\phi_1,\phi_2\rangle + \langle\phi_2,\phi_1\rangle)$ denotes the *real part* of $\langle\phi_1,\phi_2\rangle$. Hence $\nabla^3 u(p) = 0$ follows. Differentiating once again we obtain

$$\begin{aligned}\nabla^4 u(U,Z,X,Y) \quad = \quad & \langle X,(\nabla_U\nabla_Z L)(Y)\rangle u + \langle X,(\nabla_Z L)(Y)\rangle\langle U,\nabla u\rangle + \\ & \langle X,(\nabla_U L)(Y)\rangle\langle Z,\nabla u\rangle + \langle X,L(Y)\rangle\nabla^2 u(Z,U) + \\ & \frac{2}{n}\langle X,Y\rangle \mathrm{Re}\,(\langle(\nabla_U L)(Z)\phi, D\phi\rangle + \langle L(Z)\nabla_U\phi, D\phi\rangle + \\ & \langle L(Z)\phi,\nabla_U D\phi\rangle)\end{aligned}$$

Using again Equation 4 and the twistor equation 1 we obtain:

$$\begin{aligned}\nabla^4 u(U,Z,X,Y) \quad = \quad & \langle X,(\nabla_U\nabla_Z L)(Y)\rangle u + \langle X,(\nabla_Z L)(Y)\rangle\langle U,\nabla u\rangle + \\ & \langle X,(\nabla_U L)(Y)\rangle\langle Z,\nabla u\rangle + \langle X,L(Y)\rangle\nabla^2 u(Z,U) + \\ & \frac{1}{n^2}\langle X,Y\rangle\left(2\langle L(Z),U\rangle\|D\phi\|^2 + n^2\langle L(Z),L(U)\rangle u^2\right) + \\ & 2n\mathrm{Re}\langle(\nabla_U L)(Z)\phi, D\phi\rangle\end{aligned}$$

Hence we obtain at the zero $p$ of the twistor spinor:

$$\nabla^4 u(p)(U,Z,X,Y) = \frac{2}{n^2}\langle U,L(Z)\rangle\|D\phi(p)\|^2\langle X,Y\rangle$$

Hence $\nabla^4(p) = 0$ follows provided $\mathrm{Ric}(p) = 0$. This proves Lemma 3.3.

Furthermore, we will use the following observation due to K.Habermann:

**Lemma 3.4** *Let $p$ be a zero of the twistor spinor $\phi$ which does not vanish identically. Then the Weyl tensor $W$ vanishes at $p$.*

*Proof.* The Equation [BFGK, (1.40)] simplifies at the point $p$ as follows:

$$W(X,Y)Z\cdot D\phi(p) = 0$$

for all vectors $X,Y,Z$ at $p$. The pair $(\phi, D\phi)$ can be seen as a parallel section of the sum $E = \Sigma M \oplus \Sigma M$ of two copies of the spinor bundle with respect to a connection $\nabla^E$. Therefore $D\phi(p) \neq 0$ which implies that $W(p) = 0$.

With these prerequisites we are in the position to prove the main result Theorem 1.2.

PROOF OF THEOREM 1.2:

(a) Since the twistor equation is conformally invariant we can assume without loss of generality that $\mathrm{Ric}(p) = 0$, cf. Lemma 3.2. From Lemma 3.4 it follows that $W(p) = 0$, hence also the full Riemann curvature tensor $R(p) = 0$ vanishes at $p$. This implies that the metric coefficients in the normal coordinates $x$ around $p$ with $r^2 = \sum_k x_k^2$ satisfy:

$$g_{ij} = \delta_{ij} + O(r^3)\,.$$

Lemma 3.3 implies that $r^{-4}u^2 = c^{-2}(1 + O''(\rho^{-3}))$, i.e. $\partial_k(r^{-4}u^2) = O(\rho^{-4})$ and $\partial_k\partial_j(r^{-4}u^2) = O(\rho^{-5})$ where $\partial_j$ denotes the derivative $\partial/\partial z_j$. This and Lemma 2.2 imply that the metric coefficients $\overline{g}_{ij}$ of the conformally equivalent metric

$$\overline{g} = c^2 \frac{g}{u^2} = \frac{r^4}{u^2}\hat{g}$$

with respect to the inverted normal coordinates $z$ with $z_k = r^{-2}x_k$ and $\rho = r^{-1} = \sum_k z_k^2$ satisfy:

$$\overline{g}_{ij} = \delta_{ij} + O''(\rho^{-3})\,.$$

(b) From the last equation it follows that there is $\rho_1 \geq \rho_0$ such that for all $\rho \geq \rho_1$ all principal curvatures of the distance spheres $\{\sum_k z_k^2 = \rho\}$ are $\geq (2\rho)^{-1}$. This implies that all geodesics $\gamma$ which start from a distance sphere in the direction of growing $\rho$, i.e. $\overline{g}(\gamma'(0), \delta_\rho) \geq 0$ and $\rho(\gamma(0)) \geq \rho_1$ are defined for all positive real numbers and $\lim_{t\to\infty} \rho(\gamma(t)) = \infty$. In geodesic normal coordinates the Ricci–flat metric $\overline{g}$ is analytic, hence the Riemann curvature tensor does not vanish on an open set unless the metric itself is flat.
Now we assume that $\overline{g}$ is non–flat and locally reducible. Then by the above consideration we can choose a geodesic $\gamma : [0,\infty) \to M - \{p\}$ with $\lim_{t\to\infty} \rho(\gamma(t)) = \infty$ which in an open neigborhood $U$ of $\gamma(0)$ lies in the factor $U_1$ of the Riemannian product $U = U_1 \times U_2$ and such that the Riemann curvature tensor $R_2$ at $\gamma(0)$ of the factor $U_2$ does not vanish. Hence we can choose analytic parallel vector fields $X(t), Y(t)$ along $\gamma$ tangential to $U_2$ which span a tangent plane with non–zero sectional curvature $K(X,Y) = K_2(X,Y)$. Hence the function $t \mapsto K(X,Y)$ is a non–zero constant $\kappa$ for small $t$. Then the analyticity implies that $\lim_{t\to\infty} K(X(t), Y(t)) = \kappa$. But the asymptotic behaviour of the coordinate system $z$ implies that $K(X(t),Y(t)) = O(\rho^{-5}(\gamma(t))) = 0$, i.e. we obtain a contradiction.

For the proof of Theorem 1.3 we need the following

**Lemma 3.5** *Let $M$ be a connected manifold carrying two conformally equivalent and Ricci-flat Riemannian metrics $g$ and $\overline{g} = f^{-2}g$. Then $M$ is an open subset of a product $(0,\infty) \times M_*$, and the metrics are cone-like metrics $dt^2 + t^2 g_*$ with a codimension 1 Einstein metric $g_*$ on $M_*$ which is independent of $t$ and which has the same scalar curvature as the standard unit sphere. In addition the function $f$ is a function of the coordinate $t$ only and $f(t) = t^2/2$.*

Proof of the Lemma: Let $\overline{g} = f^{-2}g$ with a non-constant function $f$. From equation (3) we obtain the equations $\nabla^2 f = \frac{\Delta f}{n} \cdot g$ and $2f \cdot \Delta f = n \cdot g(\nabla f, \nabla f)$. The first equation implies that $g$ is a warped product $dt^2 + (f'(t))^2 g_*$, the second equation implies $2ff'' = f'^2$. Here $t$ denotes the arc length on the trajectories of $\nabla f$ and $f' = \frac{\partial f}{\partial t}$. Up to a shift of the parameter and the choice of constants we have $f'(t) = t$. The metric $g_*$ must be Einstein if $g$ is Einstein, and the scalar curvatures behave as in the coordinatization of the euclidean metric in polar coordinates. It follows that $\overline{g} = 4t^{-4} \cdot g$. In some sense the transformation $g \mapsto \overline{g}$ corresponds to a sphere inversion in euclidean space.

PROOF OF THEOREM 1.3
Let $\phi$ be a twistor spinor with non–empty set of zeros $Z_\phi$. Assume that $\psi$ is a twistor spinor with a zero $q \notin Z_\phi$. Then it follows from Proposition 3.1 that there is a non–constant function $f$, such that $\overline{g}$ and $f^{-2}\overline{g}$ are both Ricci–flat. Here $f = \|\psi\|^2/\|\phi\|^2$, i.e., $f(q) = 0$. It follows from Lemma 3.5 that $g$ is a cone–like metric $dt^2 + t^2 g_*$ on an open subset of $(0, \infty) \times M_*$ and $f(t) = t^2/2$. Hence $q$ is the apex. Since $q$ is not a singular point of the manifold it follows that $(M_*, g_*)$ is the standard sphere.
But this can happen only if $\overline{g}$ is flat. By Theorem 1.2 and Proposition 3.1 we obtain that $(\overline{M}, \overline{g})$ is locally irreducible unless it is flat and it carries a parallel spinor. Then the rest of the statement follows from the holonomy classification of Berger and Simons, as in [Hi, p.8, footnote p.54] and [Wa, Prop.].

In the following corollary we obtain an elementary proof of Lichnerowicz's Theorem1.1. More precisely, the proof does not depend on global solutions of partial differential equations like the solution of the Yamabe problem.

**Corollary 3.6** *A compact Riemannian spin manifold carrying a non–trivial twistor spinor with zero is conformally equivalent to the standard sphere.*

PROOF. Since the zero set of the twistor spinor $\phi$ is a discrete set, there are only finitely many zeros $p_1, \ldots, p_m$. We conclude from Theorem 1.2 that there are open neighborhoods $U_1, \ldots, U_m$ of $p_1, \ldots, p_m$ which are pairwise disjoint and which satisfy the following: For the conformally equivalent metric $\overline{g} = u^{-2}g$ which is defined on $\overline{M} = M - \{p_1, \ldots, p_m\}$ there is an asymptotically euclidean coordinate system $z^i = (z^i_1, \ldots, z^i_n)$ of order 3 in every neighborhood $U_i$. Since for sufficiently large $t_0$ the levels $\rho^i = t_0$ with $(\rho^i)^2 = \sum_j (z^i_j)^2$ are convex and since $M - (U_1 \cup \ldots U_m)$ is compact the Riemannian manifold $(\overline{M}, \overline{g})$ is complete and Ricci flat. It follows from the *splitting theorem* due to Cheeger and Gromoll that the manifold can only have one end, i.e., $m = 1$. An 'elementary' proof of the splitting theorem was given by Eschenburg and Heintze in [EH]. Hence $(\overline{M}, \overline{g})$ is an asymptotically euclidean manifold as defined in Definition 2.1 of order 3. Then the preceding Lemma implies that $(M - \{p_1\}, \overline{g})$ is isometric to $\mathbb{R}^n$, hence $(M, g)$ is conformally equivalent to the standard sphere.

# References


[ACDS] D.V.Alekseevsky, V.Cortés, C.Devchand, U.Semmelmann: *Killing spinors are Killing vector fields in Riemannian supergeometry.* Preprint MPI Bonn 1997

[Ba] C.Bär: *Real Killing spinors and holonomy.* Commun.Math.Phys. **154** (1993) 509–521

[BFGK] H.Baum, T.Friedrich, R.Grunewald and I.Kath: *Twistors and Killing spinors on riemannian manifolds.* Teubner Texte zur Math., vol. **124**, B.G.Teubner Stuttgart, Leipzig 1991

[Ca] E.Calabi: *Métriques kählériennes et fibrés holomorphes.* Ann.Ecol.Norm.Sup. **12** (1979) 269–294

[E] J.-H.Eschenburg: *Comparison theorems in Riemannian geometry.* Lect.Notes Ser. Univ.Trento, Math. vol. **3** 1994

[EH] J.-H.Eschenburg and E.Heintze: *An elementary proof of the Cheeger–Gromoll splitting theorem.* Ann.Global Anal.Geom. **2** (1984) 141–151

[FG] D.Z.Freedman and G.W.Gibbons: *Remarks on supersymmetry and Kähler geometry.* In: *Superspace and Supergravity,* Proc. Nuffield workshop, Cambridge 1980 (S.W.Hawking, M.Roček, eds.) Cambridge Univ.Press 1981

[Fr] T.Friedrich: *Dirac–Operatoren in der riemannschen Geometrie.* Advanced lectures in Math., Vieweg Verlag Braunschweig, Wiesbaden 1997

[Ha] K.Habermann: *Twistor–Spinoren auf Riemannschen Mannigfaltigkeiten und deren Nullstellen.* Thesis, Humboldt–Universität Berlin 1992, *Twistor spinors and their zeroes.*J.Geom.Phys. **14** (1994) 1–24

[He] M.Herzlich: *Compactification conforme des variétés asymptotiquement plates.* Bull.Soc.Math.France **125** (1997) 55–92

[Hi] N.Hitchin: *Harmonic spinors.* Adv.Math. **14** (1974) 1–55

[KR1] W.Kühnel and H.-B.Rademacher: *Twistor Spinors with zeros.* Int.J.Math. **5** (1994) 877–895

[KR2] W.Kühnel and H.-B.Rademacher: *Twistor Spinors and Gravitational Instantons.* Lett.Math.Phys. **38** (1996) 411–419

[KR3] W.Kühnel and H.-B.Rademacher: *Conformal completion of $\mathbb{U}(n)$–invariant Ricci flat Kähler metrics at infinity.* Zeitschr.Anal.Anwend. **16** (1997) 113–117

[LP] J.M.Lee and T.H.Parker: *The Yamabe problem.* Bull.(N.S.) Amer.Math.Soc. **17** (1987) 37–91

[Li] A.Lichnerowicz: *Killing spinors, twistor–spinors and Hijazi inequality.* J.Geom.Physics **5** (1988) 2–18

[Wa] M.Wang : *Parallel spinors and parallel forms.* Ann.Global Anal.Geom. **7** (1989) 59–68

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