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by

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Abstract

An example is given of a quasiconvex $f : M^{2\times 3} \to \mathbb{R}$ such that the transposed function $\tilde{f} : M^{3\times 2} \to \mathbb{R}$ given by $\tilde{f}(F) = f(F^T)$ is not quasiconvex. For $\tilde{f}$ one can take Šverák's quartic polynomial that is rank-one convex but not quasiconvex. The proof is closely related to the observation that the map $v \mapsto v^1 v^2 v^3$ is weakly continuous from $L^3(\mathbb{R}^3;\mathbb{R}^3)$ into distributions provided that $A(Pv) = (\partial_2 v^1, \partial_3 v^1, \partial_1 v^2, \partial_3 v^2, \partial_1 v^3, \partial_2 v^3)$ is compact in $W^{-1,3}(\mathbb{R}^3;\mathbb{R}^6)$.

1 Introduction

Quasiconvexity is the natural notion of convexity for variational problems for multiple integrals

$$I(u) = \int_\Omega f(Du)\,dx, \quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m.$$

In his pioneering work Morrey ([Mo 52], [Mo 66])showed that weak lower semicontinuity of I in Sobolev spaces is essentially equivalent to quasiconvexity of the integrand f (see [AF 84], [Ma 85] for technically nearly optimal statements). An integral $f : M^{m\times n} \to \mathbb{R}$ is called quasiconvex if

$$\int_Q f(F + D\varphi) \geq f(F),$$

for all $F \in M^{m\times n}$ and all Lipschitz functions $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ that are periodic with cell $Q = (0,1)^n$. (For the equivalence with other definitions see e.g. [Sv 92]).

Quasiconvexity is still poorly understood, partly because it is a nonlocal condition. Therefore algebraic sufficient and necessary conditons were introduced. A function f is rank-1 convex if it is convex on rank-1 lines in $M^{m\times n}$, and it is called polyconvex if it can be written as a convex function of the minors. For $n = 1$ or $m = 1$ all these notions coincide with ordinary convexity. For $n \geq 2$, $m \geq 2$ one has the implications ([Mo 52], [Mo 66], [Da 89])

$$f \text{ convex} \overset{\not\Leftarrow}{\Rightarrow} f \text{ polyconvex} \overset{\not\Leftarrow}{\Rightarrow} \text{quasiconvex} \Rightarrow f \text{ rank-1 convex}.$$

Šverák [Sv 92] solved a long standing conjecture by showing that rank-1 convexity does not imply quasiconvexity if $m \geq 3$, $n \geq 2$. The case $m = 2$, $n \geq 2$ is open. Šverák's example is reminiscent of a counterexample by Tartar [Ta 79], pp. 185–186, in trilinear compensated compactness.

Rank-1 convexity and polyconvexity are invariant under transposition, i.e. if $f : M^{m\times n} \to \mathbb{R}$ does have one of these properties so does $\tilde{f} : M^{n\times m} \to \mathbb{R}$ given by

$$\tilde{f}(F) = f(F^T).$$

Alberti thus raised the question whether quasiconvexity is also invariant under transposition. Using Šverak's counterexample Kružik [Kr 97] recently showed that this is not the case if one allows f to take the value ∞. Here we refine this analysis and show that Šverak's (finite-valued) functions provide already a counterexample.

Theorem 1 *There exists a quasiconvex function $f : M^{2\times 3} \to \mathbb{R}$ such that $\tilde{f}$ is not quasiconvex.*

2 Proofs

Proof. We will choose $\tilde{f}$ as in Šverák's counterexample. Let

$$L = \left\{ \begin{pmatrix} r & 0 & t \\ 0 & s & t \end{pmatrix} : r, s, t \in \mathbb{R} \right\} \subset M^{2\times 3}$$

and let

$$g(F) = -rst \text{ for } F \in L.$$

Denote by π the orthogonal projection onto L and consider the functions

$$f_{\varepsilon,k}(F) = g(\pi F) + \varepsilon(|F|^2 + |F|^4) + k|F - \pi F|^2, \ \varepsilon \geq 0, \ k \geq 0.$$

Šverák showed that for small enough $\varepsilon > 0$ (and all k) the function $\tilde{f}_{\varepsilon,k}$ is not quasiconvex. Indeed it suffices to note that the periodic map

$$\psi(x) = \frac{1}{2\pi} \begin{pmatrix} \sin 2\pi x^1 \\ \sin 2\pi x^2 \\ \sin 2\pi(x^1 + x^2) \end{pmatrix}$$

satisfies $(D\psi)^T \in L$ and

$$\int\limits_{(0,1)^2} g(D\psi^T)\, dx = -\frac{1}{4} < g(0) = 0.$$

Šverák also showed that for any given $\varepsilon > 0$ the function $f_{\varepsilon,k}$ is rank-one convex for large enough $k > k_0(\varepsilon)$.

We claim that for each $\varepsilon > 0$ there exists a $k(\varepsilon)$ such that for $k \geq k(\varepsilon)$ the function $f_{\varepsilon,k}$ is quasiconvex. First note that it suffices to show that

$$\int\limits_Q f_{\varepsilon,k}(F + D\varphi) - f_{\varepsilon,k}(F) - Df_{\varepsilon,k}(F)D\varphi \; dx \geq 0, \tag{1}$$

for all $\varphi \in W^{1,\infty}(T^3;\mathbb{R}^2)$ and all $F \in M^{2\times 3}$, since $\int_Q D\varphi = 0$. One easily checks that there exists $c > 0$ such that

$$|F + G|^4 - |F|^4 - 4|F|^2 F : G \geq c(|F|^2|G|^2 + |G^4|).$$

Here $F : G = \sum F_{ij}G_{ij}$. Indeed, by homogeneity we may assume $|F| = 1$ (the case $F = 0$ is trivial), and since the function $F \mapsto |F|^4$ is strictly convex it suffices to consider the cases $|G| \to 0$ or $|G| \to \infty$. The latter is obvious and for the former it suffices to compute the Hessian. Since g is a polynominal of degree three, expansion of $g(\pi F + \pi D\varphi)$ yields

$$\begin{aligned} & f_{\varepsilon,k}(F + D_\varphi) - f_{\varepsilon,k}(F) - Df_{\varepsilon,k}(F)D\varphi \\ \geq \; & \frac{1}{2}D^2 g(\pi F)(\pi D\varphi, \pi D\varphi) + g(\pi D\varphi) \\ & + \varepsilon|D\varphi|^2 + c\varepsilon(|F|^2|D\varphi|^2 + |D\varphi|^4) + k|D\varphi - \pi D\varphi|^2. \end{aligned} \tag{2}$$

Let

$$v^1 = \partial_1 \varphi^1, \quad v^2 = \partial_2 \varphi^2, \quad v^3 = \tfrac{1}{2}\partial_3(\varphi^1 + \varphi^2),$$

$$w^1 = \partial_2 \varphi^1, \quad w^2 = \partial_1 \varphi^2, \quad w^3 = \tfrac{1}{2}\partial_3(\varphi^1 - \varphi^2)$$

and $h(v) = v^1 v^2 v^3$. Then

$$\pi D\varphi = \begin{pmatrix} v^1 & 0 & v^3 \\ 0 & v^2 & v^3 \end{pmatrix}, \quad D\varphi - \pi D\varphi = \begin{pmatrix} 0 & w^1 & w^3 \\ w^2 & 0 & -w^3 \end{pmatrix},$$
$$g(\pi D\varphi) = -h(v).$$

If $w = 0$ (i.e. $D\varphi \in L$) then one easily deduces that $v^1 = v^1(x^1)$, $v^2 = v^2(x^2)$, $v^3 = v^3(x^3)$ and thus $\int_Q h(v) = 0$ since $\int_Q v = 0$. To obtain an estimate for $\int_Q h(v)$ if $w \neq 0$ let

$$A(Dv) = (\partial_2 v^1, \partial_3 v^1, \partial_1 v^2, \partial_3 v^2, \partial_1 v^3, \partial_2 v^3).$$

A short calculation shows that $A(Dv)$ can be expressed as a linear combination of first derivatives of w. Hence

$$\|A(Dv)\|_{W^{-1,2}(Q)} \leq C\|D\varphi - \pi D\varphi\|_{L^2(Q)}.$$

Application of Lemma 2 below with $p = q = 2$ yields

$$\begin{aligned} \left| \int_Q g(\pi D\varphi)\, dx \right| &= \left| \int_Q h(v)\, dx \right| \\ &\leq C\|D\varphi\|_{L^4}^2 \|D\varphi - \pi D\varphi\|_{L^2} \\ &\leq \frac{\varepsilon c}{4}\|D\varphi\|_{L^4}^4 + \frac{C}{\varepsilon}\|D\varphi - \pi D\varphi\|_{L^2}^2. \end{aligned}$$

Similarly we obtain with $a = (F_{11}, F_{22}, (F_{31} + F_{32})/2)$

$$\begin{aligned} \left| \int_Q D^2 g(\pi F)(\pi D\varphi, \pi D\varphi)\, dx \right| &= \left| \int_Q D^2 h(a)(v, v) \right| \\ &\leq C|F|\|D\varphi\|_{L^2}\|D\varphi - \pi D\varphi\|_{L^2} \\ &\leq \frac{\varepsilon c}{4}|F|^2\|D\varphi\|_{L^2}^2 + \frac{C}{\varepsilon}\|D\varphi - \pi D\varphi\|_{L^2}^2. \end{aligned}$$

In combination with (2) this yields (1), provided that $k \geq k(\varepsilon) = \frac{2C}{\varepsilon}$. $\square$

Lemma 2. Consider the function $h : \mathbb{R}^3 \to \mathbb{R}$, $h(y) = y_1 y_2 y_3$ and assume that $v \in W^{1,\infty}(\mathbb{R}^3; \mathbb{R}^3)$ is periodic with cell $Q = (0,1)^3$ and $\int_Q v = 0$.
Let

$$A(Dv) = (\partial_2 v^1, \partial_3 v^1, \partial_1 v^2, \partial_3 v^2, \partial_1 v^3, \partial_2 v^3)$$

and assume that $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, $a \in \mathbb{R}^3$.
Then

$$\left| \int_Q h(v)\, dx \right| \leq C(p) \|v\|^2_{L^{2p}(Q)} \, \|A(Dv)\|_{W^{-1,q}(Q)} \tag{3}$$

$$\left| \int_Q D^2 h(a)(v,v)\, dx \right| \leq C(p) |a| \, \|v\|_{L^p(Q)} \, \|A(Dv)\|_{W^{-1,q}(Q)}. \tag{4}$$

Proof. We split v into a part Qv that is controlled by $A(Dv)$ and a part Pv whose Fouriertransform is supported near the axes and then show that $\int_Q h(Pv) = 0$. Let $a_1 \in C^\infty(S^2)$ with

$$\begin{array}{lll} \operatorname{supp} a_1 & \subset & \left\{\xi \in S^2 : \xi_1^2 \geq 1 - \delta^2\right\}, \\ a_1 = 1 & \text{on} & \left\{\xi \in S^2 : \xi_1^2 \geq 1 - \delta^2/2\right\}. \end{array}$$

Let $b_1 = 1 - a_1$, extend a_1 and b_1 by homogeneity to $\mathbb{R}^3 \backslash \{0\}$ and define operators P_1 and Q_1 that act on periodic functions with mean zero by

$$P_1 v_1 = \mathcal{F}^{-1}(a_1 \mathcal{F} v_1), \quad Q_1 v_1 = \mathcal{F}^{-1}(b_1 \mathcal{F} v_1),$$

where $\mathcal{F}$ denotes the discrete Fourier transform, i.e.

$$(\mathcal{F} v_1)(k) = \int_Q v_1 e^{-2\pi i k \cdot x}\, dx, \ k \in \mathbb{Z}^3.$$

Note that $\mathcal{F} v_1(0) = 0$ since v_1 has mean zero. Now b_1 can be written as

$$b_1(\xi) = \frac{\xi_2}{|\xi|} m_2(\xi) + \frac{\xi_3}{|\xi|} m_3(\xi),$$

where

$$m_2(\xi) = \frac{\xi_2 |\xi|}{\xi_2^2 + \xi_3^2} b_1(\xi) \quad \text{and} \quad m_3(\xi) = \frac{\xi_3 |\xi|}{\xi_2^2 + \xi_3^2} b_1(\xi).$$

Standard results on Fourier multipliers (see [SW 71] Cor. 3.16, p. 263) yield

$$\begin{array}{rcl} \|Q_1 v_1\|_q & \leq & C(q) \|(\partial_2 v_1, \partial_3 v_1)\|_{-1,q}, \\ \|P_1 v_1\|_p + \|Q_1 v_1\|_p & \leq & C(p) \|v_1\|_p. \end{array}$$

(Here we used the abbreviations $\|\cdot\|_p = \|\cdot\|_{L^p(Q)}$ and $\|\cdot\|_{-1,q} = \|\cdot\|_{W^{-1,q}(Q)}$
Analogously we define P_2, Q_2, P_3 and Q_3 and we let

$$Pv = \begin{pmatrix} P_1 v_1 \\ P_2 v_2 \\ P_3 v_3 \end{pmatrix}, \quad Qv = \begin{pmatrix} Q_1 v_1 \\ Q_2 v_2 \\ Q_3 v_3 \end{pmatrix}.$$

Then $P+Q=id$ and

$$\begin{aligned}\|Qv\|_q &\le C(q)\|A(Dv)\|_{-1,q}, &(5)\\ \|Pv\|_p+\|Qv\|_p &\le C(p)\|v\|_p. &(6)\end{aligned}$$

To prove (3) we expand $h(v)=h(Pv+Qv)$. In view of (5) and (6) it suffices to verify that

$$\int_Q h(Pv)=0. \tag{7}$$

By construction $\mathcal{F}P_iv_i$ is supported on the cone $\Lambda_i=\{\xi\in\mathbb{R}^3:\xi_i^2\ge(1-\delta^2)|\xi|^2\}$ and thus

$$\int_Q h(Pv)\,dx=\sum_{\substack{k^{(i)}\in\Lambda_i\cap\mathbf{Z}^3\\ k^{(1)}+k^{(2)}+k^{(3)}=0}}\mathcal{F}(P_1v_1)(k^{(1)})\mathcal{F}(P_2v_2)(k^{(2)})\mathcal{F}(P_3v_3)(k^{(3)}).$$

Now the assumptions $k^{(i)}\in\Lambda_i$ and $k^{(1)}+k^{(2)}+k^{(3)}=0$ imply that

$$\begin{aligned}(1-\delta^2)|k^{(1)}|^2 &\le |k_1^{(1)}|^2\le 2\left(|k_1^{(2)}|^2+|k_1^{(3)}|^2\right)\\ &\le 2\delta^2\left(|k^{(2)}|^2+|k^{(3)}|^2\right).\end{aligned}$$

Adding to this the two other inequalities obtained by cyclic permulation of the indices we see that

$$(1-\delta^2)\sum_j|k^{(j)}|^2\ge 4\delta^2\sum_j|k^{(j)}|^2.$$

Taking $\delta<\frac{1}{\sqrt{5}}$ we conclude that $k^{(1)}=k^{(2)}=k^{(3)}=0$. This implies (7) since the v_i have mean zero, so that $\mathcal{F}(P_iv_i)(0)=0$. Thus (3) is proved and the proof of (4) is similar. $\square$

3 Trilinear compensated compactness

The following consequence of Lemma 2 is not used in the proof of Theorem 1, but provides a nice example in trilinear compensated compactness (see [Ta 79], [Ta 98] for general expositions of compensated compactness). A systematic study of trilinear quantities in the context of m×m hyperbolic systems is undertaken in [JMR 95]. In this case the number m of dependent variables and of constraints is the same, and one can easily check that for $m\ge 3$ one can only expect good results for differential constraints with variable coefficients that in addition satisfy suitable genericity conditions. In the situation of Lemma 2 there are more differential constraints than dependent variables and less sophisticated methods suffice.

Corollary 3. Suppose that

$$\begin{aligned} v_k &\rightharpoonup v \quad \text{in} \quad L^3_{loc}(\mathbb{R}^3;\mathbb{R}^3), \\ A(Dv_k) &\to A(Dv) \quad \text{in} \quad W^{-1,3}_{loc}(\mathbb{R}^3;\mathbb{R}^3). \end{aligned}$$

Then

$$\begin{aligned} h(v_k) &\rightharpoonup h(v) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3), \\ v^i_k v^j_k &\rightharpoonup v^i v^j \quad \text{in} \quad L^{\frac{3}{2}}_{loc}(\mathbb{R}^3), \quad \text{for} \quad i \neq j. \end{aligned}$$

Proof. Assume first that $v = 0$. In this case it suffices to show that

$$\int\limits_{\mathbb{R}^3} h(v_k)\varphi^3\,dx \to 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3). \tag{8}$$

Indeed if (8) holds for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$ it holds by density for all $\varphi \in C^0_0(\mathbb{R}^3)$ since $\{h(v_k)\}$ is bounded in L^1_{loc}. Now every function $\psi \in \mathcal{D}(\mathbb{R}^3)$ can be written as $\psi = \varphi^3$ with $\varphi \in C^0_0(\mathbb{R}^3)$. To prove (8) we may assume after scaling and translation, that supp $\varphi \subset Q = (0,1)^3$ and we let

$$\bar{v}_k = \int_Q \varphi v_k, \quad \tilde{v}_k = \varphi v_k - \bar{v}_k.$$

Then

$$\begin{aligned} \tilde{v}_k &\rightharpoonup 0 \quad \text{in} \quad L^3(Q),\ \bar{v}_k \to 0, \\ A(D\tilde{v}_k) &\to 0 \quad \text{in} \quad W^{-1,3}(Q). \end{aligned}$$

Lemma 2 implies that

$$\begin{aligned} & \int_{\mathbb{R}^3} h(v_k)\varphi^3\,dx = \int_Q h(\varphi v_k)\,dx \\ = \quad & \int_Q h(\bar{v}_k) + Dh(\bar{v}_k)\tilde{v}_k + \frac{1}{2}D^2h(\bar{v}_k)(\tilde{v}_h,\tilde{v}_h) + h(\tilde{v}_k)\,dx \\ \to \quad & 0, \quad \text{as} \quad k \to \infty. \end{aligned}$$

This shows that

$$h(v_k) \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}' \tag{9}$$

if $v = 0$. Using (4) one shows similarly that

$$v^i_k v^j_k \rightharpoonup 0 \quad \text{in} \quad L^{3/2}_{loc} \quad \text{if} \quad i \neq j. \tag{10}$$

One first obtains convergence in $\mathcal{D}$ but the L^3_{loc} bound on v_k implies weak convergence in $L^{3/2}_{loc}$.

Finally if $v \neq 0$ let $w_k = v_k - v$. Expanding $h(v_k) = h(v + w_k)$ and using (9) and (10) for v_k we obtain the desired assertion. □

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