Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

A-quasiconvexity, lower semicontinuity and Young measures

by

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Preprint-Nr.: 18 1998

A-QUASICONVEXITY, LOWER SEMICONTINUITY AND YOUNG MEASURES

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Abstract The notion of $\mathcal{A}$-quasiconvexity is introduced as a necessary and sufficient condition for (sequential) lower semicontinuity of

$$(u,v) \mapsto \int_\Omega f(x,u(x),v(x))\,dx$$

whenever $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,+\infty)$ is a normal integrand, $\Omega \subset \mathbb{R}^N$ is open, bounded, $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^d)$ ($\overset{*}{\rightharpoonup}$ if $p=+\infty$), and $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega)$ ($\mathcal{A}v_n = 0$ if $p=+\infty$). Here $\mathcal{A}v = \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}$ is a constant-rank partial differential operator, $A^{(i)} \in \mathrm{Lin}(\mathbb{R}^d;\mathbb{R}^l)$, and $f(x,u,\cdot)$ is $\mathcal{A}$-quasiconvex if

$$f(v) \le \int_Q f(v+w(x))\,dx$$

for all $v \in \mathbb{R}^d$ and all $w \in C^\infty(Q;\mathbb{R}^d)$ such that $\mathcal{A}w = 0$, $\int_Q w(x)\,dx = 0$, and w is Q-periodic, $Q := (0,1)^N$. The characterization of Young measures generated by such sequences $\{v_n\}$ is obtained for $1 \le p < +\infty$, thus recovering the well known results for the framework $\mathcal{A} = \mathrm{curl}$, i.e. when $v_n = \nabla\varphi_n$ for some $\varphi_n \in W^{1,p}(\Omega;\mathbb{R}^m)$, $d = N \times m$. In this case $\mathcal{A}$-quasiconvexity reduces to Morrey's notion of quasiconvexity.

1991 Mathematics subject classification (Amer. Math. Soc.): 35D99, 35E99, 49J45
Key Words : $\mathcal{A}$-quasiconvexity, equi-integrability, Young measure, lower semicontinuity

§1. Introduction

Recently there has been extensive research on minimization and relaxation of nonconvex energies relevant to the study of equilibria for materials exhibiting interesting, and technologically powerful, elastic and magnetic behaviors. Often a starting point for this study addresses directly minimization of the energy, leading to the search for necessary and sufficient conditions ensuring sequential weak lower semicontinuity of integrals of the form

$$(u,v) \mapsto I(u,v) := \int_\Omega f(x,u(x),v(x))\,dx$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded set, $(u,v) : \Omega \to \mathbb{R}^m \times \mathbb{R}^d$, and $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a normal integrand. On the other hand, there may be situations where we need to identify $\lim_{n\to\infty} I(u_n,v_n)$ for an oscillatory sequence $\{(u_n,v_n)\}$ which does not minimize the energy. Consequently, this will entail a full characterization of the Young measures generated by the sequences under consideration, i.e. weak* measurable maps $\nu : \Omega \to \mathcal{P}$, where $\mathcal{P}$ is the space of probability measures on $\mathbb{R}^{m\times d}$, such that if $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function, and if $\{g(\cdot,u_n,v_n)\}$ is equi-integrable, then

$$\int_\Omega g(x,u_n(x),v_n(x))\,dx \to \int_\Omega \int_{\mathbb{R}^{m\times d}} g(x,y,z)\,d\nu_x(y,z)\,dx.$$

Although Young measures have been used for quite some time in the contexts of Control Theory and Optimization, they were first introduced in a Partial Differential Equations framework by Tartar (see [T1, T2, T3]) in order to relate the information obtained from the linear balance equations

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via the method of Compensated Compactness with the information resulting from pointwise nonlinear constitutive relations. One application of this method was the study of quasilinear hyperbolic equations (see [T2]), and later DiPerna [DiP1, DiP2] and DiPerna and Majda [DiP3] extended it to systems. During the last few years several questions related to the study of (nonlinear) elastic materials and certain materials instabilities have been successfully carried out via minimization techniques and through the understanding of the underlying Young measures (see [BJ1, BJ2, CK, DS, JK]). Often, in this context v is the gradient ∇u of a Sobolev function $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $d = m \times N$, and coercivity of f provides boundedness of the admissible sequences in $W^{1,p}(\Omega;\mathbb{R}^m)$. If $p > 1$ then $u_n \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ (up to extraction of a subsequence). The work of Morrey [Mo], Ball [B1], and Acerbi and Fusco [AF] shows that $W^{1,p}$ (sequential) weak lower semicontinuity of

$$u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

is equivalent to *quasiconvexity* of $f(x, u, \cdot)$ provided $0 \leq f(x, u, \xi) \leq a(x, u)(1 + |\xi|^p)$ for some locally bounded function $a : \Omega \times \mathbb{R}^d \to [0, +\infty)$ and for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Omega$. We recall that a Borel function $f : \mathbb{M}^{m\times N} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(\xi) = \inf_{\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)} \int_Q f(\xi + \nabla\varphi(x))\, dx, \tag{1.1}$$

where $Q := (0,1)^N$. If f is quasiconvex then one can show that

$$f(\xi) = \inf_{\varphi \in \mathrm{W}^{1,\infty}_{\mathrm{per}}(\mathrm{Q};\mathbb{R}^{\mathrm{m}})} \int_Q f(\xi + \nabla\varphi(x))\, dx \tag{1.2}$$

where $\mathrm{W}^{1,\infty}_{\mathrm{per}}(\mathrm{Q};\mathbb{R}^{\mathrm{m}})$ is the class of periodic functions in $W^{1,\infty}(Q;\mathbb{R}^m)$. Within this context, the characterization of all Young measures generated by sequences of gradients bounded in L^p was obtained by Kinderlehrer and Pedregal [KP1, KP2]. They show that (see Theorem 2.6) in a simply connected domain Ω a weakly measurable mapping $\nu : \Omega \to \mathcal{P}$ is a Young measure generated by a sequence of gradients ∇u_n, with $\{u_n\}$ bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$, if and only if three conditions are satisfied:

ν is p-integrable, i.e.,

$$\int_\Omega \langle \nu_x, |\operatorname{id}|^p \rangle\, dx < +\infty,$$

the first moment $x \mapsto \langle \nu_x, \operatorname{id} \rangle$ satisfies the underlying PDE, i.e.,

$$\operatorname{curl}(\langle \nu_x, \operatorname{id} \rangle) = 0,$$

and, as suggested by (1.1), Jensen's inequality is satisfied for quasiconvex functions, i.e.,

$$\langle \nu_x, f \rangle \geq f(\langle \nu_x, \operatorname{id} \rangle)$$

for all quasiconvex functions f such that $|f(\xi)| \leq C(1 + |\xi|^p)$.

As emphasized by Tartar, in the setting of continuum mechanics and electromagnetism more general linear PDEs than $\operatorname{curl} v = 0$ arise, and the theory of compensated compactness was developed in that framework (see [Mu, T1, T2, T3, T4, T5]). To fix ideas, consider a collection of linear operators $A^{(i)} \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, $i = 1, \ldots, N$, and define

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}, \qquad v : \mathbb{R}^N \to \mathbb{R}^d,$$

$$\mathbb{A}(w) := \sum_{i=1}^{N} A^{(i)} w_i \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l), \quad w \in \mathbb{R}^N,$$

where $\mathrm{Lin}(X, Y)$ is the vector space of linear mappings from the vector space X into the vector space Y. Following Murat [Mu], we will assume that $\mathcal{A}$ satisfies the *constant rank* property, which states that there exists $r \in \mathbb{N}$ such that

$$\mathrm{rank}\, \mathbb{A}(w) = r \quad \text{for all } w \in S^{N-1}.$$

It is easy to see that the curl-free case is a particular case of this general framework (see Remark 3.3 (iii)). Other examples are discussed in Remarks 3.3 and 3.5 and in Examples 3.10 and 4.4.

We prove that a necessary and sufficient condition for weak lower semicontinuity of I, along sequences that satisfy $u_n \to u$ in measure, $v_n \rightharpoonup v$ in L^p, and $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega)$, is *$\mathcal{A}$-quasiconvexity* of $f(x, u, \cdot)$ (see Theorems 3.6, 3.7). The notion of $\mathcal{A}$-quasiconvexity and its implications for the lower semicontinuity of functionals $v \mapsto \int_\Omega f(v)\, dx$ were first investigated by Dacorogna who studied in particular situations where the kernel of $\mathcal{A}$ contains the range of a suitable first order differential operator $\mathcal{B}$ [Da1, pp. 100-112] (in the general definition of $\mathcal{A}$-quasiconvexity as presented in [Da1, p. 13] one needs to add periodicity of the test functions to obtain necessity of $\mathcal{A}$-quasiconvexity; this leads to some difficulties in establishing sufficiency, which, under the assumption of constant rank, can be overcome using the methods presented below). Precisely, and by analogy with (1.2), a function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *$\mathcal{A}$-quasiconvex* if

$$f(v) \le \int_Q f(v + w(x))\, dx$$

for all $v \in \mathbb{R}^d$ and all Q-periodic $w \in C^\infty(Q, \mathbb{R}^d)$ such that $\mathcal{A}(w) = 0$ and $\int_Q w(x)\, dx = 0$. In addition, we obtain the generalization to the $\mathcal{A}$-free setting of the theorem by Kinderlehrer and Pedregal concerning the characterization of gradient Young measures (see Theorem 4.1). This issue has been independently raised by Pedregal in [P], where he studied the case of divergence free fields (see also Remarks 3.3 (iv), 3.5 (iv)).

We remark that continuity of $\mathcal{A}$-quasiconvex functions is only garanteed along directions in the *characteristic cone* $\Lambda := \cup_{w \in S^{N-1}} \ker \mathbb{A}(w)$, and $\mathcal{A}$-quasiconvex functions need not be (lower semi)continuous (see Proposition 3.4 and Remark 3.5 (ii)). In particular, it will not be true in general that the relaxed energy admits the integral representation

$$(u, v) \to \int_\Omega \mathrm{Q}_{\mathcal{A}} f(x, u, v)\, dx$$

where $\mathrm{Q}_{\mathcal{A}} f$ is the $\mathcal{A}$-quasiconvexification of f. In the curl-free case this representation was first established by Dacorogna [D2], and nowadays there is a vaste literature on the subject.

We note that the method used in this $\mathcal{A}$-free framework departs from the case curl-free mostly due to the lack of 'potential functions' associated to the v_n. Indeed, in the case of gradients we reduce to the notion of quasiconvexity by localization via covering lemmas, so that on each subdomain the target function is essentially affine, followed by matching of the boundary conditions. The latter can be easily done by simple convex combinations between the potentials and the target function, avoiding layers of high concentrations of the gradients of the v_n. Clearly, the gradient of the resulting convex combinations still satisfy curl = 0. In the general $\mathcal{A}$-free setting, we must work directly on the v_n, and we need to find a way to project back the modified fields onto $\ker \mathcal{A}$. We perform these projections via discrete Fourier multipliers (see Lemmas 2.15, 2.16, 2.17). It is at this point that the constant rank condition enters in a crucial way. Situations where the constant rank condition fails are little understood. Tartar [T1] has studied the example where $v : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{A}v = \left(\frac{\partial v^1}{\partial x_2}, \frac{\partial v^2}{\partial x_1}\right)$. He showed that in this case $\mathcal{A}$-quasiconvexity reduces to separate convexity, the Young measures generated by sequences along which $\{\mathcal{A}v_n\}$ is bounded in L^∞ are tensor products, and this class is strictly smaller than the class defined by duality with separately convex functions (see condition (iii) in Theorem 4.1). The class of Young measures generated by sequences that satisfy $\mathcal{A}v_n \to 0$ in $W^{-1,p}$ is not known (see [BM2, T6]).

§2. Preliminaries

In this section we recall the notion of Young measures generated by sequences bounded in L^p and by curl-free sequences. We discuss some properties of a constant rank linear partial differential operator $\mathcal{A}$, and we conclude with the Decomposition Lemmas 2.15, 2.16, 2.17, where we show that if $\{u_n\}$ is weakly convergent in L^p and if $\mathcal{A}u_n \to 0$ in the appropriate sense then $u_n = v_n + w_n$ where $\{v_n\} \in L^p \cap \ker \mathcal{A}$ is p-equi-integrable and $\{w_n\}$ converges to zero in measure.

In the sequel $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, $B(x,\varepsilon)$ denotes the open ball centered at $x \in \mathbb{R}^N$ with radius $\varepsilon > 0$, $Q := (0,1)^N$, $Q(x_0,r) := x_0 + rQ^*$, $Q^* := Q - (1/2,\cdots,1/2)$, and $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere in $\mathbb{R}^N$. The Lebesgue measure in $\mathbb{R}^N$ is designated by $\mathcal{L}^N$, and H^{N-1} will stand for the $N-1$-dimensional Hausdorff measure in $\mathbb{R}^N$. If $1 < p < +\infty$ then $W^{-1,p}(\Omega)$ is the dual of $W_0^{1,p'}(\Omega)$, with $1/p + 1/p' = 1$, and it is well known that $F \in W^{-1,p}(\Omega)$ if and only if $F = f + \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}$ in the sense of distributions, for some $f, g_1, \ldots, g_N \in L^p(\Omega)$. We denote by $C_0(\Omega;\mathbb{R}^d)$ the set of $\mathbb{R}^d$-valued continuous functions with compact support in Ω, endowed with the supremum norm. It is well known that the dual of the closure of $C_0(\Omega;\mathbb{R}^d)$ may be identified with the set of $\mathbb{R}^d$-valued Radon measures with finite mass, $\mathcal{M}(\Omega;\mathbb{R}^d)$, through the duality

$$\langle \mu, \varphi \rangle := \int_\Omega \varphi \cdot d\mu, \qquad \varphi \in C_0(\Omega), \mu \in \mathcal{M}(\Omega).$$

In order to simplify the notation, and when there is no ambiguity, we will abbreviate $C_0(\Omega;\mathbb{R}^d)$ and $\mathcal{M}(\Omega;\mathbb{R}^d)$ as $C_0(\Omega)$ and $\mathcal{M}(\Omega)$, respectively. If $\mu \in \mathcal{M}(\Omega)$ and $E \subset \Omega$ is a Borel set, then $\mu\lfloor E$ stands for the restriction of the measure μ to E, i.e.

$$\mu\lfloor E(X) := \mu(E \cap X) \quad \text{for all Borel set } X \subset \Omega.$$

We recall that given $\lambda, \mu \in \mathcal{M}(\Omega)$ with $\mu \geq 0$, by the Radon-Nikodym Theorem we may decompose λ relative to μ, precisely $\lambda = \lambda_a + \lambda_s$ where λ_s and μ are *mutually singular* ($\lambda_s \perp \mu$), i.e.

$$\lambda_s(X) = \lambda_s(X \cap B),\ \mu(X) = \mu(X \setminus B)$$

for all Borel sets $X \subset \Omega$ and for some Borel set $B \subset \Omega$, and where λ_a is *absolutely continuous with respect to* μ, $\lambda_a << \mu$, i.e. $\lambda_a(X) = 0$ whenever $X \subset \Omega$ is a Borel set and $\mu(X) = 0$. By Besicovitch's Differentiation Theorem we have

$$\lambda_a(X) = \int_X \frac{\partial \lambda}{\partial \mu}(x)\, d\mu, \quad \frac{\partial \lambda}{\partial \mu}(x) := \lim_{\varepsilon \to 0} \frac{\lambda(B(x,\varepsilon))}{\mu(B(x,\varepsilon))} \quad \text{for } \mu \text{ a.e. } x \in \Omega$$

and for all Borel sets $X \subset \Omega$.

If $\{z_n\}$ is a sequence bounded in $L^1(\Omega)$ then it admits a subsequence converging weakly* in the sense of measures to a measure $\mu \in \mathcal{M}(\Omega)$,

$$\int_\Omega z_{n_k} \varphi\, dx \to \int_\Omega \varphi\, d\mu$$

for all $\varphi \in C_0(\Omega)$. The *equi-integrability condition*

for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{L}^N(E) < \delta \Rightarrow \sup_n \int_E |z_n(x)|\, dx < \varepsilon$

is a necessary and sufficient condition for weak compactness in L^1 of the sequence $\{z_n\}$ (recall that Ω is bounded). If equi-integrability holds then $\mu << \mathcal{L}^N$. We will say that $\{z_n\}$ is *p-equi-integrable* if $\{|z_n|^p\}$ is equi-integrable. The following Dunford-Pettits criteria for equi-integrability are well known.

Proposition 2.1. *Let $\{z_n\}$ be a sequence bounded in $L^1(\Omega)$.*
(i) The sequence $\{z_n\}$ is equi-integrable if and only if for all $\varepsilon > 0$ there exists $M > 0$ such that

$$\sup_n \int_{\{x\in\Omega:|z_n(x)|>M\}} |z_n(y)|\, dy < \varepsilon.$$

(ii) The sequence $\{z_n\}$ is equi-integrable if there exists a continuous function $g : [0, +\infty) \to [0, +\infty)$ such that

$$\lim_{t\to+\infty} \frac{g(t)}{t} = +\infty, \quad \sup_n \int_\Omega g(|z_n(x)|)\, dx < +\infty.$$

(iii) If $\{z_n\}$ is bounded in $L^p(\Omega)$ for some $1 \le p < +\infty$ then $\{f(z_n)\}$ is equi-integrable whenever $f : \mathbb{R}^d \to [0, +\infty)$ is a continuous function such that

$$\lim_{|y|\to+\infty} \frac{f(y)}{|y|^p} = 0.$$

A map $\mu : E \to \mathcal{M}(\Omega)$ is said to be *weak* measurable* if $x \mapsto \langle \mu(x), \varphi\rangle$ are measurable for all $\varphi \in C_0(\Omega)$. In order to simplify the notation we denote $\mu(x)$ by μ_x.

Often the study of the behavior of solutions of nonconvex problems leads to the need to determine the limiting energy

$$\lim_{n\to\infty} \int_E f(z_n)\, dx$$

where E is a measurable subset of Ω, $f : \mathbb{R}^d \to \mathbb{R}$ is a nonlinear function, and $\{z_n\}$ is an oscillatory sequence of measurable functions $z_n : E \to \mathbb{R}^d$. In general, the presence of oscillations entails the inequality

$$\lim_{n\to\infty} \int_E f(z_n)\, dx \ne \int_E f(z)\, dx.$$

As it turns out, the Young measure generated by (a subsequence of) $\{z_n\}$ will provide the limiting energy.

We recall that a function $f : \Omega\times\mathbb{R}^d \to \mathbb{R}$ is said to be a *normal integrand* if f is Borel measurable and $v \mapsto f(x, v)$ is lower semicontinuous for all $x \in \Omega$. Also, f is *Carathéodory* if f and $-f$ are normal integrands.

Theorem 2.2. [Fundamental Theorem on Young Measures] [B2, BL, T1] *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and let $\{z_n\}$ be a sequence of measurable functions, $z_n : E \to \mathbb{R}^d$. Then there exists a subsequence $\{z_{n_k}\}$ and a weak* measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ such that the following hold:*
(i) $\nu_x \ge 0$, $||\nu_x||_{\mathcal{M}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} d\nu_x \le 1$ for a.e. $x \in E$;
(ii) one has (i′) $||\nu_x||_{\mathcal{M}} = 1$ for a.e. $x \in E$
if and only if

$$\lim_{M\to\infty} \sup_k \mathcal{L}^N(\{|z_{n_k}| \ge M\}) = 0; \tag{2.1}$$

(iii) if $K \subset \mathbb{R}^d$ is a compact subset and $\operatorname{dist}(z_{n_k}, K) \to 0$ *in measure then*

$$\text{supp } \nu_x \subset K \textit{ for } \text{a.e. } x \in E;$$

(iv) if (i′) holds then in (iii) one may replace 'if' by 'if and only if';
(v) if $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a normal integrand, bounded from below, then

$$\liminf_{n\to\infty} \int_\Omega f(x, z_{n_k}(x))\, dx \ge \int_\Omega \overline{f}(x)\, dx$$

where

$$\overline{f}(x) := \langle \nu_x, f(x,\cdot)\rangle = \int_{\mathbb{R}^d} f(x,y)\, d\nu_x(y);$$

(vi) if (i') holds and if $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is Carathéodory and bounded from below, then

$$\lim_{n\to\infty} \int_\Omega f(x, z_{n_k}(x))\, dx = \int_\Omega \overline{f}(x)\, dx < +\infty$$

if and only if $\{f(\cdot, z_{n_k}(\cdot))\}$ is equi-integrable. In this case

$$f(\cdot, z_{n_k}(\cdot)) \rightharpoonup \overline{f} \text{ in } L^1(\Omega).$$

The map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ is called the *Young measure generated by the sequence* $\{z_{n_k}\}$. It can be shown that every weak* measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ that satisfies (i) is generated by some sequence $\{z_n\}$. The Young measure ν is said to be *homogeneous* if there is a Radon measure $\nu_0 \in \mathcal{M}(\mathbb{R}^d)$ such that $\nu_x = \nu_0$ for a. e. $x \in E$.

Remark 2.3. (i) Condition (2.1) holds if for some $p > 0$

$$\sup_{n\in\mathbb{N}} \int_E |z_n|^p\, dx < +\infty.$$

(ii) As a consequence of (vi), if $\{z_n\}$ is bounded in L^p and if f is a continuous function in $\mathbb{R}^d$ such that $|f(y)| \le C(1 + |y|^q)$ for some $C > 0$, $0 < q < p$, then $f(z_{n_k}) \rightharpoonup \overline{f}$ in $L^{p/q}$. Also, if $\{z_n\}$ is equi-integrable, then taking $f \equiv \mathrm{id}$ we obtain

$$z_{n_k} \rightharpoonup \overline{z} \text{ in } L^1(\Omega), \quad \overline{z}(x) := \langle \nu_x, \mathrm{id}\rangle.$$

Proposition 2.4. *If $\{v_n\}$ generates a Young measure ν and if $w_n \to w$ in measure then $\{v_n + w_n\}$ generates the 'translated' Young measure*

$$\overline{\nu}_x := \Gamma_{w(x)} \nu_x$$

where

$$\langle \Gamma_a \mu, \varphi\rangle := \langle \mu, \varphi(\cdot + a)\rangle$$

for $a \in \mathbb{R}^d$, $\varphi \in C_0(\mathbb{R}^d)$. In particular, if $w_n \to 0$ in measure then $\{v_n + w_n\}$ generates the Young measure ν.

Proposition 2.5. *If $\{v_n\}$ generates a Young measure ν and $u_n \to u$* a.e. *in Ω then the pair $\{(u_n, v_n)\}$ generates the Young measure μ defined by*

$$\mu_x := \delta_{u(x)} \otimes \nu_x, \qquad \text{a. e. } x \in \Omega.$$

A Young measure ν is called a *gradient Young measure* if it is generated by a sequence of gradients; more precisely, ν is a $W^{1,p}$ gradient Young measure if it is generated by $\{\nabla u_n\}$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$. A complete characterization of such Young measures has been obtained by Kinderlehrer and Pedregal [KP1, KP2] (see also [AB, FMP, K]). A key ingredient is the notion of *quasiconvexity*: a Borel function $f : \mathbb{M}^{m\times N} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(\xi) = \inf_{\varphi \in W_0^{1,\infty}(Q;\mathbb{R}^m)} \int_Q f(\xi + \nabla\varphi(x))\, dx.$$

If f is quasiconvex then one can show that

$$f(\xi) = \inf_{\varphi \in \mathrm{W}^{1,\infty}_{\mathrm{per}}(\mathrm{Q};\mathbb{R}^{\mathrm{m}})} \int_Q f(\xi + \nabla\varphi(x))\,dx$$

where $\mathrm{W}^{1,\infty}_{\mathrm{per}}(\mathrm{Q};\mathbb{R}^{\mathrm{m}})$ is the class of periodic functions in $W^{1,\infty}(Q;\mathbb{R}^m)$. It has been established by Morrey [Mo] (see also [AF, ADM, B1, D1, D2, FM1, FM2]) that sequential weak lower semicontinuity in $W^{1,p}$ and quasiconvexity are essentially equivalent. More precisely, if $0 \le f(\xi) \le C(1+|\xi|^p)$ for some $C > 0$ and all $\xi \in \mathbb{M}^{m\times N}$ (no growth condition is necessary if $p = +\infty$) then the implication

$$u_n \rightharpoonup u \text{ in } W^{1,p} (\overset{*}{\rightharpoonup} \text{ if } p = +\infty) \Rightarrow \int_\Omega f(\nabla u)\,dx \le \liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\,dx$$

holds if and only if f is quasiconvex.

Theorem 2.6. *Let $1 \le p \le +\infty$. A weak* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{M}^{m\times N})$ is a $W^{1,p}$ gradient Young measure if and only if $\nu_x \ge 0$ a.e. $x \in \Omega$ and*
(i) there exists $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\langle \nu_x, \mathrm{id}\rangle = Du$ a.e. $x \in \Omega$;
(ii) $\int_\Omega \int_{\mathbb{M}^{m\times N}} |\xi|^p \, d\nu_x(\xi)dx < +\infty$ ($\operatorname{supp}\nu_x \subset K$ a.e. $x \in \Omega$, for some compact $K \subset \mathbb{M}^{m\times N}$ if $p = +\infty$);
(iii) $\langle \nu_x, f\rangle \ge f(\langle \nu_x, \mathrm{id}\rangle)$ for a.e. $x \in \Omega$ and for all quasiconvex $f : \mathbb{M}^{m\times N} \to \mathbb{R}$ (with $|f(\xi)| \le C(1+|\xi|^p)$ for some $C > 0$ and all $\xi \in \mathbb{M}^{m\times N}$ if $1 \le p < +\infty$).

Consider a collection of linear operators $A^{(i)} \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, $i = 1, \dots, N$, and define

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}, \qquad v : \mathbb{R}^N \to \mathbb{R}^d,$$

$$\mathbb{A}(w) := \sum_{i=1}^N A^{(i)} w_i \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l), \quad w \in \mathbb{R}^N,$$

where $\mathrm{Lin}(X,Y)$ is the vector space of linear mappings from the vector space X into the vector space Y.

In the sequel we will assume that $\mathcal{A}$ satisfies the *constant rank* property, namely there exists $r \in \mathbb{N}$ such that

$$\text{(CR)} \qquad \operatorname{rank} \mathbb{A}(w) = r \quad \text{for all } w \in S^{N-1}.$$

Fix $w \in \mathbb{R}^N$. We define

$$\mathbb{P}(w) : \mathbb{R}^d \to \mathbb{R}^d \quad \text{to be \textit{the orthogonal projection of} } \mathbb{R}^d \text{ onto } \ker \mathbb{A}(w),$$

$$\mathbb{Q}(w) : \mathbb{R}^l \to \mathbb{R}^d, \quad \mathbb{Q}(w)\mathbb{A}(w)z := z - \mathbb{P}(w)z,\ z \in \mathbb{R}^d, \quad \mathbb{Q}(w) \equiv 0 \text{ on } (\text{range } \mathbb{A}(w))^\perp.$$

Proposition 2.7. *If (CR) holds then the map $\mathbb{P} : \mathbb{R}^N \setminus \{0\} \to \mathrm{Lin}(\mathbb{R}^d;\mathbb{R}^d)$ is smooth and homogeneous of degree zero, and the map $\mathbb{Q} : \mathbb{R}^N \setminus \{0\} \to \mathrm{Lin}(\mathbb{R}^l;\mathbb{R}^d)$ is smooth and homogeneous of degree -1.*

Let $\Delta := \mathbb{Z}^N$ be the *unit lattice*, i.e. the additive group of points in $\mathbb{R}^N$ with integer coordinates. We say that $f : \mathbb{R}^N \to \mathbb{R}^d$ is Δ*-periodic* if

$$f(x+\lambda) = f(x) \quad \text{for all } x \in \mathbb{R}^N, \lambda \in \Delta.$$

A Δ-periodic function f may be identified with a function f_T on the N *torus*

$$T_N := \{(e^{2\pi i x_1}, \dots, e^{2\pi i x_N}) \in \mathbb{C}^N : (x_1, \dots, x_N) \in \mathbb{R}^N\}$$

through the relation

$$f_T(e^{2\pi i x_1}, \dots, e^{2\pi i x_N}) := f(x_1, \dots, x_N).$$

The space $L^p(T_N)$ is identified with $L^p(Q)$, and $C(T_N)$ is the set of Δ-periodic continuous functions on $\overline{Q}$.

Proposition 2.8. [BM1] *Let $w \in L^p(T_N;\mathbb{R}^d)$, $1 \le p \le +\infty$, and set $w_n(x) := w(nx), n \in \mathbb{N}$. If $E \subset \mathbb{R}^N$ is a measurable set then*

$$w_n \rightharpoonup \int_{T_N} w(y)\, dy \quad \text{in } L^p(E;\mathbb{R}^d) \quad (\overset{*}{\rightharpoonup} \ \text{if } p = +\infty).$$

In particular, $\{w_n\}$ generates the homogeneous Young measure $\nu := \overline{\delta_w}$, where

$$\langle \overline{\delta_w}, \varphi \rangle := \int_{T_N} \varphi(w(y))\, dy \quad \text{for all } \varphi \in C_0(\mathbb{R}^d).$$

We recall some results on Fourier transforms of periodic functions (see [St, SWe]). If $f \in L^1(T_N)$ then its *Fourier coefficients* are defined as

$$\hat{f}(\lambda) := \int_{T_N} f(x) \mathrm{e}^{-2\pi i x\cdot\lambda}\, dx, \qquad \lambda \in \Delta,$$

and the following hold:

Theorem 2.9. *(i) The trigonometric polynomials*

$$R(x) := \sum_{\lambda \in \Delta'} a_\lambda e^{-2\pi i x\cdot\lambda}, \quad \Delta' \text{finite subset of } \Delta, a_\lambda \in \mathbb{C},$$

are dense in $C(T_N)$ and in $L^p(T_N)$ for all $1 \le p < +\infty$.
(ii) If $\mu \in \mathcal{M}(T_N)$ and $\langle \mu, e^{-2\pi i x\cdot\lambda} \rangle = 0$ for all $\lambda \in \Delta$ then $\mu \equiv 0$.
(iii) If $f \in L^2(T_N)$ then

$$f(x) = \sum_{\lambda \in \Delta} \hat{f}(\lambda) e^{2\pi i x\cdot\lambda}, \quad \sum_{\lambda \in \Delta} |\hat{f}(\lambda)|^2 = ||f||_{L^2}.$$

Corollary 2.10. *If $f \in L^1(T_N)$ and $\sum_{\lambda\in\Delta} |\hat{f}(\lambda)| < +\infty$ then there exists a representative $\overline{f}$ of f such that $\overline{f} \in C(T_N)$ and for all $x \in T_N$*

$$\overline{f}(x) = \sum_{\lambda \in \Delta} \hat{f}(\lambda) e^{2\pi i x\cdot\lambda}.$$

Corollary 2.11. *If $f \in C^k(T_N)$ for some $k > N/2$ then*

$$\sum_{\lambda \in \Delta} |\hat{f}(\lambda)| < +\infty.$$

Let $(L^p(T_N), L^q(T_N))$ denote the class of (p,q) *Fourier multiplier operators*, i.e. the class of all bounded linear operators $T : L^p(T_N) \to L^q(T_N)$ which commute with translations,

$$\Gamma_h T = T \Gamma_h \quad \text{for all } h \in \mathbb{R}^N,$$

where $\Gamma_h f(x) := f(x-h)$.

Theorem 2.12. *If $1 \le p,q \le +\infty$ and if $T \in (L^p(T_N), L^q(T_N))$ then there exists a bounded function $\Theta : \Delta \to \mathbb{C}$ such that*

$$Tf(x) := \sum_{\lambda \in \Delta} \Theta(\lambda) \hat{f}(\lambda) e^{2\pi i x\cdot\lambda} \quad \text{if } f \in L^p(T_N) \text{is given by } f(x) = \sum_{\lambda \in \Delta} \hat{f}(\lambda) e^{2\pi i x\cdot\lambda}.$$

The collection of coefficients $\{\Theta(\lambda)\}_{\lambda\in\Delta}$ is called the Fourier multiplier associated to T.

It can be shown that a certain class of continuous functions on the unit sphere S^{N-1} are Fourier multipliers. Precisely (see [St, Example iii), pp. 94], [SWe, Corollary 3.16, pp. 263, and remark just below]),

Proposition 2.13. *If Θ is homogeneous of degree zero and if it is infinitely differentiable on S^{N-1}, then the operator $T_\Theta : L^p(T_N) \to L^p(T_N)$ defined by*

$$T_\Theta f(x) := \sum_{\lambda \in \Delta \setminus \{0\}} \Theta(\lambda)\hat{f}(\lambda)e^{2\pi i x\cdot\lambda} \quad \text{if } f \in L^p(T_N) \text{ is given by } f(x) = \sum_{\lambda\in\Delta} \hat{f}(\lambda)e^{2\pi i x\cdot\lambda},$$

is a Fourier multiplier operator for $1 < p < +\infty$.

If (CR) holds, then in light of Propositions 2.7 and 2.13 the functions

$$\Theta_{ij} : w \in \mathbb{R}^N \mapsto \mathbb{P}(w)_{ij}, \quad i, j \in \{1, \dots, d\}$$

generate the Fourier multipliers $\{\Theta_{ij}(\lambda)\}_{\lambda\in\Delta\setminus\{0\}}$ associated to the Fourier multiplier operators $T_{\Theta_{ij}}$, and we define the operators

$$(\mathbb{T}u)_i(x) := (T_{\Theta_{ij}} u_j)(x) \quad \text{for } u \in L^p(T_N; \mathbb{R}^d), \quad i = 1, \dots, N,$$

where the summation convention for repeated indices is used.

Lemma 2.14. *Suppose that (CR) holds and let $1 < p < +\infty$. Then*
(i) $\mathbb{T} : L^p(T_N; \mathbb{R}^d) \to L^p(T_N; \mathbb{R}^d)$ is a linear, bounded operator that vanishes on constant mappings;
(ii) if $u \in L^p(T_N; \mathbb{R}^d)$ then $\mathbb{T} \circ \mathbb{T}u = \mathbb{T}u$, and $\mathcal{A}(\mathbb{T}u) = 0$;
(iii) $||u - \mathbb{T}u||_{L^p} \le C_p||\mathcal{A}u||_{W^{-1,p}}$ for all $u \in L^p(T_N; \mathbb{R}^d)$ such that $\int_{T_N} u\,dx = 0$ and for some $C_p > 0$;
(iv) suppose that $\{u_n\}$ is a sequence bounded in $L^p(T_N; \mathbb{R}^d)$ and $\{|u_n|^p\}$ is equi-integrable. Then $\{|\mathbb{T}u_n|^p\}$ is still equi-integrable.

Proof. (i) follows from the definition of $\mathbb{T}$ and from Propositions 2.7 and 2.13. Property (ii) is an immediate consequence of the fact that $\mathbb{P}$ is a projection.

To prove (iii), we note that by Corollaries 2.10 and 2.11 for $u \in C^\infty(T_N; \mathbb{R}^d)$ with $\int_{T_N} u\,dx = 0$ we have

$$\begin{aligned} u - \mathbb{T}u &= \Sigma_{\lambda\in\Delta\setminus\{0\}}\, \mathbb{Q}(\lambda)\mathbb{A}(\lambda)\hat{u}(\lambda)e^{2\pi i x\cdot\lambda} \\ &= \Sigma_{\lambda\in\Delta\setminus\{0\}}\, \mathbb{Q}\left(\frac{\lambda}{|\lambda|}\right)\mathbb{A}\left(\frac{\lambda}{|\lambda|}\right)\hat{u}(\lambda)e^{2\pi i x\cdot\lambda} \end{aligned}$$

where we have used the linearity of $\mathbb{A}$ and the fact that $\mathbb{Q}$ is homogeneous of degree -1. By Proposition 2.7 the inequality in (iii) is obtained, and the result for L^p periodic functions with zero average follows via a density argument.

To prove (iv) consider the truncation $\tau_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\tau_\alpha(z) := \begin{cases} z & \text{if } |z| \le \alpha \\ \alpha\dfrac{z}{|z|} & \text{if } |z| > \alpha. \end{cases}$$

Since $\{\tau_\alpha u_n\}$ is bounded in L^∞ we have that$\{\mathbb{T}(\tau_\alpha u_n)\}$ is bounded in L^q for all $p \le q < +\infty$, and so $\{|T(\tau_\alpha u_n)|^p\}$ is equi-integrable. On the other hand, by the equi-integrability of $\{u_n\}$ we have that

$$\lim_{\alpha\to\infty} \sup_n ||u_n - \tau_\alpha u_n||_p = 0$$

and by (i) we conclude that

$$\lim_{\alpha\to\infty} \sup_n ||\mathbb{T}(u_n - \tau_\alpha u_n)||_p = 0,$$

and the assertion is proved.

□

We note that, with the exception of Lemma 2.14 (iv), the above follows closely Murat's work (see [Mu]).

Decomposition results similar to the ones obtained below may be found in [FMP] and [K] in the particular case of curl-free fields.

Lemma 2.15. $[1 < p < +\infty]$ *Let* $1 < p < +\infty$, *let* $\{u_n\}$ *be a bounded sequence in* $L^p(T_N;\mathbb{R}^d)$ *such that* $\mathcal{A}u_n \to 0$ *in* $W^{-1,p}(T_N)$, $u_n \rightharpoonup u$ *in* $L^p(T_N;\mathbb{R}^d)$, *and assume that* $\{u_n\}$ *generates the Young measure* ν. *Then there exists a p-equi-integrable sequence* $\{v_n\} \subset L^p(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ *such that*

$$\int_\Omega v_n\,dx = \int_\Omega u\,dx, \quad ||v_n - u_n||_{L^q(\Omega)} \to 0 \quad \textit{for all}\ 1 \le q < p$$

and, in particular, $\{v_n\}$ *still generates* ν.

Proof. After an affine rescaling, we may suppose that $\Omega \subset Q$. The assumptions imply that $\mathcal{A}u = 0$, and by linearity (and Proposition 2.4) we may take $u = 0$. By Theorem 2.2 (v) we have

$$\int_\Omega \int_{\mathbb{R}^d} |z|^p\,d\nu_x(z)\,dx < +\infty$$

and so, using Theorem 2.2 (vi) we obtain

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty}\int_\Omega |\tau_k(u_n)|^p\,dx &= \lim_{k\to\infty}\int_\Omega\int_{\mathbb{R}^d}|\tau_k(z)|^p\,d\nu_x(z)\,dx\\ &= \int_\Omega\int_{\mathbb{R}^d}|z|^p\,d\nu_x(z)\,dx.\end{aligned}$$

Therefore we may find an increasing sequence $\alpha_n \to +\infty$ such that the truncated sequence $\{\tau_{\alpha_n} \circ u_n\}$ satisfies

$$\lim_{n\to\infty}\int_\Omega |\tau_{\alpha_n}\circ u_n|^p\,dx = \int_\Omega\int_{\mathbb{R}^d}|z|^p\,d\nu_x(z)\,dx. \tag{2.2}$$

On the other hand, as $\{u_n\}$ is equi-integrable,

$$\tau_{\alpha_n}\circ u_n - u_n \to 0 \quad \text{in measure and weakly in } L^p(\Omega).$$

Thus, by Proposition 2.4, the sequence $\{\tilde{u}_n\} := \{\tau_{\alpha_n} \circ u_n\}$ still generates the Young measure ν. By Theorem 2.2 (vi) and (2.2) we conclude that $\{\tilde{u}_n\}$ is p-equi-integrable. Moreover, if $1 < q < p$ then

$$\begin{aligned}||\tilde{u}_n - u_n||^q_{L^q(\Omega)} &\le \int_{\{|u_n|\ge\alpha_n\}} 2^q|u_n|^q\,dx\\ &\le \alpha_n^{q-p}\,2^q\int_{T_N}|u_n|^p\,dx \to 0 \quad \text{as } n\to+\infty,\end{aligned}$$

and thus $\mathcal{A}\tilde{u}_n \to 0$ in $W^{-1,q}(\Omega)$. Also, by virtue of the compact imbedding $L^q(\Omega) \hookrightarrow W^{-1,q}(\Omega)$, we have for all $\varphi \in C_0^\infty(\Omega;[0,1])$

$$\mathcal{A}(\varphi\tilde{u}_n) = \varphi\mathcal{A}(\tilde{u}_n) + \sum_{i=1}^N A^{(i)}(\tilde{u}_n)\,\frac{\partial\varphi}{\partial x_i} \to 0 \text{ in } W^{-1,q}(\Omega).$$

Thus we may select a sequence $\{\varphi_n\} \subset C_0^\infty(\Omega;[0,1])$ with $\varphi_n \nearrow 1$, and such that, setting $\hat{u}_n := \varphi_n\,\tilde{u}_n$, $\{\hat{u}_n\}$ is p-equi-integrable,

$$\hat{u}_n \rightharpoonup 0 \quad \text{in } L^p(\Omega),\ \mathcal{A}\hat{u}_n \to 0 \text{ in } W^{-1,q}(\Omega).$$

Extend $\hat{u}_n$ by zero to $Q \setminus \Omega$ and then periodically. We define

$$\tilde{v}_n := \mathbb{T}\left(\hat{u}_n - \int_{T_N}\hat{u}_n\,dy\right).$$

By Lemma 2.14 (iv) the sequence $\{\tilde{v}_n\} \subset L^p(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ is p-equi-integrable, and we have

$$\begin{aligned} ||\tilde{v}_n - u_n||_{L^q(\Omega)} &\leq ||\tilde{v}_n - \tilde{u}_n||_{L^q(\Omega)} + ||\tilde{u}_n - u_n||_{L^q(\Omega)} \\ &\leq ||\tilde{v}_n - \hat{u}_n||_{L^q(\Omega)} + ||\hat{u}_n - \tilde{u}_n||_{L^q(\Omega)} + ||\tilde{u}_n - u_n||_{L^q(\Omega)} \\ &=: I_1^n + I_2^n + I_3^n. \end{aligned} \tag{2.3}$$

We have already seen that $I_3^n \to 0$ as $n \to \infty$, and the p-equi-integrability of $\{\tilde{u}_n\}$ entails

$$\lim_{n\to\infty} I_2^n = 0.$$

Using Lemma 2.14 (iii) and the fact that $\int_{T_N} \hat{u}_n\,dy \to 0$, we obtain

$$\begin{aligned} \lim_{n\to\infty} I_1^n &\leq \lim_{n\to\infty} \left|\left| \hat{u}_n - \int_{T_N} \hat{u}_n\,dy - \mathbb{T}\left(\hat{u}_n - \int_{T_N} \hat{u}_n\,dy\right)\right|\right|_{L^q(T_N)} \\ &\leq \lim_{n\to\infty} C_q\,||\mathcal{A}\hat{u}_n||_{W^{-1,q}(T_N)} \\ &= 0. \end{aligned}$$

In particular, by Proposition 2.4 $\{\tilde{v}_n\}$ still generates ν. Finally, it suffices to set

$$v_n := \tilde{v}_n - \int_\Omega \tilde{v}_n\,dy.$$

Note that if the initial sequence $\{u_n\}$ is p-equi-integrable, then there is no need to construct the truncated sequence $\{\tilde{u}_n\}$, and from (2.3) it follows that $||v_n - u_n||_{L^p(T_N)} \to 0$.

□

Lemma 2.16. $[p = 1]$ *Let $\{u_n\}$ be a sequence converging weakly in $L^1(\Omega;\mathbb{R}^d)$ to a function u, $\mathcal{A}u_n \to 0$ in $W^{-1,r}(T_N)$ for some $r \in (1, N/(N-1))$, and assume that $\{u_n\}$ generates a Young measure ν. Then there exists an equi-integrable sequence $\{v_n\} \in L^1(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that*

$$\int_\Omega v_n\,dx = \int_\Omega u\,dx, \quad ||v_n - u_n||_{L^1(\Omega)} \to 0$$

and, in particular, $\{v_n\}$ still generates ν.

Proof. The proof is similar to the one given above, and once again we may assume that $\Omega \subset Q$ and $u = 0$. Due to the equi-integrability of $\{u_n\}$ we do not need to truncate the sequence, so we set $\tilde{u}_n := u_n$. Also, by mollification we may assume that $\hat{u}_n \in C_0^\infty(\Omega;\mathbb{R}^d)$, where in the diagonalization argument leading to the construction of $\hat{u}_n$ we use the compact imbedding $L^1(\Omega) \hookrightarrow W^{-1,r}(\Omega)$. We have

$$||\tilde{v}_n - u_n||_{L^1(\Omega)} \leq ||\tilde{v}_n - \hat{u}_n||_{L^1(\Omega)} + ||\hat{u}_n - u_n||_{L^1(\Omega)}$$

and the last term on the right hand side converges to zero due to the equi-integrability of $\{u_n\}$. Finally,

$$\begin{aligned} \lim_{n\to\infty} ||\tilde{v}_n - \hat{u}_n||_{L^1(\Omega)} &\leq \lim_{n\to\infty} C_r \left|\left| \hat{u}_n - \int_{T_N} \hat{u}_n\,dy - \mathbb{T}\left(\hat{u}_n - \int_{T_N} \hat{u}_n\,dy\right)\right|\right|_{L^r(T_N)} \\ &\leq C_r \lim_{n\to\infty} ||\mathcal{A}\hat{u}_n||_{W^{-1,r}(T_N)} \\ &= 0, \end{aligned}$$

where we have used the fact that $\int_{T_N} \hat{u}_n\,dy \to 0$.

□

Lemma 2.17. $[p = +\infty]$ *Let $\{u_n\}$ be a sequence that satisfies $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(T_N;\mathbb{R}^d)$, $\mathcal{A}u_n \rightharpoonup 0$ in $L^p(T_N)$ for some $p > N$, and assume that $\{u_n\}$ generates a Young measure ν. Then there exists a sequence $\{v_n\} \in L^\infty(T_N;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that*

$$\int_{T_N} v_n \, dx = \int_{T_N} u \, dx, \quad ||v_n - u_n||_{L^\infty(T_N)} \to 0$$

and, in particular, $\{v_n\}$ still generates ν.

Proof. As before assume that $u = 0$ and set

$$v_n := \mathbb{T}\left(u_n - \int_{T_N} u_n \, dy\right).$$

Since $\int_{T_N} u_n \, dy \to 0$, we have

$$\sup_{n\in\mathbb{N}} ||v_n - u_n||_{W^{1,p}(T_N)} \leq C_p \sup_{n\in\mathbb{N}} ||\mathcal{A}u_n||_{L^p(T_N)} < +\infty$$

and

$$\lim_{n\to\infty} ||v_n - u_n||_{L^p(T_N)} \leq C_p \lim_{n\to\infty} ||\mathcal{A}u_n||_{W^{-1,p}(T_N)} = 0,$$

and we conclude that the functions $v_n - u_n$ converge to zero uniformly.

□

The last result of this section will enable us in Section 4 to focus our attention on the characterization of $\mathcal{A}$-1-Young measures, where a Young measure ν is said to be a *$\mathcal{A}$-p-Young measure* if it is generated by a sequence in $\ker \mathcal{A}$ which is weakly convergent in $L^p(\Omega)$.

Corollary 2.18. *Let $1 < p < +\infty$. If ν is a $\mathcal{A} - 1-$Young measure with*

$$\int_\Omega \int_{\mathbb{R}^d} |z|^p \, d\nu_x(z) \, dx < +\infty$$

then ν is a $\mathcal{A}$-p-Young measure generated by a p-equi-integrable sequence.

Proof. Assume that ν is generated by an equi-integrable sequence $\{u_n\} \subset L^1(\Omega) \cap \ker \mathcal{A}$, and

$$\int_\Omega \int_{\mathbb{R}^d} |z|^p \, d\nu_x(z) \, dx < +\infty.$$

Following the beginning of the proof of Lemma 2.15, we may find a sequence of truncations $\{\tilde{u}_n\} \subset \ker \mathcal{A}$, bounded in $L^p(\Omega;\mathbb{R}^d)$, that still generates ν since, by equi-integrability,

$$||\tilde{u}_n - u_n||_{L^1(\Omega)} \to 0.$$

The result now follows by direct application of Lemma 2.15 to the sequence $\{\tilde{u}_n\}$.

□

§3. A - Quasiconvexity : a Necessary and Sufficient Condition for Lower Semicontinuity

Using the notation introduced in Section 2, consider an operator $\mathcal{A}$ satisfying the constant rank property (CR). In this section we will prove lower semicontinuity of functionals with normal integrands with respect to weakly convergent sequences with weak limit in the kernel of $\mathcal{A}$. In what follows Ω is a bounded, open subset of $\mathbb{R}^N$.

Given a normal integrand $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$, we define

$$I(u, v) := \int_\Omega f(x, u(x), v(x)) \, dx$$

for measurable $(u, v) : \Omega \to \mathbb{R}^m \times \mathbb{R}^d$.

Definition 3.1. *A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-quasiconvex if*

$$f(v) \leq \int_Q f(v + w(x))\, dx$$

for all $v \in \mathbb{R}^d$ and all $w \in C^\infty(T_N; \mathbb{R}^d)$ such that $\mathcal{A}(w) = 0$ and $\int_{T_N} w(x)\, dx = 0$.

Definition 3.2. *Given a Borel function $f : \mathbb{R}^d \to \mathbb{R}$ we define the $\mathcal{A}$-quasiconvex envelope of f at $v \in \mathbb{R}^d$ as*

$$\mathrm{Q}_{\mathcal{A}} f(v) := \inf \left\{ \int_{T_N} f(v + w(x))\, dx : w \in C^\infty(T_N) \cap \ker \mathcal{A}, \int_{T_N} w\, dx = 0 \right\}. \tag{3.1}$$

Clearly $f = \mathrm{Q}_{\mathcal{A}} f$ when f is $\mathcal{A}$-*quasiconvex*.

Remark 3.3. (i) It follows immediately from Jensen's inequality that convex functions are $\mathcal{A}$-quasiconvex.

(ii) If f is upper semicontinuous and locally bounded from above, then $C^\infty(T_N)$ may be replaced by $L^\infty(T_N)$ in Definition 3.1. Indeed, it suffices to approximate a given function $w \in L^\infty(T_N) \cap \ker \mathcal{A}$, with $\int_{T_N} w\, dx = 0$, by the mollified sequence

$$w_\varepsilon := \rho_\varepsilon * w - \int_{T_N} \rho_\varepsilon * w\, dy,$$

where $w_\varepsilon \in C^\infty(T_N) \cap \ker \mathcal{A}$, are Q-periodic, and have zero average. The result now follows by Fatou's Lemma. If, in addition, $|f(v)| \leq C(1 + |v|^p)$ for some $C > 0$ and all $v \in \mathbb{R}^d$, then $C^\infty(T_N)$ may be replaced by $L^p(T_N)$ in (3.1).

(iii) Given a matrix-valued function $V : \Omega \subset \mathbb{R}^N \to \mathbb{M}^{m\times n} \equiv \mathbb{R}^d$, $d := mn$, $n = N + \rho$, $\rho \geq 0$, we write

$$V = (F \mid \xi), \quad F \in \mathbb{M}^{m\times N},\ \xi \in \mathbb{M}^{m\times \rho},$$

where F is the matrix of the first N columns of V, and ξ is the matrix of the remaining ρ columns. In the context of membrane or film theories, $N = 2$, $m = 3$, $\rho = 1$, and F is the gradient of the membrane deformation. In the context of general nonlinear elasticity, $N = m = 3, \rho = 0$, and F is the gradient of the deformation of the elastic solid. The underlying PDE is then

$$\operatorname{curl} F = 0, \quad \text{i.e. } \frac{\partial F_{jk}}{\partial x_i} - \frac{\partial F_{ji}}{\partial x_k} = 0, \quad 1 \leq j \leq m, 1 \leq i, k \leq N.$$

We may rewrite these PDEs as $\mathcal{A}V = 0$, where $l := N^2 m$ and

$$A^{(r)}_{(j,k,i),(q,p)} := \delta_{ri}\delta_{qj}\delta_{pk} - \delta_{rk}\delta_{qj}\delta_{pi}, \quad 1 \leq j, q \leq m,\ 1 \leq i, k, p, r \leq N,$$

$$A^{(r)}_{(j,k,i),(q,p)} = 0 \quad \text{if } p = N+1, \cdots, n.$$

The constant rank condition (CR) is satisfied, since $\dim(\ker \mathbb{A}(w)) = m + m \times \rho$ for all $w \in S^{N-1}$. Indeed,

$$\begin{aligned} \ker \mathbb{A}(w) &= \{V \in \mathbb{M}^{m\times n} : \mathbb{A}(w)V = 0\} \\ &= \{V = (F \mid \xi) \in \mathbb{M}^{m\times n} : w_i F_{jk} - w_k F_{ji} = 0,\ 1 \leq j \leq m, 1 \leq i, k \leq N\} \\ &= \{V = (F \mid \xi) \in \mathbb{M}^{m\times n} : F = a \otimes w \text{ for some } a \in \mathbb{R}^m\}. \end{aligned}$$

When $\rho = 0$ and f is locally bounded then (3.1) reduces to the usual *quasiconvex envelope* of f,

$$\begin{aligned} \mathrm{Q}_{\mathcal{A}} f(v) :&= \inf \left\{ \int_{T_N} f(v + \nabla\varphi(x))\, dx : v \in C^\infty(T_N; \mathbb{R}^m) \right\} \\ &= \inf \left\{ \int_Q f(v + \nabla\varphi(x))\, dx : v \in C_0^\infty(Q; \mathbb{R}^m) \right\}. \end{aligned}$$

(iv) Now we consider the div-free case (see also [P]). Here $d = N, l = 1$,

$$A_j^{(i)} := \delta_{ij},$$

so that

$$\mathcal{A}u = 0 \quad \text{if and only if } \operatorname{div} u = 0.$$

Once again, the constant rank condition (CR) holds, as for all $w \in S^{N-1}$

$$\begin{aligned} \ker \mathbb{A}(w) &= \left\{ v \in \mathbb{R}^N : \sum_{i=1}^N A^{(i)} w_i(v) = 0 \right\} \\ &= \{ v \in \mathbb{R}^N : v \cdot w = 0 \}. \end{aligned}$$

Therefore $\dim(\ker \mathbb{A}(w)) = N - 1$.

Proposition 3.4. *If $f : \mathbb{R}^d \to \mathbb{R}$ is upper semicontinuous then $\mathrm{Q}_{\mathcal{A}} f$ is $\mathcal{A}$-quasiconvex and upper semicontinuous. Moreover, the restriction of $\mathrm{Q}_{\mathcal{A}} f$ to each cone $a + \Lambda$, $a \in \mathbb{R}^d$, is convex, i.e.*

$$\mathrm{Q}_{\mathcal{A}} f(\theta y + (1-\theta) z) \le \theta\, \mathrm{Q}_{\mathcal{A}} f(y) + (1-\theta)\, \mathrm{Q}_{\mathcal{A}} f(z)$$

for all $\theta \in (0,1)$, $y, z \in \mathbb{R}^d$ such that $y - z \in \Lambda$, where

$$\Lambda := \cup_{w \in S^{N-1}} \ker \mathbb{A}(w).$$

Remark 3.5. (i) The *characteristic cone* Λ as defined in Proposition 3.4 was introduced in the work of Murat and Tartar (see [Mu], [T1]).

(ii) There are $\mathcal{A}$-quasiconvex functions that are not continuous. Indeed, in the degenerate case $\ker \mathcal{A} = \{0\}$ all functions are $\mathcal{A}$-quasiconvex. Furthermore, in general $\mathrm{Q}_{\mathcal{A}} f$ need not be continuous in directions that are not in $\operatorname{span} \Lambda$ even when f is smooth. As an example, let $N = 1$, $d = 2$, and $\mathcal{A}u := u_2'$. Fix $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi(0) = 1$, $\lim_{|t| \to \infty} \varphi(t) = 0$, and let

$$f(v_1, v_2) := \varphi(v_1 v_2^2).$$

Then $\mathrm{Q}_{\mathcal{A}} f$ is obtained by convexification in the first component, and $\mathrm{Q}_{\mathcal{A}} f(v_1, v_2) = 0$ if $v_2 \ne 0$, while $\mathrm{Q}_{\mathcal{A}} f(v_1, 0) = 1$. In particular, this example shows that the relaxed energy

$$\mathcal{F}(u) := \inf_{\{v_n\}} \left\{ \liminf_{n \to \infty} \int_\Omega f(v_n)\, dx : v_n \rightharpoonup v \quad \text{in } L^p(\Omega; \mathbb{R}^d),\ \mathcal{A}(v_n) = 0 \right\}$$

may not agree with

$$\int_\Omega \mathrm{Q}_{\mathcal{A}} f(v)\, dx.$$

(iii) In the curl-free case and when $\rho = 0$, by Remark 3.3 (iii) we have that $\Lambda = \{ a \otimes w : a \in \mathbb{R}^m, w \in S^{N-1} \}$. Thus Proposition 3.4 entails that a quasiconvex Borel measurable function is convex along any rank-one directions. It is then said to be *rank-one convex*. In particular, it is separately convex, and so continuous. We remark that although Proposition 3.4 is stated for upper semicontinuous functions f, in the case of gradients the statement still holds if f is only assumed to be Borel measurable (see [F]).

(iv) In the div-free case and by Remark 3.3 (iv), we have $\Lambda = \mathbb{R}^N$, and by Proposition 3.4 we conclude that $\mathrm{Q}_{\mathcal{A}} f$ is convex (see also [P]). Thus, since we have always $\mathrm{Q}_{\mathcal{A}} f \le f$, $\mathrm{Q}_{\mathcal{A}} f$ reduces to the convexification of f.

(v) It follows from the convexity of $t \mapsto \mathrm{Q}_{\mathcal{A}} f(a+tz)$, $z \in \Lambda$ (see Proposition 3.4), that $\mathrm{Q}_{\mathcal{A}} f(a) > -\infty$ if and only if $\mathrm{Q}_{\mathcal{A}} f > -\infty$ on $a + \Lambda$.

Proof of Proposition 3.4.

Case 1: Suppose that f is continuous.

For $R > 0$, $v \in \mathbb{R}^d$, define

$$\mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v) := \inf \left\{ \int_{T_N} f(v + w(x))dx : w \in C^\infty(T_N) \cap \ker \mathcal{A}, \int_{T_N} w(x)dx = 0, \text{ and } ||w||_{L^\infty(T_N)} \le R \right\}.$$

We claim that

$$\mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f \quad \text{is continuous.} \tag{3.3}$$

Let $\rho > 0$, and let ω be the modulus of uniform continuity of f on $B(0, \rho + R)$, i.e.

$$\omega(r) := \sup\{|f(v) - f(v')| : v, v' \in \overline{B}(0, \rho + R), |v - v'| \le r\}.$$

For all $v, v' \in B(0, \rho)$ and every $w \in C^\infty(T_N) \cap \mathcal{A}$, with $\int_{T_N} w(x)\, dx = 0$ and $||w||_{L^\infty(T_N)} \le R$, we have

$$\begin{aligned} \int_{T_N} f(v + w(x))\, dx &\ge \int_{T_N} f(v' + w(x))\, dx - \omega(|v - v'|) \\ &\ge \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v') - \omega(|v - v'|). \end{aligned}$$

By definition of $\mathrm{Q}_{\mathcal{A}} f(v)$ this implies that

$$\mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v) - \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v') \ge \omega(|v - v'|)$$

and the uniform continuity of $\mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f$ in $B(0, \rho)$ follows by reversing the roles of v and v'

Fix $\varepsilon > 0$, let $n \in \mathbb{N}$, and decompose Q into n^N cubes along the coordinate axes, $Q = \cup Q_{n,i}$, $Q_{n,i} = a_{n,i} + (1/n)\, Q$. Now we choose smooth cut-off functions $\varphi_{n,i}$ with the following properties: $0 \le \varphi_{n,i} \le 1$, $\varphi_{n,i} = 1$ on $a_{n,i} + (1/n - 1/n^2)Q$, and $\sum_{i=1}^{n^N} \chi_{Q_{n,i}} \varphi_{n,i} \nearrow 1$. For $w \in C^\infty(T_N) \cap \ker \mathcal{A}$ with average zero on Q, consider the piecewise constant approximations

$$w_n(x) := \sum_{i=1}^{n^N} \chi_{Q_{n,i}}\, w_{n,i}, \quad \text{where } w_{n,i} := n^N \int_{Q_{n,i}} w(x)\, dx.$$

Then $||w_n - w||_{L^\infty(Q)} \to 0$, and by the continuity of $\mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f$ (see (3.3)) we have for $n \ge n_1(\varepsilon)$

$$\begin{aligned} \int_{T_N} \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v + w(x))\, dx &\ge \int_{T_N} \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v + w_n(x))\, dx - \varepsilon \\ &= \sum_{i=1}^{n^N} \frac{1}{n^N}\, \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v + w_{n,i}) - \varepsilon. \end{aligned} \tag{3.4}$$

On the other hand, due to the uniform continuity of f on compact sets there exists $\delta > 0$ such that

$$\eta, \zeta \in L^\infty(\overline{B}(0, 5R)), ||\eta - \zeta||_{L^\infty(Q)} < \delta \Rightarrow \left| \int_Q f(v + \eta(x))\, dx - \int_{T_N} f(v + \zeta(x))\, dx \right| < \varepsilon. \tag{3.5}$$

Choose $z_{n,i} \in C^\infty(T_N) \cap \ker \mathcal{A}$, with average zero, such that $||z_{n,i}||_{L^\infty(Q)} \le R$,

$$\mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v + w_{n,i}) \ge \int_{T_N} f(v + w_{n,i} + z_{n,i}(y))\, dy - \varepsilon, \tag{3.6}$$

and set

$$y_{n,k}(x) := w(x) + \sum_{i=1}^{n^N} \varphi_{n,i}(x) z_{n,i}(kn^N(x - a_{n,i})), \quad k \in \mathbb{N}.$$

Clearly

$$||y_{n,k}||_{L^\infty(T_N)} \le R + ||w||_{L^\infty(Q)},.$$

By Proposition 2.8 $z_{n,i}(kn^N(\cdot - a_{n,i})) \overset{*}{\rightharpoonup} 0$ in $L^\infty(Q_{n,i})$ as $k \to \infty$, for all $n \in \mathbb{N}, i = 1, \dots, n^N$, and so

$$\lim_{k\to\infty} \mathcal{A} y_{n,k} = 0 \quad \text{weak-* in } L^\infty(T_N), \quad \lim_{n\to\infty} \lim_{k\to\infty} \int_{T_N} y_{n,k}\, dx = 0. \tag{3.7}$$

Choose $n = n_2(\varepsilon) \ge n_1(\varepsilon)$ such that

$$n_2(\varepsilon) \to \infty \text{ as } \varepsilon \to 0, \quad ||w_n - w||_{L^\infty(Q)} < \delta, \quad \lim_{k\to\infty} \left| \int_{T_N} y_{n,k}\, dx \right| < \delta. \tag{3.8}$$

Now (3.4), (3.5), (3.6) and (3.8) yield
(3.9)

$$\begin{aligned}
&\int_{T_N} \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v + w(x))\, dx \ge \lim_{k\to\infty} \sum_{i=1}^{n^N} \int_{Q_{n,i}} f(v + w_{n,i} + z_{n,i}(kn^N(x - a_{n,i})))\, dx - 2\varepsilon \\
&\ge \limsup_{k\to\infty} \int_{T_N} f(v + y_{n,k}(x))\, dx - 3\varepsilon - Cn^N \left[\frac{1}{n^N} - \left(\frac{1}{n} - \frac{1}{n^2} \right)^N \right] \max\{|f(z)| : z \in \overline{B}(v, 2R)\}.
\end{aligned}$$

In view of Lemma 2.17 and (3.7) we may find $u_k \in L^\infty(T_N; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$\int_{T_N} u_k\, dx = 0, \quad u_k - \left(y_{n,k} - \int_{T_N} y_{n,k}(y)\, dy \right) \to 0 \quad \text{uniformly as} \quad k \to \infty.$$

Thus, by (3.5), (3.8), (3.9), and Remark 3.3 (ii) we have

$$\begin{aligned}
\int_{T_N} \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(v + w(x))\, dx &\ge \limsup_{k\to\infty} \int_{T_N} f(v + y_{n,k}(x))\, dx - 3\varepsilon - O\left(\frac{1}{n}\right) \\
&\ge \limsup_{k\to\infty} \int_{T_N} f\left(v + y_{n,k}(x) - \int_{T_N} y_{n,k}(y)\, dy\right) dx - 4\varepsilon - O\left(\frac{1}{n}\right) \\
&\ge \limsup_{k\to\infty} \int_{T_N} f(v + u_k(x))\, dx - 5\varepsilon - O\left(\frac{1}{n}\right) \\
&\ge \mathrm{Q}_{\mathcal{A}} f(v) - 5\varepsilon - O\left(\frac{1}{n}\right).
\end{aligned} \tag{3.10}$$

For $\varepsilon \to 0$ we have, by (3.8), $n = n_2(\varepsilon) \to +\infty$. Hence taking first the limit $\varepsilon \to 0$ and then $R \to \infty$ in (3.10), and observing that $\mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f \searrow \mathrm{Q}_{\mathcal{A}} f$ as $R \to \infty$, we deduce from Lebesgue's monotone convergence theorem that

$$\int_{T_N} \mathrm{Q}_{\mathcal{A}} f(v + w(x))\, dx \ge \mathrm{Q}_{\mathcal{A}} f(v).$$

<u>Case 2</u>: f is upper semicontinuous.

Let $\{f_n\}$ be a sequence of continuous functions converging decreasingly to f. By Case 1, given $v \in \mathbb{R}^d$, $w \in C^\infty(T_N) \cap \ker \mathcal{A}$, with $\int_{T_N} w\,dx = 0$, we have

$$\int_{T_N} \mathrm{Q}_{\mathcal{A}} f_n(v + w(x))\,dx \geq \mathrm{Q}_{\mathcal{A}} f_n(v), \quad n \in \mathbb{N}.$$

In view of Lebesgue's monotone convergence theorem, $\mathcal{A}-$quasiconvexity of $\mathrm{Q}_{\mathcal{A}} f$ will follow provided we show that

$$\mathrm{Q}_{\mathcal{A}} f_n \searrow \mathrm{Q}_{\mathcal{A}} f. \tag{3.11}$$

Cleary $\{\mathrm{Q}_{\mathcal{A}} f_n\}_{n\in\mathbb{N}}$ is decreasing and larger than $\mathrm{Q}_{\mathcal{A}} f$. On the other hand, for fixed $v \in \mathbb{R}^d$ with $\mathrm{Q}_{\mathcal{A}} f(v) > -\infty$, given $\delta > 0$ there exists $\eta \in C^\infty(T_N) \cap \ker \mathcal{A}$, with $\int_{T_N} \eta\,dx = 0$, such that

$$\mathrm{Q}_{\mathcal{A}} f(v) \geq \int_{T_N} f(v + \eta(x))\,dx - \delta.$$

By Lebesgue's monotone convergence theorem it follows that

$$\begin{aligned}\mathrm{Q}_{\mathcal{A}} f(v) &\geq \lim_{n\to\infty} \int_{T_N} f_n(v + \eta(x))\,dx - \delta \\ &\geq \limsup_{n\to\infty} \mathrm{Q}_{\mathcal{A}} f_n(v) - \delta.\end{aligned}$$

It suffices to let $\delta \to 0$. The case where $\mathrm{Q}_{\mathcal{A}} f(v) = -\infty$ is treated in a similar way. As proven in Case 1, the functions $\mathrm{Q}_{\mathcal{A}} f_n$ are upper semicontinuous, so $\mathrm{Q}_{\mathcal{A}} f = \inf_{n\in\mathbb{N}} \mathrm{Q}_{\mathcal{A}} f_n$ is also upper semicontinuous.

Finally, we show that $\mathrm{Q}_{\mathcal{A}} f$ is convex on the cones $a + \Lambda$, $a \in \mathbb{R}^d$, i.e.

$$\mathrm{Q}_{\mathcal{A}} f(\theta y + (1-\theta)z) \leq \theta\, \mathrm{Q}_{\mathcal{A}} f(y) + (1-\theta)\, \mathrm{Q}_{\mathcal{A}} f(z)$$

for all $\theta \in (0,1)$, $y, z \in \mathbb{R}^d$ such that $y - z \in \Lambda$. By (3.11) it suffices to prove this inequality in the case where f is a continuous function.

Let

$$\chi(t) := \begin{cases} -(1-\theta) & \text{if } \ 0 < t < \theta \\ \theta & \text{if } \ \theta < t < 1 \end{cases}$$

and extend χ periodically to $\mathbb{R}$ with period one. Let $w \in S^{N-1}$ be such that $y - z \in \ker \mathbb{A}(w)$ and define

$$u_n(x) := (z - y)\,\chi(nx \cdot w).$$

Clearly $u_n \overset{*}{\rightharpoonup} 0$ in $L^\infty(Q)$, and if $\varphi \in C_0^\infty(Q;[0,1])$ is such that $\mathcal{L}^N(\{\varphi = 1\}) = 1 - \delta$, $\delta > 0$, then

$$\mathcal{A}(\varphi u_n) = \sum_{i=1}^N A^{(i)} u_n \frac{\partial \varphi}{\partial x_i} \overset{*}{\rightharpoonup} 0 \qquad \text{in } L^\infty(T_N).$$

Due to Lemma 2.17 we may find $\overline{u}_n \in L^\infty(T_N;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$\int_{T_N} \overline{u}_n = 0, \quad ||\overline{u}_n - \varphi\, u_n||_{L^\infty(Q)} \to 0.$$

By Remark 3.3 (ii), since $\mathrm{Q}_{\mathcal{A}} f$ is a $\mathcal{A}$-quasiconvex function, upper semicontinuous and bounded above by the locally bounded function f, by (3.3) and if $R > 0$ is large enough, we have

$$\begin{aligned} \mathrm{Q}_{\mathcal{A}} f(\theta y + (1-\theta)z) &\leq \liminf_{n\to\infty} \int_{T_N} \mathrm{Q}_{\mathcal{A}} f(\theta y + (1-\theta)z + \overline{u}_n)\, dx \\ &\leq \liminf_{n\to\infty} \int_{T_N} \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(\theta y + (1-\theta)z + \overline{u}_n)\, dx \\ &\leq \liminf_{n\to\infty} \int_{T_N} \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(\theta y + (1-\theta)z + u_n)\, dx + M\delta \\ &= \theta\, \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(\theta y + (1-\theta)z - (1-\theta)(z-y)) \\ &\quad + (1-\theta)\, \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(\theta y + (1-\theta)z + (z-y)\theta) + M\delta \\ &= \theta\, \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(y) + (1-\theta)\, \mathrm{Q}^{\mathrm{R}}_{\mathcal{A}} f(z) + M\delta \end{aligned}$$

where $M := \max\{|f(z)| : z \in \overline{B}(0,R)\}$. It suffices to let $\delta \searrow 0$ and then $R \to \infty$.

□

Next we prove that $\mathcal{A}$-quasiconvexity is a necessary condition for lower semicontinuity under the PDE constraint $\mathcal{A}u = 0$.

Theorem 3.6. [Necessity] *Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function such that*

$$\int_\Omega f(x, v(x))\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, v_n(x))\, dx$$

for all sequences $\{v_n\} \subset C^\infty(\overline{\Omega}; \mathbb{R}^d)$ that satisfy

$$v_n \overset{*}{\rightharpoonup} v \text{ in } L^\infty(\Omega) \quad \text{and } \mathcal{A}v_n = 0.$$

Assume further that

$$\{f(\cdot, u_n)\} \text{ is equi-integrable}$$

whenever $\{u_n\}$ is a sequence bounded in $L^\infty(\Omega; \mathbb{R}^d)$. Then $f(x_0, \cdot)$ is $\mathcal{A}$-quasiconvex for a.e. *$x_0 \in \Omega$.*

Proof. Without loss of generality, and using a rescaling argument, we may assume that $\Omega \subset Q$.

By the Scorza-Dragoni Theorem, for all $i \in \mathbb{N}$ there exists a compact set $K_i \subset \Omega$ such that the restriction of f to $K_i \times \mathbb{R}^d$ is continuous and $\mathcal{L}^N(\Omega \setminus K_i) < 1/i$. Let $\mathcal{S}$ be a countable, dense subset (with respect to uniform convergence) of $\mathbb{W} := \left\{ w \in C^\infty(T_N) : \mathcal{A}w = 0,\ \int_{T_N} w\, dx = 0 \right\}$. Let $x_0 \in \Omega$ be a Lebesgue point for

$$x \mapsto f(x, v), \quad x \mapsto \int_Q f(x, v + w(y))\, dy$$

for all $v \in \mathbb{Q}^d$, $w \in \mathcal{S}$, and suppose that $z \mapsto f(x_0, z)$ is continuous. Fix $v \in \mathbb{Q}^N$, $w \in \mathcal{S}$. We claim that

$$f(x_0, v) \leq \int_Q f(x_0, v + w(x))\, dx.$$

If so, by continuity of $z \mapsto f(x_0, z)$ this inequality still holds true for all $v \in \mathbb{R}^d$ and all $w \in \mathbb{W}$. To establish the inequality extend w to $\mathbb{R}^d$ periodically with period Q, fix $\varepsilon > 0$, $h \in \mathbb{N}$, and choose $i = i(h, \varepsilon) \in \mathbb{N}$ such that

$$\mathcal{L}^N \left(Q\left(x_0, \frac{1}{h}\right) \setminus K_i \right) < \frac{\varepsilon}{h^N}.$$

Let $n = n(h, \varepsilon)$ be such that

$$|x - x'| < \frac{1}{n},\ x, x' \in K_i,\ z \in \overline{B}\left(0, |v| + ||w||_{L^\infty(Q)}\right) \quad \Rightarrow |f(x,z) - f(x',z)| < \varepsilon.$$

Decompose the cube $Q\left(x_0, \frac{1}{h}\right)$ as $\cup_{j=1}^{n^N} Q\left(x_j, \frac{1}{hn}\right)$, and if $K_i \cap Q\left(x_j, \frac{1}{hn}\right) \neq \varnothing$ select a_j in this intersection. Choose a cut-off function $\varphi \in C_0^\infty(Q(x_0, 1/h))$ such that $\mathcal{L}^N(Q(x_0,1/h) \cap \{\varphi \neq 1\}) < \frac{\varepsilon}{h^N}$.

Define

$$w_m(x) := \begin{cases} \varphi(x)\, w^*(hmn(x - x_j)) & \text{if } x \in Q\left(x_j, \dfrac{1}{hn}\right), j = 1, \cdots, n^N, \\ 0 & x \in \mathbb{R}^N \setminus Q\left(x_0, \dfrac{1}{h}\right) \end{cases}$$

where $w^*(y) := w(y + (1/2, \cdots, 1/2))$ for $y \in Q^*$. By Proposition 2.8 it is clear that

$$w_m \stackrel{*}{\rightharpoonup} 0 \text{ in } L^\infty(T_N), \quad \mathcal{A} w_m \stackrel{*}{\rightharpoonup} 0 \text{ in } L^\infty(T_N).$$

Using Lemma 2.17 we may find $\eta_m \in L^\infty(Q(0,L); \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $||\eta_m - w_m||_{L^\infty(\Omega)} \to 0$, and so

$$\begin{aligned} \int_\Omega f(x,v)\, dx &\le \liminf_{m\to\infty} \int_\Omega f(x, v + \eta_m(x))\, dx \\ &= \liminf_{m\to\infty} \int_\Omega f(x, v + w_m(x))\, dx \end{aligned}$$

where we used Propositions 2.4, 2.8, and Theorem 2.2 (vi). Taking into account the estimates for $\{\varphi \neq 1\}$ and $Q(x_0, 1/h) \setminus K_i$, we deduce that

$$\begin{aligned} \int_{Q(x_0,1/h)} f(x,v)\, dx &\le \liminf_{m\to\infty} \int_{Q(x_0,1/h)} f(x, v + w_m(x))\, dx \\ &\le \liminf_{m\to\infty} \left\{ \sum_{j=1}^{n^N} \int_{Q\left(x_j, \frac{1}{hn}\right) \cap K_i} f(a_j, v + w^*(hmn(x - x_j)))\, dx \right. \\ &+ \sum_{j=1}^{n^N} \int_{Q\left(x_j, \frac{1}{hn}\right) \cap K_i} |f(x, v + w^*(hmn(x-x_j))) - f(a_j, v + w^*(hmn(x - x_j)))|\, dx \\ &\left. + \sum_{j=1}^{n^N} \int_{Q\left(x_j, \frac{1}{hn}\right) \setminus K_i} f(x, v + w^*(hmn(x - x_j)))\, dx \right\} + M\, \frac{\varepsilon}{h^N} \\ &\le \sum_{j=1}^{n^N} \frac{1}{(hn)^N} \int_Q f(a_j, v + w(y))\, dy + 3M\, \frac{\varepsilon}{h^N} + \frac{\varepsilon}{h^N} \end{aligned}$$

where $M := \operatorname{esssup} \left\{ |f(x,z)| : x \in B(x_0, R_0) \subset\subset \Omega, |z| \le |v| + ||w||_{L^\infty(T_N)} \right\}$.

Hence

$$\begin{aligned} \int_{Q(x_0,1/h)} f(x,v)\, dx &\le \sum_{j=1}^{n^N} \int_{Q\left(x_j, \frac{1}{hn}\right) \cap K_i} \int_Q f(a_j, v + w(y))\, dy\, dx + \frac{O(\varepsilon)}{h^N} \\ &\le \sum_{j=1}^{n^N} \int_{Q\left(x_j, \frac{1}{hn}\right)} \int_Q f(x, v + w(y))\, dy\, dx + \frac{O(\varepsilon)}{h^N} \qquad (3.12) \\ &= \int_{Q\left(x_0, \frac{1}{h}\right)} \int_Q f(x, v + w(y))\, dy\, dx + \frac{O(\varepsilon)}{h^N}. \end{aligned}$$

Multiplying through (3.12) by h^N, letting $h \to +\infty$, and then $\varepsilon \to 0$, we conclude that

$$f(x_0, v) \le \int_Q f(x_0, v + w(y))\, dy.$$

□

Now we prove sufficiency of the $\mathcal{A}$-quasiconvexity property.

Theorem 3.7. [Sufficiency] *Let $1 \le p \le +\infty$ and suppose that $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)$ is a normal integrand such that $z \mapsto f(x, u, z)$ is $\mathcal{A}$-quasiconvex and continuous for* a. e. $x \in \Omega$ *and for all $u \in \mathbb{R}^d$. If $1 \le p < +\infty$, then assume further that there exists a locally bounded function $a : \Omega \times \mathbb{R}^d \to [0, +\infty)$ such that*

$$0 \le f(x, u, v) \le a(x, u)(1 + |v|^p).$$

If

$$u_n \to u \textit{ in measure}$$

and

$$v_n \rightharpoonup v \textit{ in} L^p(\Omega; \mathbb{R}^d)(\overset{*}{\rightharpoonup} \;\; \textit{if} \, p = +\infty), \quad \mathcal{A}v_n \to 0 \textit{ in} \, W^{-1,p}(\Omega) \; (\mathcal{A}v_n = 0 \quad \textit{if} \, p = +\infty),$$

then

$$I(u, v) \le \liminf_{n \to \infty} I(u_n, v_n).$$

This theorem is a consequence of Propositions 3.8 and 3.9.

Proposition 3.8. *Let $1 \le p < +\infty$, let $\{v_n\}$ be a p-equi-integrable sequence in $L^p(\Omega; \mathbb{R}^d)$ such that $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega)$ if $1 < p < +\infty$, $\mathcal{A}v_n \to 0$ in $W^{-1,r}(\Omega)$ for some $r \in (1, N/(N-1))$ if $p = 1$, and $\{v_n\}$ generates the Young measure $\nu = \{\nu_x\}_{x \in \Omega}$. Let $v_n \rightharpoonup v$ in $L^p(T_N; \mathbb{R}^d)$. Then for* a. e. $a \in \Omega$ *there exists a sequence $\{\overline{v}_n\} \subset L^p(T_N; \mathbb{R}^d) \cap \ker \mathcal{A}$ that is p-equi-integrable, generates the homogeneous Young measure ν_a, and satisfies*

$$\int_{T_N} \overline{v}_n \, dx = \langle \nu_a, \mathrm{id} \rangle = v(a).$$

In particular, one has

$$\langle \nu_a, f \rangle \ge f(\langle \nu_a, \mathrm{id} \rangle) = f(v(a))$$

for a. e. $a \in \Omega$, *and for every continuous A-quasiconvex f that satisfies*

$$|f(z)| \le C(1 + |z|^p)$$

for some $C > 0$ and all $z \in \mathbb{R}^d$.

Proposition 3.9. *Let $\{v_n\}$ be a bounded sequence in $L^\infty(\Omega; \mathbb{R}^d)$ that generates a Young measure $\nu = \{\nu_x\}_{x \in \Omega}$, and satisfies $\mathcal{A}v_n = 0$. Let $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(T_N; \mathbb{R}^d)$. Then for* a. e. $a \in \Omega$ *and every subcube $Q' \subset\subset Q$ there exists a sequence $\{\overline{v}_n\} \subset L^\infty(T_N; \mathbb{R}^d)$ such that*

$$\overline{v}_n \overset{*}{\rightharpoonup} v(a) \textit{ in } L^\infty(T_N), \; \mathcal{A}\overline{v}_n = 0, \; \int_{T_N} \overline{v}_n \, dx = \langle \nu_a, \mathrm{id} \rangle = v(a),$$

and $\{\overline{v}_n\}$ generates a Young measure μ such that

$$\left| \int_Q \psi(x) \, \langle \mu_x, g \rangle \, dx - \langle \nu_a, g \rangle \int_Q \psi(x) \, dx \right| \le ||g||_{L^\infty(B(0,3M))} \int_{Q \setminus Q'} |\psi(x)| \, dx$$

for all $\psi \in L^1(Q)$, $g \in C_0(\mathbb{R}^d)$, *and where* $M := \sup_{n\in\mathbb{N}} ||v_n||_{L^\infty(\Omega)}$. *In addition, if* $f : \mathbb{R}^d \to \mathbb{R}$ *is a continuous function then*

$$\langle \nu_a, f\rangle \geq f(\langle \nu_a, \mathrm{id}\rangle) = f(v(a))$$

for a. e. $a \in \Omega$.

We leave the proofs of Propositions 3.8 and 3.9 to the end of this section, and we proceed with the proof of Theorem 3.7. We follow the argument of Kristensen (based on Balder's [Ba] reasoning for the case without constraints) in the context of the usual curl-free $\mathcal{A}$-quasiconvexity.

Proof of Theorem 3.7. Upon extracting a subsequence, we may assume that

$$\liminf_{n\to\infty} I(u_n, v_n) = \lim_{n\to\infty} I(u_n, v_n),$$

and $\{v_n\}$ generates a Young measure ν. By Proposition 2.5 the pair $\{(u_n, v_n)\}$ generates the Young measure $\{\mu_x = \delta_{u(x)} \otimes \nu_x\}_{x\in\Omega}$, and by Theorem 2.2 (v) we have

$$\begin{aligned}\lim_{n\to\infty} I(u_n, v_n) &\geq \int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^d} f(x, \eta, \xi)\, d\mu_x(\eta, \xi)\, dx \\ &= \int_\Omega \int_{\mathbb{R}^d} f(x, u(x), \xi)\, d\nu_x(\xi)\, dx.\end{aligned}$$

If $p = 1$ or $p = +\infty$ the result follows from direct application of Proposition 3.8 and Proposition 3.9, respectively, to the map $\xi \mapsto f(x, u(x), \xi)$ and integration over Ω. If $1 < p < +\infty$ then by Lemma 2.15 and by Proposition 2.4, there exists a p-equi-integrable sequence $\{y_n\}$ which generates ν and satisfies $\mathcal{A}y_n = 0$. Once again, it suffices to apply Proposition 3.8 to $\{y_n\}$ and to the map $\xi \mapsto f(x, u(x), \xi)$ for a. e. $x \in \Omega$ fixed.

□

Proof of Proposition 3.8. Let $\mathcal{E}$ and $\mathcal{C}$ be countable dense subsets of $L^1(Q)$ and $C_0(\mathbb{R}^d)$, respectively. By Theorem 2.2 (vi) we have

$$g \circ v_n \overset{*}{\rightharpoonup} \langle \nu, g\rangle \text{ in } L^\infty(\Omega)$$

for all $g \in \mathcal{C}$. Let Ω_0 be the set of points $a \in \Omega$ which are Lebesgue points for v, for the functions

$$x \mapsto \int_{\mathbb{R}^d} |\xi|^p\, d\nu_x(\xi), \quad x \mapsto \langle \nu_a, \mathrm{id}\rangle,$$

and for all functions $x \mapsto \langle \nu_x, g\rangle$, $g \in \mathcal{C}$, in the sense that

$$\lim_{R\to 0} \int_Q |\langle \nu_{a+Rx}, g\rangle - \langle \nu_a, g\rangle|\, dx = 0.$$

Consider an increasing sequence of smooth cut-off functions $\varphi_j \in C_0^\infty(Q)$, $\varphi_j \nearrow 1$. For fixed $a \in \Omega_0$, $R > 0$, we define

$$v_{j,R,n}(z) := \varphi_j(z)(v_n(a + Rz) - \langle \nu_a, \mathrm{id}\rangle),\ z \in Q.$$

Recall that $\langle \nu_a, \mathrm{id}\rangle = v(a)$. We have $v_{j,R,n} \in L^p(T_N; \mathbb{R}^d)$, and for all $\psi \in \mathcal{E}$ and $g \in \mathcal{C}$ we have

$$\begin{aligned}\lim_{j\to\infty} \lim_{R\to 0} \lim_{n\to\infty} \int_Q \psi(z) g(v_{j,R,n}(z) + v(a))\, dz &= \lim_{R\to 0} \lim_{n\to\infty} \int_Q \psi(z)\, g(v_n(a + Rz))\, dz \\ &= \lim_{R\to 0} \int_Q \psi(z) \langle \nu_{a+Rz}, g\rangle\, dz \qquad (3.13) \\ &= \langle \nu_a, g\rangle \int_Q \psi(z)\, dz.\end{aligned}$$

Moreover, as $\{|v_n|^p\}$ is equi-integrable,

(3.14)

$$\limsup_{j\to\infty}\limsup_{R\to 0}\limsup_{n\to\infty}\int_Q |v_{j,R,n}(z)+v(a)|^p\,dz \le \lim_{R\to 0}\lim_{n\to\infty}\int_Q |v_n(a+Rz)|^p\,dz = \int_{\mathbb{R}^d}|\xi|^p\,d\nu_a(\xi).$$

Also, $v_{j,R,n} \rightharpoonup 0$ in L^p as $n\to\infty$ and $R\to 0$. If $1<p<+\infty$ we have, in view of the compact imbedding $L^p(T_N)\hookrightarrow W^{-1,p}(T_N)$ and the assumption $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega)$,

$$\lim_{j\to\infty}\lim_{R\to 0}\lim_{n\to\infty}\mathcal{A}v_{j,R,n} = 0 \quad \text{in } W^{-1,p}(T_N). \tag{3.15}$$

If $p=1$ then

$$v_{j,R,n}\to 0 \text{ in } W^{-1,r}(T_N) \quad \text{for } r\in\left(1,\frac{N}{N-1}\right),$$

and so, due to (3.13), (3.14), (3.15), and by means of a diagonalization procedure, we may find a sequence of functions $\{w_j\}$ with the properties

$$w_j \rightharpoonup 0 \;\text{ in } L^p(T_N), \quad \mathcal{A}w_j\to 0 \quad \text{in } W^{-1,q}(T_N)$$

where $q=p$ if $1<p<+\infty$ and $q=r$ if $p=1$, and

(3.16)

$$\lim_{j\to\infty}\int_Q |w_j(x)+v(a)|^p\,dx = \int_{\mathbb{R}^d}|\xi|^p\,d\nu_a(\xi),\ \lim_{j\to\infty}\int_Q \psi(x)g(w_j(x)+v(a))\,dx = \langle \nu_a, g\rangle\int_Q\psi(x)\,dx$$

for all $\psi\in\mathcal{E}$ and $g\in\mathcal{C}$. By Lemmas 2.15, 2.16, and by (3.16) we conclude that ν_a is generated by a p-equi-integrable sequence $\overline{w}_j \in L^p(T_N;\mathbb{R}^d)\cap\ker\mathcal{A}$ such that $\int_{T_N}\overline{w}_j\,dx = v(a)$. Finally, if f is a continuous function such that $|f(z)|\le C(1+|z|^p)$ for some $C>0$ and all $z\in\mathbb{R}^d$, then $\{f(\overline{w}_j)\}$ is equi-integrable and by Theorem 2.2 (vi) we have

$$\langle\nu_a, f\rangle = \lim_{j\to\infty}\int_{T_N} f(\overline{w}_j)\,dx \ge f(v(a))$$

where in the last inequality we used the $\mathcal{A}$-quasiconvexity of f together with Remark 3.3 (ii).

□

Proof of Proposition 3.9. As in the previous proof, let $\mathcal{E}$ and $\mathcal{C}$ be countable dense subsets of $L^1(Q)$ and $C_0(\mathbb{R}^d)$, , respectively, and let Ω_0 be the set of points $a\in\Omega$ which are Lebesgue points for $x\mapsto\langle\nu_x,\mathrm{id}\rangle$ and for all functions $x\mapsto\langle\nu_x,g\rangle$, $g\in\mathcal{C}$. Fix $Q'\subset\subset Q$ and consider a smooth cut-off function $\varphi\in C_0^\infty(Q)$, $0\le\varphi\le 1$, $\varphi=1$ in Q'.

For $a\in\Omega_0$, $R>0$, we define

$$v_{R,n}(z) := \varphi(z)\,(v_n(a+Rz)-\langle\nu_a,\mathrm{id}\rangle)+\langle\nu_a,\mathrm{id}\rangle,\ z\in Q.$$

Then $v_{R,n}$ is bounded in $L^\infty(T_N;\mathbb{R}^d)$, and for all $\psi\in\mathcal{E}$ and $g\in\mathcal{C}$ we have

$$\begin{aligned}\lim_{R\to 0}\lim_{n\to\infty}\int_Q\psi(z)g(v_{R,n}(z))\,dz &= \lim_{R\to 0}\lim_{n\to\infty}\int_Q\psi(z)\,g(v_n(a+Rz))\,dz+\mathcal{E}(\psi,g)\\ &= \lim_{R\to 0}\int_Q\psi(z)\langle\nu_{a+Rz},g\rangle\,dz+\mathcal{E}(\psi,g)\\ &= \langle\nu_a,g\rangle\int_Q\psi(z)\,dz+\mathcal{E}(\psi,g),\end{aligned}$$

where

$$|\mathcal{E}(\psi, g)| \leq ||g||_{L^\infty(B(0,3M))} \int_{Q\setminus Q'} |\psi| \, dy.$$

Clearly, $v_n(a + R\cdot) - \langle \nu_a, \mathrm{id}\rangle \overset{*}{\rightharpoonup} 0$ in L^∞ as $n \to \infty$ and $R \to 0$, and

$$\lim_{R\to 0} \lim_{n\to\infty} \mathcal{A}v_{R,n} = 0 \quad \text{weakly-* in } L^\infty(T_N), \quad \sup_{R,n} ||\mathcal{A}v_{R,n}||_{L^\infty(T_N)} < +\infty.$$

Diagonalizing $\{v_{R,n}\}$, and extracting a further subsequence if necessary, we may find a sequence of functions $\{w_j\}$ with the properties

$$w_j \overset{*}{\rightharpoonup} v(a) \text{ in } L^\infty(T_N), \quad \mathcal{A}w_j \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(T_N),$$

and $\{w_j\}$ generates a Young measure μ such that ess supp$\mu_x \subset B(0, 3M)$ and

$$\left| \int_Q \psi(x) \, \langle \mu_x, g\rangle \, dx - \langle \nu_a, g\rangle \int_Q \psi(x) \, dx \right| \leq |\mathcal{E}(\psi, g)|$$

for all $g \in \mathcal{C}, \psi \in \mathcal{E}$. By density this inequality extends to all $\psi \in L^1(Q)$, $g \in C_0(\mathbb{R}^d)$. Due to Lemma 2.17 we may find $\overline{w}_j \in L^\infty(T_N; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $||w_j - \overline{w}_j||_{L^\infty(T_N)} \to 0$, $\int_{T_N} \overline{w}_j \, dy = v(a)$. In particular, $\{\overline{w}_j\}$ generates the Young measure μ satisfying the statement, and if f is continuous then

$$\begin{aligned} \lim_{j\to+\infty} \int_{T_N} f(\overline{w}_j) \, dx &= \int_{T_N} \langle \mu_x, f\rangle \, dx \\ &\leq \langle \nu_a, f\rangle + \mathcal{L}^N(Q \setminus Q') ||f||_{L^\infty(B(0,3R))}. \end{aligned} \tag{3.18}$$

On the other hand, since f is $\mathcal{A}$-quasiconvex and in view of Remark 3.3 (ii) we have directly from Definition 3.1

$$\int_{T_N} f(\overline{w}_j) \, dx \geq f(v(a)) \quad \text{for all } j \in \mathbb{N},$$

which, together with (3.18), and letting $\mathcal{L}^N(Q \setminus Q') \to 0$, concludes the proof.

□

We end this section with some examples of problems involving PDE constraints which fall within the scope of the present study (for further examples see [SW, T1]).

Examples 3.10. (a) [Gradients and Partial Gradients]
The case where

$$\mathcal{A}v = 0 \quad \text{if and only if } v = \nabla u$$

for some function $u : \Omega \to \mathbb{R}^m$, was already treated in Remarks 3.3 (iii) and 3.5 (iii). It can be seen easily that this framework still applies when v is not a full gradient but a list of only some of the partial derivatives of u.
(b) [Divergence Free Fields]
For the example where

$$\mathcal{A}v = 0 \quad \text{if and only if } \operatorname{div} v = 0,$$

we refer the reader to Remarks 3.3 (iv) and 3.5 (iv).
(c) [Maxwell's Equations]
In magnetostatics the *magnetization* $m : \mathbb{R}^3 \to \mathbb{R}^3$ and the *induced magnetic field* $h : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy (in suitable units) the PDE constraints

$$\mathcal{A}\begin{pmatrix} m \\ h \end{pmatrix} := \begin{pmatrix} \operatorname{div}(m + h) \\ \operatorname{curl} h \end{pmatrix} = 0.$$

For $w \in S^2$ we have

$$\begin{aligned}\ker \mathbb{A}(w) &= \left\{(a,b) \in \mathbb{R}^3 \times \mathbb{R}^3 : w \cdot (a+b) = 0, w \otimes b - b \otimes w = 0\right\}\\ &= \left\{(a,b) \in \mathbb{R}^3 \times \mathbb{R}^3 : a \cdot w = -\lambda, b = \lambda w \quad \text{for some } \lambda \in \mathbb{R}\right\},\end{aligned}$$

and so $\dim \ker \mathbb{A}(w) = 3$ and (CR) is satisfied. Note also that

$$\Lambda = \left\{(a,b) \in \mathbb{R}^3 \times \mathbb{R}^3 : (a+b) \cdot b = 0\right\},$$

and the fact that Λ imposes no restrictions on a has important consequences in micromagnetics (see [DS, JK, T5]). For the full system of Maxwell's equations we refer to [T1].
(d) [Higher Gradients]
Obviously all results remain valid if we replace the target space $\mathbb{R}^d$ by an abstract d-dimensional vector space over $\mathbb{R}$. In order to treat the case of second order derivatives, consider the smooth maps $v : T_N \to E_2^m$, where E_k^m stands for the space of symmetric k-linear maps from $\mathbb{R}^N$ into $\mathbb{R}^m$. Define

$$\mathcal{A}_2 v := \left(\frac{\partial}{\partial x_i} v_{jk} - \frac{\partial}{\partial x_k} v_{ji}\right)_{1 \le i,j,k \le N}.$$

We claim that

$$\left\{v \in C^\infty(T_N; E_2^m) : \mathcal{A}v = 0, \int_{T_N} v\, dx = 0\right\} = \{D^2 u : u \in C^\infty(T_N; \mathbb{R}^m)\}.$$

Indeed, if $\mathcal{A}v = 0$ then $v_{jk} = \frac{\partial w_j}{\partial x_k}$, where $w_j \in C^\infty(\Omega; \mathbb{M}^{m\times N})$ has average zero, and is periodic due to the periodicity of v and the fact that $\int_{T_N} v\, dx = 0$. By the symmetry of v_{jk} we have that curl $w = 0$, and we conclude that $v_{jk} = \frac{\partial^2 u}{\partial x_k \partial x_j}$, where $u \in C^\infty(T_N; \mathbb{R}^m)$.

More generally, in order to study the k-th order derivatives of functions $u \in C^\infty(T_N; \mathbb{R}^m)$, we set for $v \in C^\infty(T_N; E_k^m)$

$$\mathcal{A}_k v := \left(\frac{\partial}{\partial x_i} v_{i_1 \dots i_h\, j\, i_{h+2} \dots i_k} - \frac{\partial}{\partial x_j} v_{i_1 \dots i_h\, i\, i_{h+2} \dots i_k}\right)_{0 \le h \le k-1,\ 1 \le i,j,i_1,\dots,i_k \le N}.$$

Here $h = 0$ and $h = k-1$ correspond to the multiindices $ji_2 \dots i_k$ and $i_1 \dots i_{k-1} j$, respectively. The constant rank condition is satisfied since for $w \in S^{N-1}$

$$\begin{aligned}\ker \mathbb{A}(w) = \{X \in E_k^m : w_i\, X_{i_1 \dots i_h\, j\, i_{h+2} \dots i_k} - w_j\, X_{i_1 \dots i_h\, i\, i_{h+2} \dots i_k} = 0,\\ 1 \le h \le k,\ 1 \le i,j,i_1,\dots,i_k \le N\}\\ = \{X \in E_k^m : X = b \otimes w \dots \otimes w,\ b \in \mathbb{R}^m\},\end{aligned}$$

and so $\dim \ker \mathbb{A}(w) = m$. Moreover,

$$\left\{v \in C^\infty(T_N; E_k^m) : \mathcal{A}v = 0, \int_{T_N} v\, dx = 0\right\} = \{D^k u : u \in C^\infty(T_N; \mathbb{R}^m)\}.$$

In fact, if $\mathcal{A}v = 0$ then

$$v_{i_1 \dots i_h j i_{h+2} \dots i_k} = \frac{\partial}{\partial x_j} w_{i_1 \dots i_h i_{h+2} \dots i_k}$$

for some smooth function $w_{i_1 \dots i_h i_{h+1} \dots i_k}$ with average zero. The periodicity of v and the fact that $\int_{T_N} v\, dx = 0$ entail the periodicity of w, and the symmetries of v, together with the zero average condition we imposed on w, imply the symmetry of w, so that $w \in C^\infty(T_N; E^m_{k-1})$. Furthermore, and once again using the symmetries of v,

$$\mathcal{A}_{k-1} w := \left(\frac{\partial}{\partial x_i} w_{i_1 \ldots i_h\, j\, i_{h+2} \ldots i_{k-1}} - \frac{\partial}{\partial x_j} w_{i_1 \ldots i_h\, i\, i_{h+2} \ldots i_{k-1}} \right)_{0 \le h \le k-2,\ 1 \le i,j,i_1,\ldots,i_k \le N} = 0.$$

The argument may now be completed via induction.

(e) [Linear Elasticity]

In the framework of linear elasticity one has to deal with the symmetrized gradient, $v = e(u) := \frac{1}{2}(\nabla u + \nabla^T u)$, of the displacement $u : \Omega \to \mathbb{R}^3$, where $\Omega \subset \mathbb{R}^3$ is an open, bounded set. For $1 < p < +\infty$ one can use a local version of Korn's inequality to reduce the study of functionals

$$u \mapsto I(e(u))$$

to that of functionals

$$u \mapsto J(\nabla u), \quad \text{where } J(\xi) := I\left(\frac{1}{2}(\xi + \xi^T)\right),$$

and proceed as in (a). For $p = 1$ or $p = +\infty$ where one must avoid direct manipulation of the gradient, it is possible to adopt the present framework to treat the second order operator

$$\tilde{\mathcal{A}} v := \left(\sum_{i=1}^N \frac{\partial^2 v_{ij}}{\partial x_i \partial x_k} + \frac{\partial^2 v_{ik}}{\partial x_i \partial x_j} - \frac{\partial^2 v_{ii}}{\partial x_j \partial x_k} - \frac{\partial^2 v_{jk}}{\partial x_i \partial x_i} \right)_{1 \le j,k \le N}.$$

It turns out that $\tilde{\mathcal{A}} v = 0$ if and only if $v_{ij} = \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)/2$ for some function u. In this setting we have

$$\tilde{\mathcal{A}} v = \sum_{i=1}^N A^{(ij)} \frac{\partial^2 v}{\partial x_i \partial x_j}, \quad \tilde{\mathbb{A}}(w) := \sum_{i=1}^N A^{(ij)} w_i w_j.$$

(f) [Pseudo Differential Operators]

The examples a)–e) may be treated in a unified way using pseudo differential operators (see also [T4, T5]). For a)–d), one considers (on T_N or $\mathbb{R}^N$)

$$\mathcal{B} v := (-\Delta)^{-1/2} \mathcal{A} v = \mathcal{R}_i A^{(i)} v$$

where $\mathcal{R}_i$ denotes the Riesz transform. For e) we take

$$\tilde{\mathcal{B}} v := (-\Delta)^{-1} \tilde{\mathcal{A}} u = \left(\sum_{i=1}^N \mathcal{R}_i \mathcal{R}_k v_{ij} + \mathcal{R}_i \mathcal{R}_j v_{ik} - \mathcal{R}_j \mathcal{R}_k v_{ii} - v_{jk} \right)_{1 \le j,k \le N}. \tag{3.19}$$

The symbol of $\mathcal{B}$ is

$$b(\xi) := \frac{\xi_i}{|\xi|} A^{(i)}$$

and the constant rank condition becomes rank $b(\xi) = r$ for all $\xi \neq 0$. Similarly, for (3.19) the symbol takes values in $\mathrm{Lin}(E_2, E_2)$ and is given by

$$\tilde{b}(\xi) M := M\xi \otimes \xi + \xi \otimes M\xi - (\xi \otimes \xi)\, \mathrm{tr} M - M.$$

One can easily check that that if $|\xi| = 1$ then

$$\ker \tilde{b}(\xi) = \left\{ a \otimes \xi + \xi \otimes a : a \in \mathbb{R}^N \right\}$$

which has dimension N. Hence $\tilde{\mathcal{B}}$ satisfies the analogue of (CR).

§4. Characterization of Young Measures

The result below is the generalization to the $\mathcal{A}$-free setting of the theorem by Kinderlehrer and Pedregal for the case of gradients [KP1, KP2]. We roughly follow their strategy that relies on the Hahn-Banach separation theorem and the representation of the ($\mathcal{A}$-)quasiconvex envelope (see (3.1) and Proposition 3.4). Tartar [T1] has earlier used the Hahn-Banach separation theorem to characterize Young measures in the case without differential constraints (in a similar vein, Berliocchi and Lasry [BL] used the Krein-Milman theorem). Our presentation closely follows Kristensen's strategy for the case of gradients. We first establish the result for $p = 1$ and then deduce the assertion for $1 < p < +\infty$ by a truncation process. Some of our arguments are similar to those of Sychev [Sy] who, independently of our work, proposed an alternative approach to gradient Young measures.

Theorem 4.1. *Let $1 \leq p < +\infty$, and let $\{\nu_x\}_{x\in\Omega}$ be a weakly measurable family of probability measures on $\mathbb{R}^d$. There exists a p-equi-integrable sequence $\{v_n\}$ in $L^p(\Omega;\mathbb{R}^d)$ that generates the Young measure ν and satisfies $\mathcal{A}v_n = 0$ in Ω if and only if the following three conditions hold :*
(i) there exists $v \in L^p(\Omega;\mathbb{R}^d)$ such that $\mathcal{A}v = 0$ and

$$v(x) = \langle \nu_x, \mathrm{id}\rangle \quad \text{a.e. } x \in \Omega;$$

(ii)

$$\int_\Omega \int_{\mathbb{R}^d} |z|^p \, d\nu_x(z)\, dx < +\infty;$$

(iii) for a.e. $x \in \Omega$ *and all continuous functions g that satisfy $|g(v)| \leq C(1+|v|^p)$ for some $C > 0$ and all $v \in \mathbb{R}^d$ one has*

$$\langle \nu_x, g\rangle \geq \mathrm{Q}_{\mathcal{A}}\, g(\langle \nu_x, \mathrm{id}\rangle).$$

Remark 4.2. (i) From Lemma 2.15 it follows that if $1 < p < +\infty$ properties (i)-(iii) are still necessary if the condition $\mathcal{A}v_n = 0$ is replaced by the weaker requirement $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega)$.
(ii) In view of Theorem 2.2 (i) it suffices to assume that $\nu_x \geq 0$ a.e. $x \in \Omega$. Condition (iii) then implies $\nu_x(\mathbb{R}^d) = 1$.
(iii) A similar statement is valid for operators with variable coefficients, as long as rank $\mathbb{A}(x, w)$ is constant for all $w \in S^{N-1}$ and a.e. $x \in \Omega$. Such results are, however, more naturally discussed in the context of pseudo differential constraints and will appear elsewhere. For the quadratic case see [T5].

Proof of Theorem 4.1 - Necessity. Necessity of (i) follows immediatly from Theorem 2.2 (vi), where v is the weak limit in L^p of the sequence $\{v_n\}$. Property (ii) is deduced from Theorem 2.2 (v) with $f(z) = |z|^p$, and (iii) is a consequence of Proposition 3.8 (and Lemma 2.15 if $1 < p < +\infty$).

□

The proof of sufficiency for $1 < p < +\infty$ follows from the case $p = 1$ and Corollary 2.18.
We proceed with the proof in the case of homogeneous $\mathcal{A} - 1-$Young measures.
Let $\mathcal{P}$ be the set of probability measures on $\mathbb{R}^d$ and define

$$\begin{aligned}\mathbb{H} := \{\nu \in \mathcal{P}(\mathbb{R}^d) : &\langle \nu, \mathrm{id}\rangle = 0, \text{ there exists an equi-integrable sequence}\\ &\{w_j\} \subset L^1(T_N) \cap \ker \mathcal{A} \text{ generating the Young measure } \nu\}.\end{aligned}$$

Set

$$E := \left\{ g \in C(\mathbb{R}^d) : \lim_{|z|\to\infty} \frac{g(z)}{1+|z|} \quad \text{exists in } \mathbb{R} \right\}$$

equipped with the norm

$$||g||_E := \sup_{z\in\mathbb{R}^d} \frac{|g(z)|}{1+|z|}.$$

This space is isometrically isomorphic to the space $C(\mathbb{R}^d \cup \{\infty\}) \sim C(S^d)$ of continuous functions on the one-point compactification of $\mathbb{R}^d$, via the map

$$g \mapsto \frac{g(\cdot)}{1+|\cdot|}.$$

In particular, E is a separable Banach space, and its dual E' may be identified with the space of Radon measures on $\mathbb{R}^d \cup \{\infty\}$. Thus if $\nu \in \mathcal{P}$ is such that

$$\int_{\mathbb{R}^d} |z|\, d\nu(z) < +\infty$$

then $\nu \in E'$ since for all $g \in E$

$$\left|\int_{\mathbb{R}^d} g\, d\nu\right| \leq ||g||_E \int_{\mathbb{R}^d} (1+|z|)\, d\nu(z).$$

Proposition 4.2. *Let $\nu \in \mathcal{P}(\mathbb{R}^d)$ with $\langle \nu, \mathrm{id}\rangle = 0$. Then $\nu \in \mathbb{H}$ if*
(i)

$$\int_{\mathbb{R}^d} |z|\, d\nu(z) < +\infty;$$

(ii)

$$\langle \nu, g\rangle \geq \mathrm{Q}_{\mathcal{A}}\, g(0)$$

for all $g \in C(\mathbb{R}^d)$ such that $|g(z)| \leq C(1+|z|)$.

Proof. We follow [KP1, KP2] and use the Hahn-Banach theorem to show that measures satisfying (i) and (ii) cannot be separated from $\mathbb{H}$.

We will prove that $\mathbb{H}$ is convex and relatively closed in $\mathcal{P}$.

Claim 1: $\mathbb{H}$ is convex.

Fix $\nu, \mu \in \mathbb{H}$, $\theta \in (0,1)$. Let $\{v_j\}, \{w_j\} \subset L^1(T_N) \cap \ker \mathcal{A}$ be equi-integrable sequences generating the $\mathcal{A} - 1$-Young measures ν and μ, respectively. By means of a mollification, we may take $v_j, w_j \in C^\infty(T_N)$. Also, as

$$\int_{T_N} v_j\, dx,\ \int_{T_N} w_j\, dx \to 0,$$

without loss of generality we may assume that

$$\int_{T_N} v_j\, dx = \int_{T_N} w_j\, dx = 0.$$

Since $v_j, w_j \to 0$ in $W^{-1,p}(T_N)$ for $p < \frac{N}{N-1}$, and as for all $\varphi \in C_0^\infty((0,\theta) \times T_{N-1})$

$$||\mathcal{A}(\varphi(w_j - v_j))||_{W^{-1,p}} = \left|\left|\frac{\partial \varphi}{\partial x_i} A^{(i)}(w_j - v_j)\right|\right|_{W^{-1,p}} \to 0,$$

we may find a sequence $\{\varphi_j\} \subset C_0^\infty((0,\theta) \times T_{N-1})$ such that $\varphi_j \nearrow \chi_{(0,\theta)\times T_{N-1}}$ and

$$||\mathcal{A}(\varphi_j(w_j - v_j))||_{W^{-1,p}} \to 0.$$

Define

$$u_j := v_j + \mathbb{T}\left(\varphi_j(w_j - v_j) - \int_{T_N} \varphi_j(w_j - v_j)\,dy\right).$$

Then $u_j \in L^1(T_N) \cap \ker \mathcal{A}$, $\int_{T_N} \varphi_j(w_j - v_j)\,dy \to 0$, and by Lemma 2.14 (iii)

$$u_j = v_j + \varphi_j(w_j - v_j) + h_j, \quad h_j \to 0 \text{ in } L^p(T_N),\, p < \frac{N}{N-1}.$$

In particular, $\{u_j\}$ is equi-integrable and generates the Young measure $\{\lambda_x\}_{x \in T_N}$ given by

$$\lambda_x = \begin{cases} \nu & \text{if } x_1 \in (0, \theta) \\ \mu & \text{if } x_1 \in (\theta, 1). \end{cases}$$

Finally, let

$$\overline{u}_{j,m}(x) := u_j(mx),\ m \in \mathbb{N}.$$

Then $\overline{u}_{j,m} \in C^\infty(T_N) \cap \ker \mathcal{A}$, by periodicity $\sup_{j,m} ||\overline{u}_{j,m}||_{L^1(T_N)} < +\infty$, and due to the equi-integrability of $\{u_j\}$, for all $\psi \in C_0(\mathbb{R}^N)$, $g \in E$, we have

$$\begin{aligned} \lim_{j\to\infty} \lim_{m\to\infty} \int_{\mathbb{R}^N} \psi(x) g(\overline{u}_{j,m}(x))\,dx &= \lim_{j\to\infty} \int_{\mathbb{R}^N} \psi(x) \left(\int_{T_N} g(u_j(y))\,dy\right)\,dx \\ &= \int_{\mathbb{R}^N} \psi(x)\,dx\ (\theta\langle \nu, g\rangle + (1-\theta)\langle \mu, g\rangle). \end{aligned} \tag{4.1}$$

Extracting a diagonal subsequence and taking $g = |\cdot|$ in (4.1), by Theorem 2.2 (vi) we conclude that $\theta\nu + (1-\theta)\mu$ is generated by an equi-integrable sequence in $\ker \mathcal{A}$ and thus belongs to $\mathbb{H}$.

Claim 2: $\mathbb{H}$ is relatively closed in $\mathcal{P}$ with respect to the weak-* topology in E', i.e.

$$\overline{\mathbb{H}}^{E'} \cap \mathcal{P} = \mathbb{H}.$$

Let $\nu \in \overline{\mathbb{H}}^{E'} \cap \mathcal{P}$, let $\{f_i\}_{i\in\mathbb{N}} \subset C^\infty(T_N)$ be dense in $L^1(T_N)$, and let $\{g_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ be dense in $C_0(\mathbb{R}^d)$. We take $f_0 = 1$ and $g_0(z) = |z|$. By definition of weak-* topology in E' there exist $\nu_k \in \mathbb{H}$ such that

$$|\langle \nu - \nu_k, g_j\rangle| < \frac{1}{2k}, \quad j = 0, \cdots, k,$$

thus, by virtue of Theorem 2.2 (vi) we may find $w_k \in L^1(T_N) \cap \ker \mathcal{A}$ such that

$$\left|\langle \nu, g_j\rangle \int_{T_N} f_i\,dx - \int_{T_N} f_i g_j(w_k)\,dx\right| < \frac{1}{k}, \quad 0 \le i, j \le k. \tag{4.2}$$

In particular, setting $i = 0 = j$ we deduce that $\{w_k\}$ is bounded in $L^1(T_N)$ and so (a subsequence) generates a Young measure μ. From (4.2) and the density properties of $\{f_i\}_{i\in\mathbb{N}}$ and $\{g_j\}_{j\in\mathbb{N}}$ it follows that $\mu = \nu$, and the choice $i = 0 = j$ yields

$$\int_{T_N} |w_k|\,dx \to \langle \nu, |\cdot|\rangle.$$

By Theorem 2.2 (vi) we conclude that $\{w_k\}$ is equi-integable and so $\nu \in \mathbb{H}$. This proves Claim 2.

Consider $\nu \in \mathcal{P}$ such that $\langle \nu, \mathrm{id}\rangle = 0$ and ν satisfies (i), (ii). We want to prove that $\nu \in \mathbb{H}$. Suppose that $\nu \notin \mathbb{H}$. By Claims 1 and 2, $\nu \notin \overline{\mathrm{co}(\mathbb{H})}$ with respect to the weak-* topology of E'. Therefore, by the Hahn-Banach theorem and (ii) there exist $g \in E$, $\alpha \in \mathbb{R}$, such that

$$\langle \mu, g\rangle \ge \alpha \quad \text{for all } \mu \in \mathbb{H}, \qquad Q_{\mathcal{A}}\, g(0) \le \langle \nu, g\rangle < \alpha. \tag{4.3}$$

Given $w \in C^\infty(T_N) \cap \ker \mathcal{A}$, with $\int_{T_N} w\,dx = 0$, by Proposition 2.8 we have $\overline{\delta_w} \in \mathbb{H}$ and thus

$$\int_{T_N} g(w)\,dx = \langle \overline{\delta_w}, g\rangle \geq \alpha,$$

which, by Definition 3.1 implies that $\mathrm{Q}_{\mathcal{A}}\, g(0) \geq \alpha$, contradicting (4.3). We conclude that $\nu \in \mathbb{H}$.

□

Next we treat the case of inhomogeneous $\mathcal{A}$-1-Young measures.
We define

$$\mathbb{X} := \left\{ \nu : \Omega \to \mathcal{P}(\mathbb{R}^d) : \nu \text{ is weak* measurable}, \int_\Omega \int_{\mathbb{R}^d} |z| d\nu_x(z) dx < +\infty, \langle \nu_x, \mathrm{id}\rangle = 0 \text{ a.e. } x \in \Omega \right\},$$

$$\mathbb{Y} := \left\{ \nu \in \mathbb{X} : \nu \text{ is generated by some equi-integrable sequence } \{w_n\} \in L^1(T_N) \cap \ker \mathcal{A} \right\},$$

$$\mathbb{W} := \{ \nu \in \mathbb{X} : \langle \nu_x, g\rangle \geq \mathrm{Q}_{\mathcal{A}}\, g(0) \;\text{ a.e. } x \in \Omega \text{ and for all } g \in E \},$$

and

$$\mathcal{E} := C(\overline{\Omega}; E) \sim C(\overline{\Omega} \times (\mathbb{R}^d \cup \{\infty\})).$$

Suppose that ν satisfies (i), (ii) and (iii) of Theorem 4.1, and set $\overline{\nu}_x := \Gamma_{-v(x)}\nu_x$ (the translation of a measure was defined in Proposition 2.4). Clearly $\overline{\nu} \in \mathbb{W}$, and so if $\mathbb{W} \subset \mathbb{Y}$ then ν is generated by an equi-integrable sequence $\{v + w_j\}$ where $\mathcal{A}w_j = 0$. It thus suffices to verify the following assertion.

Proposition 4.3.

$$\mathbb{W} \subset \mathbb{Y}. \tag{4.4}$$

Proof. The strategy to prove (4.4) is as follows:
Step 1 : $\overline{\mathbb{Y}}^{\mathcal{E}'} \cap \mathbb{X} = \mathbb{Y}$ in the weak-* topology;
Step 2 : It is possible to find a 'good' subset $D \subset \mathbb{W}$ such that $\overline{D}^{\mathcal{E}'} \cap \mathbb{W} = \mathbb{W}$;
Step 3 : $D \subset \mathbb{Y}$.
The proof of Step 1 is entirely identical to that of Claim 2 in the proof of Proposition 4.2. For Step 2, we define $\mathcal{G}_k$ to be the family of cubes of the form $\left\{\frac{1}{k}(y+Q) : y \in \mathbb{Z}^N, \frac{1}{k}(y+Q) \subset \Omega\right\}$, and we set

$$G_k := \cup_{\mathcal{U} \in \mathcal{G}_k} \mathcal{U}.$$

Consider the sets of piecewise homogeneous Young measures

$$\mathbb{W}_k := \left\{ \nu \in \mathbb{W} : \nu_{|\mathcal{U}} \text{ is homogeneous if } \mathcal{U} \in \mathcal{G}_k,\ \nu_{|(\Omega \setminus G_k)} = \delta_0 \right\}$$

and let

$$D := \cup_{k \in \mathbb{N}} \mathbb{W}_k.$$

In order to show that

$$\overline{D}^{\mathcal{E}'} \cap \mathbb{W} = \mathbb{W},$$

let $\nu \in \mathbb{W}$ and define

$$\nu_x^k := \begin{cases} \dfrac{1}{\mathcal{L}^N(\mathcal{U})} \displaystyle\int_{\mathcal{U}} \nu_y\,dy & \text{if } x \in \mathcal{U},\ \mathcal{U} \in \mathcal{G}_k, \\ \delta_0 & \text{otherwise.} \end{cases}$$

It is clear that $\nu^k \in \mathbb{W}_k$, so it suffices to show that

$$\langle \nu^k, f\rangle \to \langle \nu, f\rangle \qquad \text{for all } f \in \mathcal{E}. \tag{4.5}$$

Fix $f \in \mathcal{E}$, and for each $\mathcal{U} \in \mathcal{G}_k$ denote by $x_{\mathcal{U}} \in \left(\frac{1}{k}\mathbb{Z}\right)^N$ the lower left corner of $\mathcal{U}$ so that $\mathcal{U} = x_{\mathcal{U}} + \frac{1}{k}Q$. Let ω be a modulus of uniform continuity of f, i.e.

$$\omega(\delta) := \sup\left\{||f(x,\cdot) - f(y,\cdot)||_E : x, y \in \overline{\Omega},\ |x-y| \le \delta\right\}.$$

We have

$$\begin{aligned}
&\left|\int_{\mathcal{U}}\int_{\mathbb{R}^d} f(x,z)\, d\nu_x(z)\ dx - \int_{\mathcal{U}}\int_{\mathbb{R}^d} f(x,z)\, d\nu_x^k(z)\, dx\right| \\
&\qquad \le \left|\int_{\mathcal{U}}\int_{\mathbb{R}^d} f(x_{\mathcal{U}},z)\, d\nu_x(z)\, dx - \int_{\mathcal{U}}\int_{\mathbb{R}^d} f(x_{\mathcal{U}},z)\, d\nu_x^k(z)\, dx\right| \\
&\qquad\quad + \omega\left(\frac{1}{k}\right) ||f||_{\mathcal{E}} \left(\int_{\mathcal{U}}\int_{\mathbb{R}^d} (1+|z|)\, d\nu_x(z)\, dx + \int_{\mathcal{U}}\int_{\mathbb{R}^d} (1+|z|)\, d\nu_x^k(z)\, dx\right) \\
&\qquad \le 2\omega\left(\frac{1}{k}\right) ||f||_{\mathcal{E}} \int_{\mathcal{U}}\int_{\mathbb{R}^d} (1+|z|)\, d\nu_x(z) dx.
\end{aligned}$$

Therefore,

$$|\langle \nu^k, f\rangle - \langle \nu, f\rangle| \le 2\omega\left(\frac{1}{k}\right) ||f||_{\mathcal{E}} \int_{G_k}\int_{\mathbb{R}^d} (1+|z|)\, d\nu_x(z) dx + 2||f||_{\mathcal{E}} \int_{\Omega\setminus G_k}\int_{\mathbb{R}^d} (1+|z|)\, d\nu_x(z) dx,$$

and (4.5) follows by letting $k \to \infty$ and using assertion (ii) in Theorem 4.1.

Next, we carry out Step 3 by showing that

$$\mathbb{W}_k \subset \mathbb{Y} \quad \text{for all } k \in \mathbb{N}.$$

Using a rescaling argument, we may assume that $\Omega \subset Q$. Fix $k \in \mathbb{N}$ and let $\mathcal{G}_k = \{Q_i\}_{i=1}^m$ for some $m \in \mathbb{N}$. Fix $\nu \in \mathbb{W}_k$, with $\nu_{|Q_i} = \nu^i$. By Corollary 2.18 for each $i \in \{1, \cdots, m\}$ there exists an equi-integrable sequence $\{w_j^i\} \subset L^1(T_N) \cap \ker \mathcal{A}$ generating ν^i. In particular, without loss of generality we may assume that w_j^i are smooth, and that we have

$$w_j^i \rightharpoonup 0 \quad \text{in } L^1(Q_i), \quad w_j^i \to 0 \quad \text{in } \mathrm{W}^{-1,p}_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$$

for $p < N/(N-1)$. Hence, we may find smooth cut-off functions $\varphi_j^i \in C_0^\infty(Q_i; [0,1])$ such that $\varphi_j^i \nearrow \chi_{Q_i}$ and

$$\mathcal{A}\left(\sum_{i=1}^m \varphi_j^i w_j^i\right) = \sum_{k=1}^N \sum_{i=1}^m A^{(k)} w_j^i \frac{\partial \varphi_j^i}{\partial x_k} \to 0 \quad \text{in } W^{-1,p}(\mathbb{R}^N).$$

Setting

$$u_j := \mathbb{T}\left(w_j - \int_{T_N} w_j\, dy\right), \qquad \text{where } w_j := \sum_{i=1}^m \varphi_j^i\, w_j^i,$$

then $u_j \in \ker \mathcal{A}$, $\left|\left|u_j - \sum_{i=1}^m \varphi_j^i\, w_j^i\right|\right|_{L^p(\Omega)} \to 0$. In particular $\{u_j\}$ is equi-integrable and it generates ν, so $\nu \in \mathbb{Y}$.

□

Examples 4.4. (a) [Gradients]
Using Remark 3.3 (iii) and Theorem 4.1, we recover the characterization of $W^{1,p}$ gradient Young measures as obtained by Kinderlehrer and Pedregal [KP1, KP2] (see Theorem 2.6).
(b) [Divergence Free Fields]
It follows from Remarks 3.3 (iv), 3.5 (iv), and by Theorem 4.1, that any weakly measurable family of probability measures $\{\nu_x\}_{x\in\Omega}$ satisfying

$$\operatorname{div}(\langle \nu_x, \mathrm{id}\rangle) = 0, \qquad \int_\Omega \int_{\mathbb{R}^N} |z|^p \, d\nu_x(z) dx < +\infty,$$

is generated by a p-equi-integrable sequence of divergence-free fields $v_n \in L^p(\Omega;\mathbb{R}^N)$ (see also [P]).
(c) [Micromagnetics]
In view of Example 3.10 c), we may apply Theorem 4.1 to the system of Maxwell equations. Moreover, if $1 < p < +\infty$, if ν is a $\mathcal{A}$-p-Young measure and if we define the projection λ by

$$\lambda_x(U) := \nu_x(U \times \mathbb{R}^3), \quad \text{for any open subset } U \subset \mathbb{R}^3,$$

then $\operatorname{supp} \lambda_x \subset S^2$ for a.e. $x \in \Omega$ if and only if ν is generated by a p-equi-integrable sequence $\{\tilde{m}_n, \tilde{h}_n)\} \subset \ker \mathcal{A}$ such that $|\tilde{m}_n(x)| = 1$ for a.e. $x \in \Omega$. Indeed, assuming that λ_x is supported on the unit sphere, let $\{(m_n, h_n)\} \subset \ker \mathcal{A}$ be a p-equi-integrable generating sequence, with $h_n = -\nabla u_n$, $u_n \in W_0^{1,p}(\Omega)$ ($h_n = -\nabla u_n + H_n$ with div H_n = curl $H_n = 0$ if Ω is not simply connected). Consider the projection

$$\pi(x) := \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0 \\ \\ x_0 & \text{if } x = 0, \end{cases}$$

where $x_0 \in S^2$ is fixed, and define $\tilde{m}_n := \pi m_n$. Since $\operatorname{dist}(m_n, S^2) \to 0$ as $n \to \infty$, we have that $\tilde{m}_n - m_n \to 0$ in measure, and, due to the p-equi-integrability, we conclude that $\tilde{m}_n - m_n \to 0$ in L^p. Let $\tilde{h}_n := -\nabla \tilde{u}_n$ ($\tilde{h}_n := -\nabla \tilde{u}_n + H_n$ if Ω is not simply connected) where $\tilde{u}_n \in W_0^{1,p}(\Omega)$ and $\operatorname{div}(\tilde{m}_n - \nabla \tilde{u}_n) = 0$. We have

$$\operatorname{div}((\tilde{m}_n - m_n) - (\nabla \tilde{u}_n - \nabla u_n)) = 0,$$

therefore $\Delta(\tilde{u}_n - u_n) \to 0$ in $W^{-1,p}$, and thus $\tilde{u}_n - u_n \to 0$ in $W^{1,p}$. We conclude that $\{(\tilde{m}_n, \tilde{h}_n)\}$ still generates ν.

Acknowledgments. The research of I. Fonseca was partially supported by the National Science Foundation through the Center for Nonlinear Analysis, and by the National Science Foundation under grant No. DMS–9500531, and by the Max Planck Institute for Mathematics in the Sciences, Leipzig (Germany).

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