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by

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1 Introduction

We prove maximal regularity for vector valued solutions $u : \Omega \to \mathbb{R}^m$ of the nonlinear elliptic system

$$-\operatorname{div} \sigma(x, u, Du) = \mu \quad \text{in } \mathcal{D}'(\Omega), \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

and we establish uniqueness of solutions under a few additional assumptions. Here Ω is an open set in $\mathbb{R}^n$ and μ is a Radon measure on Ω with finite mass. The prototypical problem is the n-Laplace system

$$-\operatorname{div}(|Du|^{n-2}Du) = \mu$$

which together with variants with measurable coefficients plays an important role in quasiconformal geometry. Our results on maximal regularity ($Du \in L^{n,\infty}(\Omega)$) and on existence of solutions in $\mathbb{R}^n$ seem to be new even for that system (even for the case of a single equation). We state the general assumptions on σ in a form that is suitable for both bounded and unbounded sets Ω and which allows one to treat nonhomogeneous boundary value problems by considering $\tilde{\sigma}(x, u, F) = \sigma(x, u + \tilde{u}(x), F + D\tilde{u}(x))$ (see Section 6 for details). We assume that σ satisfies the following hypotheses:

(H0) (continuity) $\sigma \colon \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $x \mapsto \sigma(x, u, p)$ is measurable for every (u, p) and $(u, p) \mapsto \sigma(x, u, p)$ is continuous for almost every $x \in \Omega$.

(H1) (monotonicity) For all $x \in \Omega$, $u \in \mathbb{R}^m$ and all F, $G \in \mathbb{M}^{m\times n}$ there holds

$$(\sigma(x, u, F) - \sigma(x, u, G)) : (F - G) \geq 0.$$

(H2) (coercivity and growth) There exist constants $\gamma_1 > 0$, $\gamma_2 \geq 0$ and functions $\gamma_3 \in L^1(\Omega)$, $\gamma_4 \in L^{n/(n-1)}(\Omega)$ such that for all $x \in \Omega$, $u \in \mathbb{R}^m$ and $F \in \mathbb{M}^{m\times n}$

$$\begin{aligned} \sigma(x, u, F) : F &\geq \gamma_1 |F|^n - \gamma_3(x), \\ |\sigma(x, u, F)| &\leq \gamma_2 |F|^{n-1} + \gamma_4(x). \end{aligned}$$

(H3) (structure condition) There exist constants $1 \leq s < n$, $\gamma_5 \geq 0$ and a function $\gamma_6 \in L^1(\Omega)$ such that for all $x \in \Omega$, $u \in \mathbb{R}^m$ and $F \in \mathbb{M}^{m\times n}$ the inequality

$$\sigma(x, u, F) : MF \geq -\gamma_5 |F|^s - \gamma_6(x)$$

holds for all matrices $M \in \mathbb{M}^{m\times m}$ of the form $M = \mathrm{Id} - a \otimes a$ with $|a| \leq 1$.

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While (H0), (H1) and (H2) are natural, (H3), which has been in common use since the work of Landes [La], is little understood. It guarantees a lower bound when the system is tested with radially symmetric truncations of u in the target. It is in particular satisfied for diagonal systems and certain perturbations thereof.

The proofs of some of our results require that $\Omega^C = \mathbb{R}^n \setminus \Omega$ is a domain of type A. Here we say that a set E has property A if there exists a constant $K > 0$ such that for all $x \in \bar{E}$ and $0 < r < \operatorname{diam}(E)$ the inequality $|Q(x,r) \cap E| \geq K r^n$ holds.

Our main results on existence, regularity and uniqueness are the following:

Theorem 1.1 (Existence and regularity) *Let Ω be a bounded, open set such that Ω^C has property A. Suppose that the hypotheses (H0)–(H3) and one of the following conditions are satisfied:*

(i) $F \mapsto \sigma(x,u,F)$ is a C^1 function.

(ii) There exists a function $W\colon \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$ and $F \mapsto W(x,u,F)$ is convex and C^1.

(iii) σ is strictly monotone, i.e., σ is monotone and $(\sigma(x,u,F)-\sigma(x,u,G)) : (F-G) = 0$ implies $F = G$.

Let μ be an $\mathbb{R}^m$-valued Radon measure on Ω with finite mass. Then the system (1.1), (1.2) has a solution $u \in \mathrm{BMO}(\Omega;\mathbb{R}^m) \cap W_0^{1,q}(\Omega;\mathbb{R}^m)$ for all $q < n$, and the solution satisfies the a priori estimate

$$\|u\|_{\mathrm{BMO}(\Omega;\mathbb{R}^m)} \leq C_2\big(\|\mu\|_{\mathcal{M}}^{\frac{1}{n-1}} + \gamma_5^{\frac{1}{n-s}}|\Omega|^{\frac{1}{n}} + \big\||\gamma_3| + |\gamma_4|^{\frac{n}{n-1}} + |\gamma_6|\big\|_{L^1(\Omega)}\big). \tag{1.3}$$

Moreover, Du belongs to the weak Lebesgue space $L^{n,\infty}(\Omega;\mathbb{M}^{m\times n})$ and

$$\|Du\|_{L^{n,\infty}(\Omega;\mathbb{M}^{m\times n})} \leq C_1\big(\|\mu\|_{\mathcal{M}}^{\frac{1}{n-1}} + \gamma_5^{\frac{1}{n-s}}|\Omega|^{\frac{1}{n}} + \big\||\gamma_3| + |\gamma_4|^{\frac{n}{n-1}} + |\gamma_6|\big\|_{L^1(\Omega)}\big). \tag{1.4}$$

Here the constant C_1 depends only on γ_1, γ_2, K, and n. The constant C_2 depends in addition on $|\Omega|$.

We prove Theorem 1.1 at the end of Section 3.

Remarks. 1) A local version of the BMO estimate was proven in [DHM], and (1.3) follows by adapting the methods used in the interior situation to the boundary situation. We give the proof in Section 2 for the convenience of the reader and in order to derive the Caccioppoli estimates in Lemma 2.2 that are important ingredients in the proof of (1.4).
2) It is often convenient to extend u by zero to $\mathbb{R}^n$. In particular $\|u\|_{\mathrm{BMO}(\Omega;\mathbb{R}^m)}$ refers to the norm of that extension. If we use the seminorm $[u]_{\mathrm{BMO}(\Omega;\mathbb{R}^m)}$ in (1.3) instead of the the norm $\|u\|_{\mathrm{BMO}(\Omega;\mathbb{R}^m)}$, then the constant C_2 does not depend on $|\Omega|$.
3) Clearly (1.4) implies (1.3). However, we use the BMO estimate for u in the proof of (1.4) which we give in Section 3.
4) The example of the nonlinear Green's function $G_n(x) = c(n)\ln(|x|)$ for the n-Laplace equation $\operatorname{div}(|DG_n|^{n-2}DG_n) = \delta_0$ shows that our results are optimal.
5) Independently and with different techniques involving a nonlinear Hodge decomposition it was shown in [GIS] that the nonhomogeneous n-harmonic equation $\operatorname{div}(|Du|^{n-2}Du) = \mu$ has a unique solution $u \in W_0^{1,n)}$ for all Radon measures μ. A function u belongs to the grand Sobolev space

$W_0^{1,n)}$ if $u \in W_0^{1,s}(\Omega)$ for all $1 \le s < n$ and if

$$\sup_{0<\varepsilon\le n-1} \left(\varepsilon \int_\Omega |Du|^{n-\varepsilon} dx \right)^{1/(n-\varepsilon)} < \infty.$$

These results can be extended to the case of systems and more general operators of the form $\mathcal{A}(x, Du)$. Note that $u \in W_0^{1,n)}(\Omega)$ does not imply $Du \in L^{n,\infty}(\Omega; \mathbb{R}^n)$, while, for Lipschitz domains Ω, $u \in W_0^{1,1}(\Omega)$ and $Du \in L^{n,\infty}(\Omega; \mathbb{R}^n)$ imply $u \in W_0^{1,n)}(\Omega)$.
6) A BMO estimate for uniformly elliptic n-Laplace type equations has independently been proved in [FF]. Moreover, it is shown in [FF] that these equations admit a solution in VMO, provided the measure μ has no atoms (see also our Corollary 3.6 for the corresponding statement for systems).

Theorem 1.2 (Uniqueness) *Let Ω be a bounded, open set in $\mathbb{R}^n$ such that Ω^C is of type A. Suppose that the hypotheses (H0), (H2) and (H3) are satisfied and that σ is independent of u and uniformly monotone, i.e., there exists a constant $\gamma_0 > 0$ such that*

$$(\sigma(x, F) - \sigma(x, G)) : (F - G) \ge \gamma_0 |F - G|^n$$

for all F, $G \in \mathbb{M}^{m\times n}$ and all $x \in \Omega$. Assume that $u, v \in W^{1,1}(\Omega; \mathbb{R}^m)$ satisfy $u - v \in W_0^{1,1}(\Omega; \mathbb{R}^m)$, Du, $Dv \in L^{n,\infty}(\Omega; \mathbb{M}^{m\times n})$ and

$$\operatorname{div} \sigma(x, Du) = \operatorname{div} \sigma(x, Dv) \quad \text{in } \mathcal{D}'(\Omega). \tag{1.5}$$

Then $u \equiv v$ in Ω.

Remark. The regularity assumption on one of the solutions can be relaxed: it suffices that $Dv \in L^{n-\varepsilon}(\Omega)$ for an $\varepsilon > 0$ which depends only on Ω and $\gamma_0, \gamma_1, \gamma_3$ (see Theorem 4.3).

The problem $\Omega = \mathbb{R}^n$ is of particular interest in connection with the theory of quasiconformal maps. We prove the following theorem:

Theorem 1.3 (Solutions in $\mathbb{R}^n$) *Let μ be an $\mathbb{R}^m$-valued Radon measure on $\Omega = \mathbb{R}^n$ of finite mass. Suppose that the hypotheses (H0)–(H3) and one of the conditions (i)–(iii) of Theorem 1.1 are satisfied and that σ does not depend on u. If in addition $\gamma_5 = 0$, then the system*

$$-\operatorname{div} \sigma(x, Du) = \mu \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \tag{1.6}$$

has a distributional solution u which satisfies the a priori estimate

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} + \|Du\|_{L^{n,\infty}(\mathbb{R}^n;\mathbb{M}^{m\times n})} \le C_2 \big(\|\mu\|_{\mathcal{M}}^{\frac{1}{n-1}} + \big\| |\gamma_3| + |\gamma_4|^{\frac{n}{n-1}} + |\gamma_6| \big\|_{L^1(\mathbb{R}^n)} \big).$$

For existence results on general unbounded domains see Theorem 5.1.

The rest of this paper is organized as follows: In Section 2 we derive the BMO-estimate through Caccioppoli inequalities and a blow up argument (inspired by L. Simon's proof of $C^{0,\alpha}$ estimates for the Poisson equation in [Si1] and [Si2]).

In Section 3 we derive the weak-L^n estimate for the gradient of solutions. We first use a Caccioppoli inequality and the BMO-estimate to derive a reverse Hölder inequality for $q < n$. A careful analysis of the distribution function of Du then yields the desired estimate. Roughly speaking, we exploit the fact that the solution of the free system ($\mu = 0$) has slight additional regularity properties while the influence of μ on the solution is locally controlled by the maximal function of μ up to a solution of the free system.

In Section 4 we prove uniqueness under a few additional assumptions. The difficulty is that although the operator is uniformly monotone, the solution is not an admissible test function. The key idea is to use Lipschitz test functions that agree with u on a large set.

We conclude by existence results on unbounded domains (in Section 5) and a discussion of the nonhomogeneous Dirichlet problem and local regularity in Section 6.

2 A BMO estimate for the solution u

A function u is said to belong to $\mathrm{BMO}(\mathbb{R}^n)$, the space of functions of bounded mean oscillation, if $u \in L^n(\mathbb{R}^n)$ and

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} = \Big(\sup_{a\in\mathbb{R}^n}\sup_{r>0} r^{-n}\int_{Q(a,r)} |u-(u)_{a,r}|^n dx\Big)^{1/n} < \infty,$$

where $Q(a,r)$ is the cube $\{x \in \mathbb{R}^n : |x_i - a_i| < \frac{r}{2} \text{ for } i = 1,\dots,n\}$ and $(u)_{a,r}$ denotes the mean value of u on $Q(a,r)$. In the following we will always assume that all functions under consideration are extended by zero to $\mathbb{R}^n$. Equipped with the norm

$$\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} = \|u\|_{L^n(\mathbb{R}^n)} + [u]_{\mathrm{BMO}(\mathbb{R}^n)}$$

$\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space. If $u = 0$ on Ω^C and $a \in \Omega^C$ then the mean value of u on $Q(a,r)$ is estimated by the mean oscillation:

$$|(u)_{a,r}||Q(a,r)\cap\Omega^C| \le \int_{Q(a,r)} |u-(u)_{a,r}|dx \le |Q(a,r)|[u]_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{2.1}$$

In particular, if Ω^C has property A and $0 < r < \mathrm{diam}(\Omega^C)$, then

$$|(u)_{a,r}| \le C(K)[u]_{\mathrm{BMO}(\mathbb{R}^n)}, \tag{2.2}$$

where K is the constant that appears in the definition of property A, and

$$\Big(r^{-n}\int_{Q(a,r)} |u|^n dx\Big)^{1/n} \le C(n,K)\Big(r^{-n}\int_{Q(a,r)} |u-(u)_{a,r}|^n dx\Big)^{1/n}. \tag{2.3}$$

It is also convenient to define the following local version of the BMO norm:

$$[u]_{\mathrm{BMO}(Q)} = \sup_{Q(a,r)\subset Q}\Big(\frac{1}{r^n}\int_{Q(a,r)} |u-(u)_{a,r}|^n dx\Big)^{1/n}.$$

The following important property of functions of bounded mean oscillation was proved in the fundamental paper of John and Nirenberg [JN].

Lemma 2.1 *Assume that $u \in \mathrm{BMO}(\mathbb{R}^n)$. Then there exist constants b, $B > 0$ such that*

$$|\{x \in Q(a,r) : |u(x)-(u)_{a,r}| > \sigma\}| \le B e^{-b\sigma/[u]_{\mathrm{BMO}(Q(a,r))}}|Q(a,r)| \tag{2.4}$$

for all cubes $Q(a,r)$.

As a consequence (or directly from the definition of the BMO-norm) we obtain the following inequality: if $Q(a_1,r_1) \subset Q(a_2,r_2)$ then

$$
|(u)_{a_1,r_1} - (u)_{a_2,r_2}| \le c(b,B)(1+\ln\frac{r_2}{r_1})[\,u\,]_{\mathrm{BMO}(Q(a_2,r_2))}. \tag{2.5}
$$

In the next lemma we establish Caccioppoli estimates for solutions $u \in \mathcal{D}^{1,n}(\Omega;\mathbb{R}^n)$ of system (1.1) with smooth right hand side f. Here, $\mathcal{D}^{1,n}(\Omega;\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\Omega;\mathbb{R}^n)$ in the seminorm $\|Du\|_{L^n(\Omega)}$. For the moment we allow Ω to be unbounded. These estimates are crucial in the proof of the BMO–estimate for u.

Lemma 2.2 *Let $u \in \mathcal{D}^{1,n}(\Omega;\mathbb{R}^m)$ be a solution of system (1.1) with $f \in L^1(\Omega;\mathbb{R}^m) \cap C^\infty(\Omega;\mathbb{R}^m)$ in place of μ. Let $g = |\gamma_3| + |\gamma_4|^{n/(n-1)} + \gamma_5|Du|^s + |\gamma_6|$, and $0<\rho<r$. There exists a constant C_3, which depends only on γ_1, γ_2 and n, such that the following inequalities hold:*
i) (Interior estimate) We have for all cubes $Q(a,r) \subset \Omega$, all $\beta \in \mathbb{R}^m$ and all $\alpha > 0$

$$
\int\limits_{\substack{\{|u-\beta|<\alpha\}\\ \cap Q(a,\rho)}} |Du|^n dx \le \frac{C_3}{(r-\rho)^n} \int\limits_{Q(a,r)\setminus Q(a,\rho)} |u-\beta|^n dx + C_3 \int\limits_{Q(a,r)} (\alpha|f| + g)dx. \tag{2.6}
$$

ii) (Boundary estimate) We have for all cubes $Q(a,r)$ and all $\alpha > 0$

$$
\int\limits_{\substack{\{|u|<\alpha\}\\ \cap Q(a,\rho)}} |Du|^n dx \le \frac{C_3}{(r-\rho)^n} \int\limits_{Q(a,r)\setminus Q(a,\rho)} |u|^n dx + C_3 \int\limits_{Q(a,r)} (\alpha|f| + g)dx. \tag{2.7}
$$

iii) (Estimate on annuli) We have for all cubes $Q(a,r)$ such that $Q(a,r)\setminus Q(a,\frac{r}{8}) \subset \Omega$, all $\beta \in \mathbb{R}^m$ and all $\alpha > 0$

$$
\int\limits_{\substack{\{|u-\beta|<\alpha\}\\ \cap Q(a,\frac{r}{2})\setminus Q(a,\frac{r}{4})}} |Du|^n dx \le \frac{C_3}{r^n} \int\limits_{Q(a,r)\setminus Q(a,\frac{r}{8})} |u-\beta|^n dx + C_3 \int\limits_{Q(a,r)} (\alpha|f| + g)dx. \tag{2.8}
$$

iv) (Global estimates) Assume in addition that $Du \in L^s(\Omega;\mathbb{M}^{m\times n})$ or $\gamma_5 = 0$. Then we have for all $\alpha > 0$

$$
\int\limits_{\{|u|<\alpha\}} |Du|^n dx \le C_3 \int\limits_{\Omega} (\alpha|f| + g)dx. \tag{2.9}
$$

If, moreover, $\Omega^C \subset Q(a,\rho)$, then we have for all $\beta \in \mathbb{R}^m$

$$
\int\limits_{\substack{\{|u-\beta|<\alpha\}\\ \cap(\mathbb{R}^n\setminus Q(a,r))}} |Du|^n dx \le \frac{C_3}{(r-\rho)^n} \int\limits_{Q(a,r)\setminus Q(a,\rho)} |u-\beta|^n dx + C_3 \int\limits_{\mathbb{R}^n\setminus Q(a,\rho)} (\alpha|f| + g)dx. \tag{2.10}
$$

Proof. We first prove *ii)*. Let $\eta \in C_0^\infty(Q(a,r))$ be a cut-off function such that $\eta \equiv 1$ on $Q(a,\rho)$, $0 \le \eta \le 1$ and $|D\eta| \le C/(r-\rho)$. Choose a smooth function $g_\alpha : \mathbb{R} \to \mathbb{R}$ with the following properties: $g_\alpha \equiv \mathrm{Id}$ on $[0,\alpha]$, $0 \le g_\alpha \le n\alpha$, $g'_\alpha \le 1$ and

$$
0 < c\left(\frac{g_\alpha(s)}{s}\right)^{n/(n-1)} \le g'_\alpha(s) \le \frac{g_\alpha(s)}{s} \le 1 \quad \text{on } (0,\infty). \tag{2.11}
$$

Define the cut-off function φ_α in the target by

$$\varphi_\alpha(z) = \frac{g_\alpha(|z|)}{|z|}\, z.$$

Then

$$D(\varphi_\alpha \circ u) = \frac{g_\alpha(|u|)}{|u|}\left(\mathrm{Id} - \frac{u}{|u|} \otimes \frac{u}{|u|}\right) Du + g_\alpha'(|u|)\left(\frac{u}{|u|} \otimes \frac{u}{|u|}\right) Du,$$

and by (2.11), (H2) and (H3) with $M = \mathrm{Id} - \frac{u}{|u|} \otimes \frac{u}{|u|}$ we deduce

$$\begin{aligned}\sigma(Du) : D(\varphi_\alpha \circ u) &= g_\alpha'(|u|)\sigma(Du) : Du + \Big(\frac{g_\alpha(|u|)}{|u|} - g_\alpha'(|u|)\Big)(\sigma(Du) : MDu) \\ &\geq g_\alpha'(|u|)(\gamma_1|Du|^n - \gamma_3) - (\gamma_5|Du|^s + \gamma_6).\end{aligned}$$

Notice that we frequently drop the first two variables of σ in the notation if no confusion arises. If we multiply equation (1.1) by $\eta^n \varphi_\alpha \circ u \in W_0^{1,n}(\mathbb{R}^n;\mathbb{R}^m) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^m)$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} \eta^n \sigma(Du) : D(\varphi_\alpha \circ u)dx \\ &\qquad = -\int_{\mathbb{R}^n} n\eta^{n-1}\sigma(Du) : \varphi_\alpha \circ u \otimes D\eta dx + \int_{\mathbb{R}^n} \eta^n f \varphi_\alpha \circ u dx.\end{aligned}$$

It follows by (H2), Hölder's inequality and (2.11) on the right hand side that

$$\begin{aligned}&\gamma_1 \int_\Omega \eta^n |Du|^n g_\alpha'(|u|)\, dx \\ &\leq \frac{C}{r-\rho}\left(\int_{Q(a,r)} \eta^n g_\alpha'(|u|)(\gamma_2|Du|^{n-1} + |\gamma_4|)^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \left(\int_{Q(a,r)\setminus Q(a,\rho)} |u|^n dx\right)^{\frac{1}{n}} \\ &+ \int_{Q(a,r)} |\gamma_3| dx + \int_{Q(a,r)} (\gamma_5|Du|^s + |\gamma_6|)dx + C\alpha \int_{Q(a,r)} |f| dx.\end{aligned}$$

Application of Young's inequality yields

$$\begin{aligned}&\int_\Omega \eta^n g_\alpha'(|u|)|Du|^n dx \\ &\qquad \leq \frac{C}{(r-\rho)^n}\int_{Q(a,r)\setminus Q(a,\rho)} |u|^n dx + C\Big(\int_{Q(a,r)} g dx + \alpha \int_{Q(a,r)} |f| dx\Big),\end{aligned}$$

and inequality (2.7) follows from the definition of g_α.

The proof of *i)* and *iii)* is analogous, see also Lemma 15 in [DHM]. Finally, the proofs of the two estimates in *iv)* follow as the proofs for *ii)* and *i)*, since in view of Lemma A.2 and A.3 in the appendix $g_\alpha \circ u$ and $\eta^n g_\alpha \circ (u - \beta)$ are admissible test functions. Here $\eta \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function such that $0 \leq \eta \leq 1$, $\eta \equiv 0$ on $Q(a,\rho)$, $\eta \equiv 1$ on $\mathbb{R}^n \setminus Q(a,r)$, and $|D\eta| \leq C/(r-\rho)$. $\square$

The next two lemmas summarize well–known facts which will be used in the proof of the BMO–estimate in Lemma 2.5.

Lemma 2.3 *Assume that $v \in \mathrm{BMO}(\mathbb{R}^n;\mathbb{R}^m)$, $R > 0$ and $\alpha_0 > 0$ are such that*

$$\int_{\substack{\{|v|<\alpha\} \\ \cap Q(0,R)}} |Dv|^n dx \leq C(1+\alpha) \text{ for all } \alpha \geq \alpha_0,$$

and that either $(v)_{0,1} = 0$ or there exists $r > 0$, $K > 0$ such that $|\{v = 0\} \cap Q(0,r)| \geq Kr^n$. Then $\|v\|_{W^{1,s}(Q(0,R);\mathbb{R}^m)} \leq C'(s, R, |\ln \frac{R}{r}|, C, \alpha_0, [v]_{\mathrm{BMO}})$ for all $1 \leq s < n$.

Proof. Since the assumptions are invariant under the rescaling $x \mapsto Rx$, and the scaling of the $W^{1,s}$-norm is known we may assume that $R = 1$. The assertion then follows from the estimate

$$\|v\|_{W^{1,s}(Q)} \leq C'(C, \alpha_0) + \fint_Q |v| dx$$

in connection with (2.1), (2.5) and the obvious inequality $(|v|)_{0,1} \leq [v]_{\mathrm{BMO}} + |(v)_{0,1}|$. The proof of the $W^{1,s}$-estimate is standard (see [Ta] or [DHM], proof of Lemma 10). Indeed the inequality $|D|v|| \leq |Dv|$ and an application of the Sobolev-Poincaré inequality to the truncated function $v_\alpha = \min(\alpha, |v|)$ yields

$$\|v_\alpha\|_p \leq C_p \big(C(1+\alpha)\big)^{1/n} + \fint_Q v_\alpha dx \leq \tilde{C}_p (1+\alpha)^{1/n} + \fint_Q |v| dx,$$

for all $p < \infty$. Splitting the set $\{|Dv| > t\}$ into regions where $\{|v| \geq \alpha\}$ (and hence $v_\alpha = \alpha$) and $\{|v| < \alpha\}$ one easily concludes by taking $\alpha = t^{n-\tilde{s}}$, $p = \frac{n-1}{n} \frac{\tilde{s}}{n-\tilde{s}}$ with $\tilde{s} \in (s, n)$. □

Lemma 2.4 *Assume that $v_k \rightharpoonup v$ in $W^{1,s}(Q(0,R);\mathbb{R}^n)$ for all $1 \leq s < n$ and that*

$$\int_{\substack{\{|v_k - \beta| < \alpha\} \\ \cap Q(0,R)}} |Dv_k|^n dx \leq h_k + \alpha \delta_k$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^m$, h_k, $\delta_k \in \mathbb{R}^+$ with $\limsup_{k\to\infty} h_k = h$ and $\lim_{k\to\infty} \delta_k = 0$. Then

$$\int_{\substack{\{|v - \beta| < \alpha\} \\ \cap Q(0,R)}} |Dv|^n dx \leq \int_{Q(0,R)} |Dv|^n dx \leq h.$$

Proof. This follows easily from the weak lower semicontinuity of the L^n-norm and the monotone convergence theorem (c.f. also the proof of Lemma 15 in [DHM]). □

We are now in a position to prove that a function satisfying (2.6) – (2.10) is a function of bounded mean oscillation.

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^n$ be an open domain and assume that $\Omega = \mathbb{R}^n$ or that Ω^C has property A. Let $u \in \mathcal{D}^{1,n}(\Omega)$ and suppose that there exist f, $g \in L^1(\Omega)$ such that the estimates (2.6) – (2.10) hold. Then $u \in \mathrm{BMO}(\mathbb{R}^n)$ and*

$$[\, u\,]_{\mathrm{BMO}(\mathbb{R}^n)} \leq C_4 \cdot (\|f\|_{L^1(\Omega)}^{1/(n-1)} + \|g\|_{L^1(\Omega)}^{1/n}),$$

where C_4 depends only on C_3, n, and the constant K in the definition of property A.

Proof. The proof is inspired by Simon's beautiful proof of $C^{0,\alpha}$ estimates for the Poisson equation in [Si1] (see also [Si2]). We argue by contradiction and use a scaling and blow up argument to construct a sequence v_k such that $[\, v_k\,]_{\mathrm{BMO}} = 1$. This sequence converges to a limit $v \in \mathcal{D}^{1,n}(\mathbb{R}^n)$

which corresponds to a solution of the homogeneous problem, i.e., satisfies (2.6) and (2.7) with $f = 0$ and $g = 0$. We will deduce $v \equiv const$ and this leads to a contradiction. We will later distinguish different cases which correspond to a limit problem on $\mathbb{R}^n$, on a domain with unbounded complement or a domain with bounded complement. In the first two situations (Cases 1 and 2 below) we use the local inequalities (2.6) and (2.7) as well as condition A to bound the sequence v_k while we employ the global estimates (2.9) and (2.10) if the complement is bounded (Cases 3 and 4 below).

Suppose the assertion of the lemma were false. Then there exists a sequence of functions $u_k \in \mathcal{D}^{1,n}(\Omega)$, f_k, $g_k \in L^1(\Omega)$ such that (2.6) – (2.10) hold (with u replaced by u_k), but

$$\lim_{k\to\infty} \frac{[\,u_k\,]_{\mathrm{BMO}(\mathbb{R}^n)}}{\|f_k\|_{L^1(\Omega)}^{1/(n-1)} + \|g_k\|_{L^1(\Omega)}^{1/n}} = \infty. \tag{2.12}$$

(In view of (2.9) we may assume that $\|f_k\|_{L^1(\Omega)} + \|g_k\|_{L^1(\Omega)} \neq 0$.) By definition of the BMO norm there exist $x_k \in \mathbb{R}^n$ and $r_k > 0$ such that

$$\frac{1}{r_k^n} \int_{Q(x_k,r_k)} |u_k - (u_k)_{x_k,r_k}|^n \, dx \geq \frac{1}{2} [\,u_k\,]^n_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Let $d_k = \operatorname{dist}(x_k, \Omega^C)$ ($d_k = \infty$ if $\Omega = \mathbb{R}^n$).
Case 1: Suppose that $\Omega = \mathbb{R}^n$ or that there exists a subsequence (not relabeled) such that $\frac{d_k}{r_k} \to \infty$. The rescaled functions

$$v_k(z) = \frac{u_k(x_k + r_k z) - (u_k)_{x_k,r_k}}{[\,u_k\,]_{\mathrm{BMO}}}$$

satisfy

$$(v_k)_{0,1} = 0, \quad \int_{Q(0,1)} |v_k|^n dx \geq \frac{1}{2}, \quad [\,v_k\,]_{\mathrm{BMO}} = 1. \tag{2.13}$$

Changing coordinates in (2.6) yields

$$\int_{\substack{\{|v_k-\beta|<\alpha\}\\ \cap Q(0,\rho)}} |Dv_k|^n dz \leq \frac{C_3}{(r-\rho)^n} \int_{Q(0,r)\setminus Q(0,\rho)} |v_k - \beta|^n dz + C_3 \Big(\frac{\alpha \|f_k\|_{L^1(\Omega)}}{[\,u_k\,]^{n-1}} + \frac{\|g_k\|_{L^1(\Omega)}}{[\,u_k\,]^n} \Big)$$

whenever $Q(0,r) \subset \Omega_k = \frac{1}{r_k}(-x_k + \Omega)$. By assumption $\Omega_k = \mathbb{R}^n$ or $\Omega_k \to \mathbb{R}^n$ as $k \to \infty$ (i.e., for all $R > 0$ there exists a k_0 such that $Q(0,R) \subset \Omega_k$ for all $k \geq k_0$). It follows from (2.13) and (2.5) that $|(v_k)_{0,r}| \leq \gamma(r) = C(|\ln(r)| + 1)$. Since $\{|v_k| < \alpha - \gamma(r)\} \subset \{|v_k - (v_k)_{0,r}| < \alpha\}$ we obtain with $\rho = r/2$, $\beta = (v_k)_{0,r}$ and $[\,v_k\,]_{\mathrm{BMO}} = 1$

$$\int_{\substack{\{|v_k|<\alpha-\gamma(r)\}\\ \cap Q(0,r/2)}} |Dv_k|^n dz \leq 2^n C_3 + C_3 \Big(\frac{\alpha \|f_k\|_{L^1(\Omega)}}{[\,u_k\,]^{n-1}} + \frac{\|g_k\|_{L^1(\Omega)}}{[\,u_k\,]^n} \Big),$$

and hence we may apply Lemma 2.3 for v_k with $\alpha_0(r) = 2\gamma(r)$ and $k \geq k_0(r)$. We deduce that $\{v_k\}$ is bounded in $W^{1,s}_{\mathrm{loc}}(\mathbb{R}^n)$ and thus (for a subsequence) $v_k \rightharpoonup v$ in $W^{1,s}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $s < n$. In view of (2.12) and Lemma 2.4 we conclude $v \in \mathcal{D}^{1,n}(\mathbb{R}^n)$. Another application of Lemma 2.4 in connection with Poincaré's inequality then shows that

$$\int_{Q(0,r/2)} |Dv|^n dz \leq \inf_{\beta\in\mathbb{R}} \frac{2^n C_3}{r^n} \int_{Q(0,r)\setminus Q(0,r/2)} |v - \beta|^n dz \leq C \int_{Q(0,r)\setminus Q(0,r/2)} |Dv|^n dz.$$

For $r \to \infty$ we infer $Dv \equiv 0$. On the other hand, the strong convergence of v_k together with (2.13) implies $(v)_{0,1} = 0$ and therefore, again by Poincaré,

$$\frac{1}{2} \leq \int_{Q(0,1)} |v|^n dz \leq C \int_{Q(0,1)} |Dv|^n dz = 0.$$

This contradiction finishes the proof in case 1.
Case 2: Suppose now that $\Omega \neq \mathbb{R}^n$ and that for a subsequence (not relabeled) $\frac{d_k}{r_k} \leq C$ and $\frac{\operatorname{diam}\Omega^C}{r_k} \to \infty$ ($\operatorname{diam}\Omega = \infty$ if Ω is unbounded). In this case there exist points $\tilde{x}_k \in \Omega^C$ such that $r_k^{-1}|\tilde{x}_k - x_k| \leq C$. Passing to a further subsequence if necessary, we may assume

$$z_k = \frac{x_k - \tilde{x}_k}{r_k} \to \bar{z} \in \mathbb{R}^n \quad \text{for } k \to \infty. \tag{2.14}$$

Let

$$v_k(z) = \frac{u_k(\tilde{x}_k + r_k z)}{[u_k]_{\mathrm{BMO}}}, \quad \Omega_k = \frac{1}{r_k}(-\tilde{x}_k + \Omega).$$

Then

$$[v_k]_{\mathrm{BMO}} = 1, \quad \int_{Q(z_k,1)} |v_k - (v_k)_{z_k,1}|^n dx \geq \frac{1}{2}, \tag{2.15}$$

and (2.7) implies

(2.16)
$$\int_{\substack{|v_k|<\alpha \\ \cap Q(0,\rho)}} |Dv_k|^n dx \leq \frac{C_3}{(r-\rho)^n} \int_{Q(0,r)\setminus Q(0,\rho)} |v_k|^n dx + C_3\Big(\frac{\alpha\|f_k\|_{L^1(\Omega)}}{[u_k]^{n-1}} + \frac{\|g_k\|_{L^1(\Omega)}}{[u_k]^n}\Big).$$

Since $0 \in \Omega_k^C$ and Ω_k^C has property A we obtain

$$Q(0,r) \cap \Omega_k^C \geq Kr^n \text{ for all } r \leq R_k = \operatorname{diam}(\Omega_k^C) = r_k^{-1}\operatorname{diam}(\Omega^C) \to \infty.$$

It follows from (2.16), (2.2) and $[v_k]_{\mathrm{BMO}} = 1$ that

$$\int_{\substack{\{|v_k|<\alpha\} \\ \cap Q(0,r)}} |Dv_k|^n dx \leq C + \delta_k\alpha \text{ for } r \leq R_k/2,$$

and that $\delta_k \to 0$ as $k \to \infty$. By Lemma 2.3 there exists a subsequence (not relabeled) such that $v_k \rightharpoonup v$ in $W^{1,s}_{\mathrm{loc}}(\mathbb{R}^n)$. In view of Lemma 2.4 we deduce $Dv \in L^n(\mathbb{R}^n)$ and from (2.16) we infer in addition

$$\int_{Q(0,\rho)} |Dv|^n dx \leq \frac{C_3}{(r-\rho)^n} \int_{Q(0,r)\setminus Q(0,\rho)} |v|^n dx. \tag{2.17}$$

Let θ be such that $\theta^n = K/2 < 1$. We obtain from (2.17) with $\rho = \theta r$ and Poincaré's inequality (see, e.g., [Mo], Theorem 3.6.5; we justify the application below)

$$\int_{Q(0,\theta r)} |Dv|^n dx \leq \frac{C}{r^n} \int_{Q(0,r)\setminus Q(0,\theta r)} |v|^n dx \leq C \int_{Q(0,r)\setminus Q(0,\theta r)} |Dv|^n dx$$

and consequently $Dv \equiv 0$. This leads to a contradiction with (2.15) since by (2.14) we have $z_k \to \bar{z}$ for $k \to \infty$. It remains to justify the use of Poincaré's inequality. Choose k_0 such that $r \le \operatorname{diam}(\Omega_k)$ for all $k \ge k_0$. Then we have by property A that

$$|(Q(0,r) \setminus Q(0,\theta r)) \cap \Omega_k^C| \ge K r^n - \theta^n r^n = \frac{K}{2} r^n.$$

Consider the characteristic function $\chi_k = \chi_{\Omega_k^C}$. It follows from the estimate above that

$$\chi_k \rightharpoonup \eta \text{ weakly* in } L^\infty(\mathbb{R}^n), \qquad \int_{Q(0,r)\setminus Q(0,\theta r)} \eta dx \ge \frac{K}{2} r^n.$$

The strong convergence of v_k implies

$$0 = \int_{Q(0,r)\setminus Q(0,\theta r)} \chi_k |v_k| dx \to \int_{Q(0,r)\setminus Q(0,\theta r)} \eta |v| dx.$$

Since $0 \le \eta \le 1$ we conclude

$$|\{v = 0\} \cap (Q(0,r) \setminus Q(0,\theta r))| \ge |\{\eta > 0\} \cap (Q(0,r) \setminus Q(0,\theta r))| \ge \frac{K}{2} r^n.$$

This concludes the proof of the second case.
Case 3: Suppose that for a subsequence $\frac{d_k}{r_k} \le C$ and $0 < c \le \frac{\operatorname{diam} \Omega^C}{r_k} \le C$. Clearly $\operatorname{diam} \Omega^C < \infty$ and $r_k \le C$. Define the rescaled functions v_k and the rescaled domains Ω_k as in Case 2. Since $0 \in \Omega_k^C$ we may choose a further subsequence such that $\Omega_k^C \to \Omega_\infty^C$ with $|\Omega_\infty^C| > 0$. From (2.9) and property A we deduce as before that $v_k \rightharpoonup v$ in $W^{1,s}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ for $s < n$ and $Dv \equiv 0$. This contradicts (2.15).
Case 4: Assume that for a subsequence $\frac{d_k}{r_k} \le C$ and $\frac{\operatorname{diam} \Omega^C}{r_k} \to 0$. Define the rescaled functions v_k as in Case 1. In this case, $\operatorname{diam}(\Omega^C) < \infty$ and $r_k \to \infty$. Therefore we may choose a subsequence such that $r_k^{-1} \tilde{x}_k \to \bar{a}$ and $\Omega^C \to 0$. Using the global estimate (2.10) we conclude as in Case 1. □

Corollary 2.6 *Let $\Omega \subset \mathbb{R}^n$ be an open domain and assume that $\Omega = \mathbb{R}^n$ or that Ω^C has property A. Let $u \in \mathcal{D}^{1,n}(\Omega)$ and suppose that there exist $f, g \in L^1(\Omega)$ such that the estimates (2.6) – (2.10) hold. Then we have the local* BMO*–estimate*

$$[u]_{\mathrm{BMO}(Q(a,r))} \le C_4 \Big(\|f\|^{1/(n-1)}_{L^1(Q(a,2r))} + \|g\|^{1/n}_{L^1(Q(a,2r))} + \frac{1}{(2r)^n} \int_{Q(a,2r)} |u - (u)_{a,2r}|^n dx \Big),$$

where C_4 depends only on C_3, n, and the constant K in the definition of property A.

Proof. This follows with an indirect argument similar to the one used in the global BMO estimate. □

3 Weak L^n-estimate for the gradient Du

Let $\Omega \subset \mathbb{R}^n$ be measurable and $1 < p < \infty$. We define the weak Lebesgue space $L^{p,\infty}(\Omega)$ by

$$\begin{aligned} L^{p,\infty}(\Omega) \quad = \quad & \{u : \Omega \to \mathbb{R} \text{ measurable} : \exists M \ge 0 \text{ such that} \\ & \quad |\{|u| > \lambda\}| \le M^p \lambda^{-p} \ \forall \lambda > 0\}. \end{aligned}$$

This is a Banach space with the norm

$$\|u\|^*_{L^{p,\infty}(\Omega)} = \sup_{t>0} t^{1/p}\Big(\frac{1}{t}\int_0^t u^*(s)ds\Big),$$

where u^* denotes the nonincreasing rearrangement of u. Let $E_\lambda = \{|u| > \lambda\}$. We will use in the sequel the quasinorm

$$\|u\|_{L^{p,\infty}(\Omega)} = \inf\{M : \lambda|E_\lambda|^{1/p} \leq M \ \forall\lambda > 0\}$$

which is equivalent to the norm $\|u\|^*_{L^{p,\infty}(\Omega)}$ (see [Hu] for more information on weak Lebesgue spaces). A useful property of weak Lebesgue spaces is the following Hölder inequality: if $u \in L^{p,\infty}(\Omega)$, $E \subset \Omega$, and $q < p$ then

$$\|u\|_{L^q(E)} \leq \big(\frac{p}{p-q}\big)^{1/q} |E|^{(1/q)-(1/p)}\|u\|_{L^{p,\infty}(\Omega)}. \tag{3.1}$$

We define for a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ the Hardy-Littlewood maximal function Mf by

$$Mf(a) = \sup_{r>0} r^{-n}\int_{Q(a,r)} |f|dx.$$

The following lemma is a well-known result in real analysis and can be found for example in [St].

Lemma 3.1 *i) If $f \in L^1(\mathbb{R}^n)$ then there exists a constant $A > 0$ which depends only on n such that the following estimate holds:*

$$|\{x : Mf(x) > \alpha\}| \leq \frac{A}{\alpha}\int_{\mathbb{R}^n} |f|dx. \tag{3.2}$$

ii) If $f \in L^p(\mathbb{R}^n)$ then $Mf \in L^p(\mathbb{R}^n)$ and there exists a constant $A > 0$ which depends only on n and p such that the following estimate holds:

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq A\|f\|_{L^p(\mathbb{R}^n)}.$$

Since the weak L^p-spaces can be characterized as interpolation spaces (see, e.g., [BuB]) we obtain by a slight generalization of Marcinkiewicz's interpolation theorem that the maximal function operator M maps $L^{p,\infty}$ to $L^{p,\infty}$ and there exists a constant $\tilde{A}$ which depends only on n and p such that

$$\|Mf\|_{L^{p,\infty}(\Omega)} \leq \tilde{A}\|f\|_{L^{p,\infty}(\Omega)}.$$

Finally we use the following version of the Sobolev–Poincaré inequality:

$$\Big(\int_{Q(a,r)} |u - (u)_{a,r/2}|^n dx\Big)^{1/n} \leq c(n)\Big(\int_{Q(a,r)} |Du|^{n/2}dx\Big)^{2/n}, \tag{3.3}$$

where the constant $c(n)$ depends only on n.

We split the proof of Theorem 1.1 into a series of lemmas. The first lemma yields a quantitative estimate of the L^q-norm of Du for $q < n$.

Lemma 3.2 *Assume that Ω^C has property A and that $u \in \mathrm{BMO}(\mathbb{R}^n) \cap \mathcal{D}^{1,n}(\Omega)$ satisfies the Caccioppoli inequalities (2.6) – (2.8) with $f, g \in L^1(\Omega)$. Then there exists a constant C_5, which* *depends only on K, q, n, and C_3, such that for all $q \in [\,\frac{n}{2}, n)$ and all $Q(a,r)$ the following estimate holds:*

$$\begin{aligned}\Big(\int_{Q(a,r)} |Du|^q dx\Big)^{1/q} &\leq C_5 r^{\frac{n}{q}-1}\Big\{[\,u\,]_{\mathrm{BMO}(Q(a,2r))} + \\ &\quad + \Big(\frac{1}{n-q}\int_{Q(a,2r)} |f| dx\Big)^{1/n-1} + \Big(\int_{Q(a,2r)} |g| dx\Big)^{1/n}\Big\}.\end{aligned}$$

Remark. If $Q(a,2r) \subset \Omega$ or $\operatorname{diam}(\Omega^C) \sim r$, then the constant C_5 is in fact independent of q.

Proof. Case 1: Assume that $Q(a, \frac{3}{2}r) \subset \Omega$. Let

$$S(\beta; k, M) = \{x \in \Omega : \ kM \leq |u - \beta| < (k+1)M\}.$$

By Hölder's inequality and the John-Nirenberg estimate (2.4) we have

$$\begin{aligned}\int_{\substack{Q(a,r)\\ \cap S((u)_{a,r};k,M)}} |Du|^q dx &\leq |Q(a,r) \cap S((u)_{a,r}; k, M)|^{1-\frac{q}{n}} \Big(\int_{\substack{Q(a,r)\\ \cap S((u)_{a,r};k,M)}} |Du|^n dx\Big)^{\frac{q}{n}} \\ &\leq C \exp\Big(\frac{-(n-q)bkM}{n[u]_{\mathrm{BMO}(Q(a,2r))}}\Big) r^{n-q} \Big(\int_{\substack{Q(a,r)\\ \cap S((u)_{a,r};k,M)}} |Du|^n dx\Big)^{\frac{q}{n}}.\end{aligned}$$

On the other hand, we have by (2.6) and (2.5)

$$\int_{\substack{Q(a,r)\\ \cap S((u)_{a,r};k,M)}} |Du|^n dx \leq C\Big([\,u\,]^n_{\mathrm{BMO}(Q(a,2r))} + (k+1)M \int_{Q(a,2r)} |f| dx + \int_{Q(a,2r)} g dx\Big).$$

Thus we obtain

$$\begin{aligned}\int_{\substack{Q(a,r)\\ \cap S((u)_{a,r};k,M)}} |Du|^q dx &\leq C \exp\Big(\frac{-(n-q)bkM}{n[u]_{\mathrm{BMO}(Q(a,2r))}}\Big) r^{n-q} \,\cdot \\ &\quad \cdot \Big([\,u\,]^n_{\mathrm{BMO}(Q(a,2r))} + (k+1)M \int_{Q(a,2r)} |f| dx + \int_{Q(a,2r)} g dx\Big)^{\frac{q}{n}}.\end{aligned}$$

If we choose $M = \frac{n[u]_{\mathrm{BMO}(Q(a,2r))}}{(n-q)b}$ and take the sum for $k = 0, 1, \ldots$ we get

$$\begin{aligned}\int_{Q(a,r)} |Du|^q dx &\leq C r^{n-q} \sum_{k=0}^{\infty} e^{-k} \Big\{[\,u\,]^q_{\mathrm{BMO}(Q(a,2r))} + \Big(\int_{Q(a,2r)} g dx\Big)^{q/n}\Big\} \\ &+ C r^{n-q} \sum_{k=0}^{\infty} e^{-k} (k+1)^{q/n} \Big(\frac{n[u]_{\mathrm{BMO}(Q(a,2r))}}{(n-q)b} \int_{Q(a,2r)} |f| dx\Big)^{q/n}.\end{aligned}$$

The sums in this estimate are easily computed:

$$\begin{aligned}\sum_{k=0}^{\infty} e^{-k} &= \frac{e}{e-1},\\ \sum_{k=0}^{\infty}(k+1)^{q/n}e^{-k} &= \Phi(\frac{1}{e},-\frac{q}{n},1) \leq \\ &\leq \Phi(\frac{1}{e},-1,1) = (\frac{e}{e-1})^2 \quad \text{for } \frac{n}{2} < q < n,\end{aligned}$$

where Φ is the Lerch function (see, e.g., [GR]). The assertion of the lemma follows now with Young's inequality.

Case 2: Assume that $Q(a,\frac{3}{2}r) \cap \Omega^C \neq \emptyset$ and $\mathrm{diam}(\Omega^C) \geq \frac{1}{10}r$. In this case, let $b \in Q(a,\frac{3}{2}r) \cap \Omega^C$. Then, by (2.1) we have

$$|(u)_{b,r/2}| \leq C[u]_{\mathrm{BMO}(Q(b,r/2))} \leq C[u]_{\mathrm{BMO}(Q(a,2r))}. \tag{3.4}$$

On the other hand, from (2.5) we infer

$$|(u)_{b,r/2} - (u)_{a,2r}| \leq C[u]_{\mathrm{BMO}(Q(a,2r))} \tag{3.5}$$

and

$$|(u)_{a,r} - (u)_{a,2r}| \leq C[u]_{\mathrm{BMO}(Q(a,2r))}. \tag{3.6}$$

Combination of (3.4)–(3.6) yields

$$|(u)_{a,r}| \leq \eta_1[\,u\,]_{\mathrm{BMO}(Q(a,2r))}, \tag{3.7}$$

where η_1 depends only on K and n. Therefore we have for $M \geq \eta_1[\,u\,]_{\mathrm{BMO}(Q(a,2r))}$ the implication

$$|u| \geq kM \implies |u-(u)_{a,r}| \geq k(M - \eta_1[\,u\,]_{\mathrm{BMO}(Q(a,2r))}),$$

and we may apply the John–Nirenberg estimate with $k(M-\eta_1[\,u\,]_{\mathrm{BMO}(Q(a,2r))})$ instead of kM. Similarly we may replace (2.6) by (2.7) and use (2.5) to conclude as before. Finally we choose $M = (\frac{n}{(n-q)b} + \eta_1)[\,u\,]_{\mathrm{BMO}(Q(a,2r))}$ and obtain the assertion of the lemma.

Case 3: Assume that $Q(a,\frac{3}{2}r) \cap \Omega^C \neq \emptyset$ and $\mathrm{diam}(\Omega^C) < \frac{1}{10}r$. Before we consider this case in full generality, we consider two special cases:

Case 3a: We first consider the case $\Omega^C = \{a\}$. The idea is to use a dyadic decomposition of the cube into annuli and to prove first an inequality on a singe annulus. To this end, let $r_j = r2^{-j+1}$, $A_j = Q(a,r_j) \setminus Q(a,r_{j+1})$ and $B_j = A_{j-1} \cup A_j \cup A_{j+1}$. We assert that

$$\begin{aligned}\int_{A_j} |Du|^q dx &\leq Cr_j^{n-q}\Big\{[\,u\,]^q_{\mathrm{BMO}(Q(a,2r))}\\ &+ \Big(\frac{n[u]_{\mathrm{BMO}(Q(a,2r))}}{(n-q)b}\int_{B_j}|f|dx\Big)^{q/n} + \Big(\int_{B_j} g dx\Big)^{q/n}\Big\}.\end{aligned}$$

Since $A_j \subset Q(a,r_j)$ we obtain from Hölder's inequality

$$\begin{aligned}\int_{\substack{A_j\cap\\ S((u)_{a,r_j};k,M)}} |Du|^q dx &\leq |Q(a,r_j)\cap S((u)_{a,r_j};k,M)|^{1-\frac{q}{n}}\Big(\int_{\substack{A_j\cap\\ S((u)_{a,r_j};k,M)}} |Du|^n dx\Big)^{\frac{q}{n}}\\ &\leq \exp\big(\frac{-(n-q)bkM}{n[u]_{\mathrm{BMO}(Q(a,2r))}}\big) r_j^{n-q}\Big(\int_{\substack{A_j\cap\\ S((u)_{a,r_j};k,M)}} |Du|^n dx\Big)^{\frac{q}{n}}.\end{aligned}$$

Using the Caccioppoli estimate on an annulus and (2.5) we obtain the assertion above as before. Finally, by definition

$$\sum_{j=1}^{\infty} r_j^{n-q} \leq \frac{r^{n-q}}{1-2^{n-q}}$$

and the assertion of the lemma in the case $\Omega^C = \{a\}$ follows easily.

Case 3b: We assume that $\Omega^C \subset Q(a,\rho)$ with $\rho = 2^{-l}r$. Then we combine the estimate in case 2 for $Q(a,\rho)$ with a finite dyadic decomposition using only the annuli A_j, $j = 1, \cdots, l-1$.

Case 3c: After having prepared the two special cases 3a and 3b, we now consider the full case 3 and assume $Q(a, \frac{3}{2}r) \cap \Omega^C \neq \emptyset$ and $\operatorname{diam}(\Omega^C) < \frac{1}{10}r$. In this case, $\Omega^C \subset Q(a_0, \frac{1}{10}r) \subset Q(a, 2r)$, and we can easily find cubes $Q(a_i, \rho) \subset \Omega$, $i = 1, 2, \ldots, l$, (where l depends only on n, but not on r) with $\rho \geq \frac{1}{20}r$ such that

$$\bigcup_{i=0}^{l} Q(a_i, 2\rho) = Q(a, 2r) \text{ and } Q(a,r) \subset \bigcup_{i=0}^{l} Q(a_i, \rho).$$

Then, applying the estimate from case 3a or 3b to $Q(a_0, \rho)$ and the estimate from case 1 to the cubes $Q(a_i, \rho)$, $i = 1, 2, \ldots, l$, we obtain the desired estimate by summation. □

The next lemma is an estimate in the spirit of reversed Hölder inequalities with increasing support. It gives at the same time an estimate for the rate with which the L^q-norm of the gradient Du diverges to infinity as q tends to n if $Du \notin L^n(\Omega; \mathbb{M}^{m\times n})$.

Lemma 3.3 *Assume that Ω^C has property A and that $u \in \mathrm{BMO}(\mathbb{R}^n) \cap \mathcal{D}^{1,n}(\Omega)$ satisfies the Caccioppoli inequalities (2.6) – (2.8) with $f, g \in L^1(\Omega)$. Then there exists a constant C_6, which depends only on K, q, n, and C_3, such that for all $q \in [\frac{n}{2}, n)$ and all $Q(a,r)$ the following estimate holds:*

$$\begin{aligned}(3.8)\qquad \Big(\fint_{Q(a,r)} |Du|^q dx\Big)^{1/q} &\leq C_6 \Big\{ \Big(\fint_{Q(a,2r)} |Du|^{n/2} dx\Big)^{2/n} \\ &+ \Big(\frac{[u]_{\mathrm{BMO}(\mathbb{R}^n)}}{n-q}\Big)^{1/n} \Big(\fint_{Q(a,2r)} |f| dx\Big)^{1/n} + \Big(\fint_{Q(a,2r)} |g| dx\Big)^{1/n} \Big\}.\end{aligned}$$

Proof. The proof is analogous to the proof of Lemma 3.2. We include the necessary modifications in case 1. We use the Sobolev-Poincaré inequality (3.3)

$$\int_{Q(a,2r)\setminus Q(a,r)} |u - (u)_{a,r}|^n dx \leq c \Big(\int_{Q(a,2r)} |Du|^{n/2} dx\Big)^2$$

and (2.6) to estimate

$$\begin{aligned}\int_{\substack{Q(a,r)\\ \cap S((u)_{a,r};k,M)}} |Du|^n dx \quad &\leq \quad Cr^{-n} \Big(\int_{Q(a,2r)} |Du|^{n/2} dx\Big)^2 \\ &+ \quad Cr^{-n}(k+1)M \int_{Q(a,2r)} |f| dx + \int_{Q(a,2r)} g dx.\end{aligned}$$

Choosing M as in the proof of the previous Lemma, we see that

$$r^{-n}\int_{\substack{Q(a,r)\\ \cap S((u)_{a,r};k,M\}}}|Du|^q dx \le Ce^{-k}\Big\{\big(\fint_{Q(a,2r)}|Du|^{n/2}dx\big)^{2q/n}+$$

$$(k+1)^{q/n}\big(\frac{n[u]_{\mathrm{BMO}}}{(n-q)b}\fint_{Q(a,2r)}|f|dx\big)^{q/n}+\big(\fint_{Q(a,2r)}|g|dx\big)^{q/n}\Big\},$$

and we conclude as before. □

Lemma 3.4 *Assume that Ω^C has property A and that $u\in \mathrm{BMO}(\mathbb{R}^n;\mathbb{R}^m)\cap\mathcal{D}^{1,n}(\Omega;\mathbb{R}^m)$ satisfies the Caccioppoli inequalities (2.6) – (2.8) with $f,g\in L^1(\Omega)$. For $\lambda>0$, let E_λ denote the set $\{|Du|>\lambda\}$. Then there exists a constant C_7 such that for all $\lambda>0$ and all $\delta>0$ there exists a measurable set $F_{\lambda,\delta}$ such that*

$$|F_{\lambda,\delta}|\le c_\delta\lambda^{-n}+\delta|E_\lambda| \tag{3.9}$$

and

$$\int_{E_{2\lambda}\setminus F_{\lambda,\delta}}|Du|^n dx\le C_7(\lambda^n|E_\lambda|+1). \tag{3.10}$$

Here C_7 depends only on K, n, C_3, $[\,u\,]_{\mathrm{BMO}(\mathbb{R}^n)}$, $\|f\|_{L^1(\mathbb{R}^n)}$, and $\|g\|_{L^1(\mathbb{R}^n)}$, while c_δ depends in addition on δ and A.

Proof. Recall that u is extended by zero to $\mathbb{R}^n$. Let G be the set of Lebesgue points of Du. Fix $\lambda>0$ and define for $a\in E_{2\lambda}\cap G$

$$R_\lambda(a)=\{\rho>0\,:\,\fint_{Q(a,\rho)}|Du|^{n/2}dx\le(2\lambda)^{n/2}\}$$

and $r(a)=\frac{1}{2}\inf\{\rho\in R_\lambda(a)\}$. We will often suppress the argument a in the notation if there is no confusion. By assumption $R_\lambda(a)\neq\emptyset$ and $r(a)>0$ since $a\in E_{2\lambda}\cap G$. Moreover the continuity of the map $\rho\mapsto\fint_{Q(a,\rho)}|Du|^{n/2}dx$ implies

$$\fint_{Q(a,2r(a))}|Du|^{n/2}dx=(2\lambda)^{n/2}. \tag{3.11}$$

The constants c_i in the estimates below depend only on K, n, C_3, $[\,u\,]_{\mathrm{BMO}(\mathbb{R}^n)}$, $\|f\|_{L^1(\mathbb{R}^n)}$, and $\|g\|_{L^1(\mathbb{R}^n)}$.

Step 1: There exists a constant c_0 such that

$$\lambda r(a)\le c_0. \tag{3.12}$$

Indeed, it follows from Lemma 3.2 with $q=\frac{n}{2}$ and (3.11) that

$$\begin{aligned}2\lambda(2r)^2 &= (2r)^2\big(\frac{1}{(2r)^n}\int_{Q(a,2r)}|Du|^{n/2}dx\big)^{2/n}\\ &\le 2rC_5\Big\{[\,u\,]_{\mathrm{BMO}(Q(a,4r))}+\big(\frac{2}{n}\int_{Q(a,4r)}|f|dx\big)^{1/(n-1)}+\big(\int_{Q(a,4r)}|g|dx\big)^{1/n}\Big\}\end{aligned}$$

which proves (3.12).

Step 2: There exists a constant c_1 such that for all a which satisfy (3.16) below

$$c_1|E_\lambda \cap Q(a,r)| \geq |Q(a,r)|. \tag{3.13}$$

We prove the claim by deriving a lower bound for the density

$$\Theta := \frac{|Q(a,r)\cap E_\lambda|}{|Q(a,r)|}.$$

We fix $q \in (\frac{n}{2}, n)$. By the definition of $r(a)$ we have

$$\begin{aligned}
&(2\lambda)^{n/2}|Q(a,r)| \leq \\
&\leq \int_{Q(a,r)} |Du|^{n/2}dx \\
&\leq \lambda^{n/2}|Q(a,r)\setminus E_\lambda| + \int_{Q(a,r)\cap E_\lambda} |Du|^{n/2}dx \\
&\leq \lambda^{n/2}|Q(a,r)\setminus E_\lambda| + |Q(a,r)\cap E_\lambda|^{1-n/2q}\Big(\int_{Q(a,r)\cap E_\lambda} |Du|^q dx\Big)^{n/2q}.
\end{aligned}$$

By Lemma 3.3 there exist a constant C_6 such that

$$\begin{aligned}
\Big(\fint_{Q(a,r)} |Du|^q dx\Big)^{1/q} &\leq C_6\Big\{\Big(\fint_{Q(a,2r)} |Du|^{n/2}dx\Big)^{2/n} \\
&+ \Big(\frac{1}{n-q}[\,u\,]_{\mathrm{BMO}(\mathbb{R}^n)} Mf(a)\Big)^{1/n} + \big(Mg(a)\big)^{1/n}\Big\}.
\end{aligned} \tag{3.14}$$

Since by (3.2)

$$|\{a\in\Omega:\ \max\{Mf(a), M(g)(a)\} > (\eta\lambda)^n\}| \leq \frac{A}{(\eta\lambda)^n}(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Omega)}) \tag{3.15}$$

we may assume

$$\max\{Mf(a), Mg(a)\} \leq (\eta\lambda)^n \tag{3.16}$$

($0<\eta\leq 1$ will be chosen later, see below). By (3.14) and (3.16) we obtain with $\beta = 1-\frac{n}{2q}$

$$\begin{aligned}
2^{n/2} &\leq (1-\Theta) + \Theta^\beta\lambda^{-n/2}\Big(\fint_{Q(a,r)} |Du|^q dx\Big)^{n/2q} \\
&\leq (1-\Theta) + \Theta^\beta\lambda^{-n/2}\cdot \\
&\quad \cdot\Big(C_6\Big(\fint_{Q(a,2r)} |Du|^{n/2}dx\Big)^{2/n} + C_6\Big(\Big(\frac{1}{n-q}[\,u\,]_{\mathrm{BMO}(\mathbb{R}^n)}\Big)^{1/n} + 1\Big)\eta\lambda\Big)^{n/2} \\
&\leq (1-\Theta) + \Theta^\beta\Big(2C_6 + C_6\Big(\Big(\frac{1}{n-q}[\,u\,]_{\mathrm{BMO}(\mathbb{R}^n)}\Big)^{1/n} + 1\Big)\eta\Big)^{n/2},
\end{aligned}$$

where we used (3.11) in the last step. As $\Theta \to 0$, the right hand side of (3.17) converges to 1. Hence, Θ must be bounded from below and we conclude (3.13) with a constant which is independent of η as long as $\eta \leq 1$.

Step 3: Let $\alpha \geq 2^{n+2}K^{-1}[u]_{\mathrm{BMO}(\mathbb{R}^n)}$ (see Step 4 below). Assume that either $Q(a,\frac{3}{2}r) \subset \Omega$ or $Q(a,\frac{3}{2}r) \cap \Omega^C \neq \emptyset$ and $\mathrm{diam}(\Omega^C) \geq r$. Let

$$F = \{|u-(u)_{a,r}| > \frac{\alpha}{2}\}. \tag{3.17}$$

Then there exists a constant c_2 such that

$$\int\limits_{Q(a,r)\setminus F} |Du|^n dx \leq c_2(1+\alpha\eta^n)\lambda^n |E_\lambda \cap Q(a,r)|, \tag{3.18}$$

and

$$|F \cap Q(a,r)| \leq Bc_1 |E_\lambda \cap Q(a,r)| \exp(-\frac{b\alpha}{2[u]_{\mathrm{BMO}(\mathbb{R}^n)}}). \tag{3.19}$$

The second estimate is an immediate consequence of the John–Nirenberg lemma and Step 2. To prove (3.18) assume first that $Q(a,\frac{3}{2}r) \subset \Omega$. We get by (2.6) and the Sobolev-Poincaré inequality (3.3)

$$\begin{aligned}
\frac{1}{|Q(a,r)|}\int\limits_{Q(a,r)\setminus F} |Du|^n dx &\leq c(n)C_3\Big(\;\;\fint\limits_{Q(a,\frac{3}{2}r)} |Du|^{n/2}dx\Big)^2 + C_3\big(\frac{\alpha}{2}Mf(a) + Mg(a)\big) \\
&\leq c(n)C_3(2\lambda)^n + C_3(\alpha+1)(\eta\lambda)^n
\end{aligned}$$

and by (3.13)

$$\int\limits_{Q(a,r)\setminus F} |Du|^n dx \leq c_2(1+\alpha\eta^n)\lambda^n |E_\lambda \cap Q(a,r)|. \tag{3.20}$$

Assume now that $Q(a,\frac{3}{2}r) \cap \Omega^C \neq \emptyset$ and $\mathrm{diam}(\Omega^C) \geq r$. In this situation we may apply property A to some cube $Q(a_0,\frac{r}{2}) \subset Q(a,2r)$ and deduce

$$Q(a,2r) \cap \{u=0\} \geq K\big(\frac{r}{2}\big)^n. \tag{3.21}$$

Since $|(u)_{a,r}| \leq 2^n|(u)_{a,2r}| \leq 2^{n+1}K^{-1}[u]_{\mathrm{BMO}}$ we conclude as before $F = \{|u-(u)_{a,r}| > \frac{\alpha}{2}\} \supseteq \{|u| \geq \alpha\}$ and the proof of the claim follows now with (2.7) instead of (2.6). We may apply the Sobolev–Poincaré inequality in view of (3.21).

Step 4: Proof of the lemma with a covering argument.

If $d = \mathrm{diam}(\Omega^C) \leq \frac{1}{\lambda}$ we choose a cube $Q(a_0,\frac{1}{\lambda})$ such that $\Omega^C \subset Q(a_0,\frac{1}{\lambda})$ and define

$$A = \big(E_{2\lambda} \setminus \Big(\{Mf > (\eta\lambda)^n\} \cup Q(a_0,\frac{2c_0}{\lambda})\Big)\big) \cap G.$$

Otherwise we define

$$A = \big(E_{2\lambda} \setminus \{Mf > (\eta\lambda)^n\}\big) \cap G.$$

For each $a \in A$ there exists a cube $Q(a,r(a))$ as above and in view of Step 1 it is easy to see that the estimate in Step 3 holds for all these cubes. By Besicovitch's covering theorem there exists a fixed number L of families $\mathcal{F}^{(i)} = \{Q_j^{(i)}\}_{j\in\mathbb{N}}$ of disjoint cubes such that the union of these families covers A. Let $F_j^{(i)}$ be the exceptional set for the cube $Q_j^{(i)}$ as defined in (3.17) and let

$$\tilde{F}_{\lambda,\eta} = \bigcup_{i=1}^{L}\bigcup_{j\in\mathbb{N}} F_j^{(i)} \cup \{Mf > (\eta\lambda)^n\} \cup Q(a_0,\frac{2c_0}{\lambda}) \cup G^C$$

if $\operatorname{diam}(\Omega^C) \le \frac{1}{\lambda}$ and

$$\tilde{F}_{\lambda,\eta} = \bigcup_{i=1}^{L} \bigcup_{j\in\mathbb{N}} F_j^{(i)} \cup \{Mf > (\eta\lambda)^n\} \cup G^C$$

else. Clearly, $\tilde{F}_{\lambda,\eta}$ is measurable and by (3.18)

$$\begin{aligned} \int\limits_{E_{2\lambda}\setminus F_{\lambda,\eta}} |Du|^n dx &\le \sum_{i=1}^{L}\sum_{j=1}^{\infty} \int\limits_{Q_j^{(i)}\setminus F_j^{(i)}} |Du|^n dx \\ &\le \sum_{i=1}^{L} c_2(1+\alpha\eta^n)\lambda^n \sum_{j=1}^{\infty} |E_\lambda \cap Q_j^{(i)}| \end{aligned}$$

and hence

$$\int\limits_{E_{2\lambda}\setminus F_{\lambda,\eta}} |Du|^n dx \le Lc_2(1+\alpha\eta^n)\lambda^n |E_\lambda|.$$

Similarly by (3.16) and by (3.19)

$$|\tilde{F}_{\lambda,\eta}| \le A(\eta\lambda)^{-n} + BLc_1 \exp\bigl(-\frac{b\alpha}{2[u]_{\mathrm{BMO}(\Omega)}}\bigr)|E_\lambda| + \bigl(\frac{2c_0}{\lambda}\bigr)^n.$$

Now, given $\delta > 0$, choose

$$\alpha(\delta) = \max\Bigl\{\frac{3[u]_{\mathrm{BMO}(\mathbb{R}^n)}}{b}\ln\bigl(\frac{c_1 BL}{\delta}\bigr), \frac{2^{n+1}}{K^{1/n}}\Bigr\}, \quad \eta(\delta) = \min\{\frac{1}{\alpha^{1/n}}, 1\}.$$

Then $F_{\lambda,\delta} = \tilde{F}_{\lambda,\eta(\delta)}$ satisfies

$$|F_{\lambda,\delta}| \le c_\delta \lambda^{-n} + \delta |E_\lambda|$$

with $c_\delta = A\max\{1,\alpha\} + (2c_0)^n$ and

$$\int\limits_{E_{2\lambda}\setminus F_{\lambda,\delta}} |Du|^n dy \le 2c_2 L\lambda^n |E_\lambda|.$$

This completes the proof. $\square$

Lemma 3.5 *Assume that Ω^C is a domain of type A and that $u \in \mathcal{D}^{1,n}(\Omega;\mathbb{R}^m)$ satisfies the Caccioppoli inequalities (2.6) – (2.8). Then $Du \in L^{n,\infty}(\Omega;\mathbb{M}^{m\times n})$ and*

$$\|Du\|_{L^{n,\infty}(\Omega;\mathbb{M}^{m\times n})} \le C_8\bigl(\|f\|_{L^1(\Omega)}^{1/(n-1)} + \|g\|_{L^1(\Omega)}^{1/n}\bigr), \tag{3.22}$$

where C_8 depends on n, K and C_3.

Proof. Since the inequalities (2.4) and (2.5) are invariant under the rescaling $u \mapsto \lambda u$, $f \mapsto \lambda^{n-1} f$ and $g \mapsto \lambda^n g$, we may assume that $\|f\|_{L^1(\Omega)}^{1/(n-1)} + \|g\|_{L^1(\Omega)}^{1/n} = \min\{C_4^{-1}, 1\}$ and thus $[\,u\,]_{\mathrm{BMO}} \le 1$.

We begin by proving the following assertion from which the proof of the theorem follows by iteration: For $M > 1$ fixed there exists a $\bar{\lambda} \in (2\lambda, 2M\lambda)$ such that

$$\bar{\lambda}^n |E_{\bar{\lambda}} \setminus F_{\lambda,\delta}| \le \frac{C_7}{n\log M}\bigl(\lambda^n |E_\lambda| + 1\bigr). \tag{3.23}$$

Indeed, let $h := |Du|(1-\chi_{F_{\lambda,\delta}})$. Then, $\alpha(\nu) := |E_\nu \setminus F_{\lambda,\delta}|$ is the distribution function of h. If $\alpha(\nu)\nu^n \geq \beta$ for all $\nu \in (2\lambda, 2M\lambda)$, then

$$\begin{aligned}
\int\limits_{E_{2\lambda}\setminus F_{\lambda,\delta}} |Du|^n dx &= -\int\limits_{2\lambda}^{\infty} s^n d\alpha(s) \\
&= (2\lambda)^n \alpha(2\lambda) + \int\limits_{2\lambda}^{\infty} n s^{n-1}\alpha(s) ds \\
&\geq \beta + \int\limits_{2\lambda}^{2M\lambda} \frac{n}{s}\beta ds = \beta(1+n\log M).
\end{aligned}$$

Together with (3.10) this implies

$$\beta \leq \frac{1}{1+n\log M} \int\limits_{E_{2\lambda}\setminus F_{\lambda,\delta}} |Du|^n dx \leq \frac{C_7}{n\log M}\big(\lambda^n |E_\lambda| + 1\big)$$

and this establishes (3.23).

Combination of (3.9) and (3.23) yields

$$|E_{\bar\lambda}| \leq \big(\frac{C_7}{n\log M}\big(\frac{\lambda}{\bar\lambda}\big)^n + \delta\big)|E_\lambda| + \big(\frac{C_7}{n\log M} + c_\delta\big)\lambda^{-n}.$$

Now, we choose $M := \exp\frac{4C_7}{n}$ (we may assume $M > 1$) and a fixed $\delta < \frac{1}{4}(\frac{1}{2M})^n = \frac{1}{4}\,2^{-n}e^{-4C_7}$. Hence we have

$$|E_{\bar\lambda}| \leq \frac{1}{2}\big(\frac{\lambda}{\bar\lambda}\big)^n |E_\lambda| + c_3\lambda^{-n}$$

(with $c_3 = (\frac{C_7}{n\log M} + c_\delta)$). Iterated application of this inequality with $\lambda_{i+1} := \bar\lambda_i$, $i \in \mathbb{N}$, and $a_i := \lambda_i^n |E_{\lambda_i}|$ gives

$$a_{i+1} \leq \frac{1}{2}\,a_i + (2M)^n c_3$$

and hence

$$a_i \leq \big(\frac{1}{2}\big)^i a_0 + 2(2M)^n c_3 \leq \Big(\frac{\lambda_0}{\lambda_i}\Big)^\delta a_0 + 2(2M)^n c_3,$$

where $\delta \leq \ln 2/\ln(2M) < 1$. Thus, we have for all $i \in \mathbb{N}$

$$|E_{\lambda_i}| \leq a_0\lambda_0^\delta \lambda_i^{-(n+\delta)} + 2(2M)^n c_3 \lambda_i^{-n}.$$

Since $\lambda_{i+1} \leq 2M\lambda_i$, we deduce for all $\lambda \geq \lambda_0$

$$|E_\lambda| \leq 2(2M)^{2n} c_3\lambda^{-n} + (2M)^{n+\delta} a_0\Big(\frac{\lambda_0}{\lambda}\Big)^\delta \lambda^{-n}.$$

Since $Du \in L^n$ we have

$$a_0 \leq \int_{E_{\lambda_0}} |Du|^n dx \leq \|Du\|_{L^n}^n.$$

Hence the assertion follows in the limit $\lambda_0 \to 0$. $\square$

Proof of Theorem 1.1: The hypotheses (H2) and (H3) of the present paper are slightly weaker than in [DHM]. However, in the same way as in [DHM], it is still possible to construct approximating solutions $u_k \in W_0^{1,n}(\Omega)$ of the regularised system

$$-\operatorname{div}\sigma(x,u_k,Du_k) = f_k$$

for smooth and bounded $L^1(\Omega)$ functions f_k with $f_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}$. In order to pass to the limit $k \to \infty$, we have to check that the crucial Lemma 11 of [DHM] still holds if we replace condition (5.2) of [DHM] by the weaker condition (H3) of the present paper. For the reader's convenience, we specify the relevant changes: estimate (5.12) does not hold any more and has to be replaced by

$$\sigma(x,u_k,Du_k) \,:\, D\psi_1 \circ (u_k - v)\, Du_k \geq -\gamma_5|Du_k|^s - \gamma_6(x).$$

Thus the conclusion that $(h_k)^-$ is equiintegrable is still valid. In (5.21) of [DHM], we get an additional term:

$$\text{(3.24)} \quad \text{LHS}(r) \geq \int_\Omega \varphi_r(x) g^M_{ijlj}(x)(D\psi_r)_{il}(u-v)(x)dx \\ - \int_\Omega \varphi_r(x)\int_{\mathbb{M}^{m\times n}} \big(1-\eta(\frac{|\lambda|}{M})\big)(\gamma_5|\lambda|^s + \gamma_6(x))d\nu_x(\lambda)dx.$$

We are free to assume that x_0 is (in addition) a Lebesgue point of the functions $\int_{\mathbb{M}^{m\times n}}(1 - \eta(\frac{|\lambda|}{M}))(\gamma_5|\lambda|^s + \gamma_6(x))d\nu_x(\lambda) \in L^1(\Omega)$, $M \in \mathbb{N}$. Thus, as $r \to 0$, the second term on the right hand side of (3.24) converges (for fixed $M \in \mathbb{N}$) to $-\int_{\mathbb{M}^{m\times n}}(1-\eta(\frac{|\lambda|}{M}))(\gamma_5|\lambda|^s + \gamma_6(x_0))d\nu_{x_0}(\lambda)$. Since ν_x is (for almost every $x \in \Omega$) a probability measure with finite s-th moment, the additional term vanishes as $M \to \infty$ and we conclude as in [DHM].

Now, the approximating solutions u_k satisfy the weak $L^{n,\infty}$ estimate in Lemma 3.5. In view of the weak* lower semicontinuity of the $L^{n,\infty}$-norm we obtain the same estimate for the solution u. Since in this situation $g = |\gamma_3| + |\gamma_4|^{n/(n-1)} + \gamma_5|Du|^s + |\gamma_6|$ it remains to estimate $\|Du\|_{L^s}$. By Hölder's inequality in the weak Lebesgue spaces we deduce for $\varepsilon > 0$

$$\begin{aligned} \gamma_5^{1/n}\Big(\int_\Omega |Du|^s dx\Big)^{1/n} &\leq \gamma_5^{1/n}(\frac{n}{n-s})^{1/n}|\Omega|^{(n-s)/n^2}\|Du\|^{s/n}_{L^{n,\infty}} \\ &\leq c_\varepsilon\gamma_5^{1/(n-s)}(\frac{n}{n-s})^{1/(n-s)}|\Omega|^{1/n} + \varepsilon\|Du\|_{L^{n,\infty}}. \end{aligned}$$

This estimate implies first the weak $L^{n,\infty}$ estimate for the gradient and Theorem 1.1 is an immediate consequence. $\square$

It is possible to improve the BMO estimate if the measure μ has no atoms, i.e., if

$$\lim_{r\to 0}\mu(Q(a,r)) = 0 \quad \text{for all } a \in \bar\Omega.$$

Here we say that a function belongs to the space $\text{VMO}(\mathbb{R}^n)$, the space of functions of vanishing mean oscillation, if

$$\lim_{r\to 0} r^{-n}\int_{Q(a,r)} |u-(u)_{a,r}|^n dx = 0$$

for all $a \in \mathbb{R}^n$.

Corollary 3.6 *Suppose that the assumptions in Theorem 1.1 are satisfied and that in addition μ has no atoms. Then the system (1.1), (1.2) has a solution $u \in \text{VMO}(\Omega;\mathbb{R}^m)$.*

Proof. This follows with an indirect argument similar to the one used in the BMO estimate. $\square$

4 Uniqueness results

The uniqueness result would be an immediate consequence of the uniform monotonicity if the difference $w = u - v$ were an admissible test function. The idea here is to use techniques developed in [AF] to approximate a given $W^{1,p}$ function w by a function $\tilde{w} \in W^{1,\infty}$ which agrees with w on a large set (see also [L], [EG], [MZ]). We obtain a sharp estimate for the measure of the set on which the two functions do not agree if $Dw \in L^{p,\infty}(\Omega; \mathbb{R}^n)$. For uniqueness of entropy solutions of nonlinear elliptic equations with measures which vanish on sets of p-capacity zero, see [BGO], and compare also the results in [BB], [BG] and [KX].

Lemma 4.1 *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with complement Ω^C of type A. Then there exist constants C_8 and C_9 which depend on Ω and n such that the following is true: If $w \in W_0^{1,1}(\Omega; \mathbb{R}^m)$ with $Dw \in L^{p,\infty}(\Omega; \mathbb{M}^{m\times n})$ then there exists for all $\lambda > 0$ a function $w_\lambda \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that $\|w_\lambda\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)} \le C_8\lambda$ and*

$$|\{x \in \Omega : w(x) \neq w_\lambda(x)\}| \le C_9 \lambda^{-p} \|Dw\|^p_{L^{p,\infty}(\Omega;\mathbb{M}^{m\times n})}. \tag{4.1}$$

If $w \in W^{1,p}(\Omega; \mathbb{R}^m)$ then

$$|\{x \in \Omega : w(x) \neq w_\lambda(x)\}| = o(\lambda^{-p}) \quad \text{as } \lambda \to \infty. \tag{4.2}$$

Proof. We extend w by zero to $\mathbb{R}^n$ and define

$$R^\lambda = \{x \in \mathbb{R}^n : M(Dw)(x) > \lambda\} \cup \{x \in \mathbb{R}^n : x \text{ not Lebesgue point of } Dw\}.$$

Since $M : L^{p,\infty} \to L^{p,\infty}$ we obtain

$$|R^\lambda| \le C\lambda^{-p}\|Dw\|^p_{L^{p,\infty}(\Omega;\mathbb{M}^{m\times n})}.$$

It follows from Lemma 1 in [AF] that there exists a constant $c(n)$ such that

$$|w(x) - w(y)| \le c(n)\lambda|x-y| \text{ on } \mathbb{R}^n \setminus R^\lambda$$

and

$$|w(x) - (w)_{x,r}| \le c(n)r\lambda \text{ on } \mathbb{R}^n \setminus R^\lambda.$$

If we choose $x \in \Omega \setminus R^\lambda$ and $r = 2\,\mathrm{dist}(x, \Omega^C)$, then condition A implies that $|Q(x,r) \cap \Omega^C| \ge C(K)r^n$ and hence Poincaré's inequality yields

$$|(w)_{x,r}| \le cr \fint_{Q(x,r)} |Dw|\,dx \le C\,\mathrm{dist}(x,\Omega^C)\lambda.$$

Thus

$$|w(x)| \le C\,\mathrm{dist}(x,\Omega^C)\lambda \text{ on } \mathbb{R}^n \setminus R^\lambda.$$

It follows that

$$\tilde{w}_\lambda = \begin{cases} w(x) & \text{on } \Omega \setminus R^\lambda, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

is Lipschitz continuous on its domain of definition and thus there exists an extension w_λ to $\mathbb{R}^n$ with the same Lipschitz constant and $\{w \neq w_\lambda\} \subset R^\lambda$. In particular $w_\lambda \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$. This proves the assertion of the lemma if $Dw \in L^{p,\infty}(\Omega;\mathbb{M}^{m\times n})$. If $Dw \in L^p(\Omega;\mathbb{M}^{m\times n})$ the assertion follows with the same arguments as above since in this case it is possible to use Hölder's inequality in the standard L^p spaces. □

Proof of Theorem 1.2: Let $w = u - v$ and $\lambda > 0$, and define w_λ as in Lemma 4.1. Then $w_\lambda \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ is an admissible test function in (1.5), $Dw_\lambda = Du - Dv$ a.e. on $\Omega\setminus E^\lambda$, where $E^\lambda = \{x \in \Omega : w(x) \neq w_\lambda(x)\}$, and we obtain

$$\int_{\Omega\setminus E^\lambda} (\sigma(x,Du) - \sigma(x,Dv)) : (Du - Dv)dx = -\int_{E^\lambda} (\sigma(x,Du) - \sigma(x,Dv)) : Dw_\lambda dx.$$

We deduce by Hölder's inequality in weak L^n and estimate (4.1)

$$\begin{aligned}\gamma_0 \int_{\Omega\setminus E^\lambda} |Du - Dv|^n dx &\leq C\lambda \int_{E^\lambda} (|Du|^{n-1} + |Dv|^{n-1} + 1)dx \\ &\leq C\lambda |E^\lambda|^{1/n} \big\| |Du| + |Dv| \big\|^{n-1}_{L^{n,\infty}(\Omega;\mathbb{M}^{m\times n})} + C\lambda|E^\lambda| \\ &\leq C.\end{aligned}$$

We may pass to the limit $\lambda \to \infty$ and obtain

$$Dw = Du - Dv \in L^n(\Omega;\mathbb{M}^{m\times n}).$$

Thus by (4.2) $|E^\lambda| = o(\lambda^{-n})$ and the result follows from the inequality above as $\lambda \to \infty$. □

Now, we prove the following stronger uniqueness result:

Theorem 4.2 *Under the same assumptions on Ω and σ as in Theorem 1.2, there exists a number $p \in (n-1,n)$ which depends only on Ω and γ_0, γ_1, γ_3, such that the following is true: Assume that u, $v \in W^{1,1}(\Omega;\mathbb{R}^m)$ satisfy $u - v \in W_0^{1,1}(\Omega;\mathbb{R}^m)$, $Du \in L^{n,\infty}(\Omega;\mathbb{M}^{m\times n})$, $Dv \in L^p(\Omega;\mathbb{M}^{m\times n})$ and*

$$\operatorname{div}\sigma(x,Du) = \operatorname{div}\sigma(x,Dv) \quad \text{in } \mathcal{D}'(\Omega). \tag{4.3}$$

Then $u \equiv v$ in Ω.

Proof. Let $w := u - v$ and consider w_λ and E^λ as above in the proof of Theorem 1.2. By testing (4.3) by w_λ we get

$$\begin{aligned}\gamma_0 \int_{\Omega\setminus E^\lambda} |Du - Dv|^n dx &\leq \\ &\leq C_8\lambda \int_{E^\lambda} |\sigma(Dv) - \sigma(Du)|\, dx \\ &\leq C\lambda \int_{E^\lambda} (|Du|^{n-1} + |Dw|^{n-1} + 1)dx \\ &\leq C\lambda|E^\lambda| + C\lambda|E^\lambda|^{1/n}\|Du\|^{n-1}_{L^{n,\infty}} + C\lambda \int_{E^\lambda} |Dw|^{n-1}dx.\end{aligned}$$

Thus, we have

$$\int_{\Omega\setminus E^\lambda} |Dw|^n dx \leq C(1 + \lambda^n|E^\lambda|) + C\lambda \int_{E^\lambda} |Dw|^{n-1}dx.$$

This is almost a reverse Hölder inequality, and to conclude we adapt some arguments of the theory of reverse Hölder inequalities (see, e.g., [G]). Let $f = |Dw|$, then the above inequality gives

$$\int\limits_{|f|\le\lambda} f^n dx \le \int\limits_{\Omega\setminus E^\lambda} f^n dx + \lambda^n |E^\lambda| \le C(1+\lambda^n|E^\lambda| + \lambda \int\limits_{E^\lambda} f^{n-1}dx). \tag{4.4}$$

Now we claim that

$$\lambda^n|E^\lambda| \le C\lambda \int\limits_{f>\frac{\lambda}{2}} f^{n-1}dx, \tag{4.5}$$

$$\lambda \int\limits_{E^\lambda} f^{n-1}dx \le C\lambda \int\limits_{f>\frac{\lambda}{2}} f^{n-1}dx. \tag{4.6}$$

We postpone the proof of (4.5) and (4.6) for the moment. Combining (4.4) with (4.5) and (4.6) we get

$$\int\limits_{f\le\lambda/2} f^n dx \le \int\limits_{f\le\lambda} f^n dx \le C(1+\lambda \int\limits_{f\ge\lambda/2} f^{n-1}dx). \tag{4.7}$$

Let $\alpha(s) = |\{f > s\}|$ denote the distribution function of f and let

$$G(t) = \int\limits_{f\le t} f^n dx = -\int_0^t s^n d\alpha.$$

Formally one has

$$\begin{aligned}\int\limits_{f\ge\lambda} f^{n-1}dx &= -\int_\lambda^\infty s^{n-1}d\alpha = \int_\lambda^\infty \frac{G'(s)}{s}ds \\ &= \frac{G(s)}{s}\Big|_\lambda^\infty + \int_\lambda^\infty \frac{G(s)}{s^2}ds = -\frac{G(\lambda)}{\lambda} + \int_\lambda^\infty \frac{G(s)}{s^2}ds.\end{aligned} \tag{4.8}$$

This formal calculation is correct for simple functions $f \ge 0$ and hence by monotone convergence for nonnegative functions $f \in L^p(\Omega)$ provided $p \ge n-1$. Thus replacing $\frac{\lambda}{2}$ by λ we get from (4.7)

$$G(\lambda) \le C(1 - G(\lambda) + \lambda\int_\lambda^\infty \frac{G(s)}{s^2}ds)$$

and hence

$$G(\lambda) \le \theta(1+\lambda h(\lambda))$$

with $\theta = \frac{C}{C+1} < 1$ and $h(\lambda) := \int_\lambda^\infty \frac{G(s)}{s^2}ds$. Thus we have

$$-s^2 h'(s) \le \theta + \theta s h(s)$$

and hence

$$(s^\theta h(s))' = \frac{\theta h(s) + sh'(s)}{s^{1-\theta}} \ge \frac{-\theta s^{-1}}{s^{1-\theta}} = (\frac{\theta}{1-\theta}s^{\theta-1})'.$$

Suppose now that $p > n-1+\theta$. Since $f \in L^{p,\infty}$ we have $\alpha(s) \leq Cs^{-p}$, $G(s) \leq Cs^{n-p}$ and $h(s) \leq Cs^{n-p-1}$. Thus, $s^\theta h(s) \to 0$ as $s \to \infty$. Hence, integration from λ to ∞ yields

$$\lambda^\theta h(\lambda) \leq \frac{\theta}{1-\theta}\lambda^{\theta-1}.$$

Thus we have $h(\lambda) \leq \frac{\theta}{1-\theta}\lambda^{-1}$ and hence

$$G(\lambda) \leq \frac{\theta}{1-\theta}$$

which implies $Dw \in L^n(\Omega)$ and hence $Dv \in L^{n,\infty}(\Omega)$ and the uniqueness follows from Theorem 1.2 up to the proof of (4.5) and (4.6) which we give now.

By the Vitali covering theorem we may choose almost disjoint cubes Q_i which cover R^λ (and hence E^λ) such that

$$\int\limits_{Q_i} f\,dx \geq \lambda|Q_i|.$$

Since

$$\int\limits_{Q_i\cap\{f<\frac{\lambda}{2}\}} f\,dx \leq \frac{\lambda}{2}|Q_i|$$

we have

$$|Q_i| \leq \frac{2}{\lambda}\int\limits_{Q_i\cap\{f\geq\frac{\lambda}{2}\}} f\,dx$$

and summation gives (4.5) for R^λ:

$$|R^\lambda| \leq \frac{C}{\lambda}\int\limits_{f\geq\frac{\lambda}{2}} f\,dx \leq \frac{C'}{\lambda^{n-1}}\int\limits_{f\geq\frac{\lambda}{2}} f^{n-1}dx.$$

Similarly

$$\frac{1}{|Q_i|}\int\limits_{Q_i} f^{n-1}dx \geq \Big(\frac{1}{|Q_i|}\int\limits_{Q_i} f\,dx\Big)^{n-1} \geq \lambda^{n-1}$$

and

$$\frac{1}{|Q_i|}\int\limits_{Q_i\cap\{f\leq\frac{\lambda}{2}\}} f^{n-1}dx \leq \Big(\frac{\lambda}{2}\Big)^{n-1}$$

imply

$$\int\limits_{Q_i} f^{n-1}dx \leq C\int\limits_{Q_i\cap\{f\geq\frac{\lambda}{2}\}} f^{n-1}dx$$

and summations yields

$$\int\limits_{R^\lambda} f^{n-1}dx \leq C\int\limits_{f\geq\frac{\lambda}{2}} f^{n-1}dx$$

which is (4.6) for R^λ. □

As a corollary we immediately obtain the following regularity result:

Corollary 4.3 *Under the same assumptions on Ω and σ as in Theorem 1.2, there exists a number $p \in (n-1,n)$ which depends only on Ω and γ_0, γ_1, γ_3 such that the following is true: Assume that $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ satisfies (1.1). Then $u \in \mathrm{BMO}(\Omega;\mathbb{R}^m)$ and $Du \in L^{n,\infty}(\Omega)$ with a priori estimates for the norms of u and Du in these spaces.*

5 Unbounded domains

In this section, we first want to show, that Theorem 1.1 still holds for an unbounded domain Ω if $\gamma_5 = 0$ or if Ω has finite measure.

Theorem 5.1 *Let $\Omega \neq \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ such that Ω^C has property A, and let μ be an $\mathbb{R}^m$-valued Radon measure on Ω with finite mass. Suppose that the hypotheses (H0)–(H3) and one of the conditions (i)–(iii) of Theorem 1.1 are satisfied. Then, if in addition $\gamma_5 = 0$ or $|\Omega| < \infty$, the system (1.1), (1.2) has a solution $u \in \mathrm{BMO}(\Omega; \mathbb{R}^m)$ in the sense of distributions and the solution satisfies the a priori estimate*

$$[u]_{\mathrm{BMO}(\Omega;\mathbb{R}^m)} \leq C_2\big(\|\mu\|_{\mathcal{M}}^{\frac{1}{n-1}} + \gamma_5^{\frac{1}{n-s}}|\Omega|^{\frac{1}{n}} + \big\||\gamma_3| + |\gamma_4|^{\frac{n}{n-1}} + |\gamma_6|\big\|_{L^1(\Omega)}\big) \tag{5.1}$$

(with the convention $0 \cdot \infty = 0$). Moreover, Du belongs to the weak Lebesgue space $L^{n,\infty}(\Omega; \mathbb{M}^{m\times n})$ and

$$\|Du\|_{L^{n,\infty}(\Omega;\mathbb{M}^{m\times n})} \leq C_1\big(\|\mu\|_{\mathcal{M}}^{\frac{1}{n-1}} + \gamma_5^{\frac{1}{n-s}}|\Omega|^{\frac{1}{n}} + \big\||\gamma_3| + |\gamma_4|^{\frac{n}{n-1}} + |\gamma_6|\big\|_{L^1(\Omega)}\big). \tag{5.2}$$

Here the constants C_2, C_1 depend only on γ_1, γ_2, K, and n. In case $|\Omega| < \infty$ we have $u \in W_0^{1,q}(\Omega, \mathbb{R}^m)$ for all $q < n$ and we can replace $[u]_{\mathrm{BMO}(\Omega;\mathbb{R}^m)}$ by $\|u\|_{\mathrm{BMO}(\Omega;\mathbb{R}^m)}$ in (5.1) if we allow C_2 to depend on $|\Omega|$.

Proof. First, we solve (1.1), (1.2) on $\Omega_R := \Omega \cap B(0, R)$. From Theorem 1.1 we infer the estimates (5.1) and (5.2) on Ω_R in place of Ω for the solution u_R. Notice that Ω_R is of type A with the same constant K as Ω and hence the constants C_2 and C_1 are independent of R (compare the remark after Theorem 1.1). Hence the sequence u_R (extended to zero outside Ω_R) is bounded in $\mathrm{BMO}_{\mathrm{loc}}(\mathbb{R}^n) \cap W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ and we may extract a sequence which converges weakly on every Ω_ρ in the corresponding spaces to a limit function u satisfying the estimates (5.1) and (5.2). In order to prove that u is a distributional solution of (1.1), we proceed as in the proof of Theorem 1.1 noticing that condition (5.6) in Lemma 11 of [DHM] can be replaced by the condition that $-\operatorname{div}\sigma_k$ is a fixed Radon measure. □

The problem in the case $\Omega = \mathbb{R}^n$ is that no boundary data prevent a sequence of approximating solutions from diverging to infinity. We use our BMO-estimate to overcome this difficulty and to obtain Theorem 1.3 which we prove now.

Proof of Theorem 1.3. As in the proof of Theorem 5.1 we start constructing a solution $\tilde{u}$ on $B(0, k)$ with zero boundary values. Then, we add a suitable constant such that $u := \tilde{u} + \gamma$ (which is still a solution of (1.6) on $B(0, k)$) satisfies $\fint_{B(0,1)} u\,dx = 0$. Now, u has boundary value γ and we extend u by 0 outside $B(0, k)$ and denote this function by u_k. Thus, the sequence $\{u_k\}_k$ is bounded in $L^n(B(0, R))$ for all R and in fact in $W^{1,p}(B(0, R))$ as long as $p < n$ because of $\fint_{B(0,1)} u\,dx = 0$. Hence we have (at least for a subsequence)

$$u_k \rightharpoonup u \quad \text{in } W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$$

and, as in the proof of Theorem 1.1,

$$-\operatorname{div}\sigma(x, Du) = \mu \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Also, by locality of the BMO seminorm, we have

$$[u]_{\mathrm{BMO}(\mathbb{R}^n)} \leq C_2\big(\|\mu\|_{\mathcal{M}}^{\frac{1}{n-1}} + \big\||\gamma_3| + |\gamma_4|^{\frac{n}{n-1}} + |\gamma_6|\big\|_{L^1(\Omega)}\big)$$

and

$$\fint_{B(0,1)} u\,dx = 0$$

by just passing to the limit in the expressions

$$\fint_{Q(x,r)} |u_k - (u_k)_{x,r}|^n dx \quad \text{and} \quad \fint_{B(0,1)} u_k\,dx = 0$$

for fixed r and x. The weak-L^n bound for the gradient follows by weak lower semicontinuity of the norm. □

6 The nonhomogeneous Dirichlet problem and local regularity

In this section, we allow boundary values $\tilde{u} \in W^{1,n+\varepsilon}(\mathbb{R}^n)$ on $\partial\Omega$ with $\varepsilon > 0$. Consider $\tilde{\sigma}(x,u,F) := \sigma(x, u+\tilde{u}, F + D\tilde{u})$. It is easy to verify that $\tilde{\sigma}$ satisfies hypotheses (H0)–(H3), with possibly different $\tilde{\gamma}_i$ and $\tilde{s}$ in place of γ_i and s (for $\varepsilon = 0$ hypothesis (H3) may no longer be satisfied). Thus, we may apply Theorem 1.1 and obtain a solution u of $-\operatorname{div}\tilde{\sigma}(x,u,Du) = \mu$ in Ω with $u = 0$ on $\partial\Omega$. Hence $v := u + \tilde{u}$ solves

$$-\operatorname{div}\sigma(x,v,Dv) = \mu \quad \text{in } \mathcal{D}'(\Omega), \tag{6.1}$$

$$v = \tilde{u} \quad \text{on } \partial\Omega. \tag{6.2}$$

We summarise this result in the following proposition:

Proposition 6.1 *Under the same assumptions as in Theorem 1.1 there exists a distributional solution v of (6.1), (6.2) for boundary values $\tilde{u} \in W^{1,n+\varepsilon}(\mathbb{R}^n)$. The estimates (1.3) and (1.4) hold for v with an additional term $\|\tilde{u}\|_{\mathrm{BMO}(\mathbb{R}^n)}$ and $\|D\tilde{u}\|_{L^{n,\infty}(\mathbb{R}^n)}$ on the right hand side of (1.3) and (1.4), respectively.*

As an immediate consequence of the previous result combined with the uniqueness results of Section 4 we obtain:

Proposition 6.2 *Under the same assumptions on Ω and on σ as in Theorem 1.2 there exists a number $p \in (n-1, n)$ which depends only on Ω and $\gamma_0, \gamma_1, \gamma_3$ such that the following is true: If $u \in W^{1,q}(\Omega)$, for some $q > p$, with boundary data $u = \tilde{u}$ on $\partial\Omega$, $\tilde{u} \in W^{1,n+\varepsilon}(\mathbb{R}^n)$, and if $\operatorname{div}\sigma(x,Du)$ is a bounded Radon measure on Ω, then $u \in \mathrm{BMO}(\Omega)$ and $Du \in L^{n,\infty}(\Omega)$.*

A Some properties of $\mathcal{D}^{1,n}(\Omega)$

For the convenience of the reader we briefly recall some facts about $\mathcal{D}^{1,n}(\Omega)$. Let $\Omega \subset \mathbb{R}^n$ be an open domain and define the semi-norm $|\cdot|_{1,n;\Omega}$ by

$$|u|_{1,n;\Omega} = \Big(\int_\Omega |Du|^n dx\Big)^{1/n}.$$

We define $\mathcal{D}^{1,n}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to $|\cdot|_{1,n;\Omega}$. The elements of $\mathcal{D}^{1,n}(\Omega)$ can be identified with equivalence classes of $W^{1,n}_{\text{loc}}(\Omega)$-functions where two functions are identified if their difference is a.e. constant.

Lemma A.1 *Assume that Ω^C has property A. Then for each $\varphi \in \mathcal{D}^{1,n}(\Omega)$ there exists a sequence $\{\varphi_k\} \in C_0^\infty(\Omega)$ such that $D\varphi_k \to D\varphi$ in $L^n(\Omega)$ and $\varphi_k \to \varphi$ in $L^n_{\text{loc}}(\Omega)$. If, in addition, $\varphi \in L^\infty(\Omega)$, then we may choose the sequence φ_k uniformly bounded in $L^\infty(\Omega)$.*

Proof. Let $\pi : S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection from the north pole N onto $\mathbb{R}^n$ and define $U = \pi^{-1}(\Omega)$. Since π is conformal and the n-Dirichlet integral is conformally invariant, we may identify $\mathcal{D}^{1,n}(\Omega)$ and $\mathcal{D}^{1,n}(U)$ by $\varphi \in \mathcal{D}^{1,n}(\Omega) \mapsto \hat{\varphi} = \varphi \circ \pi \in \mathcal{D}^{1,n}(U)$. Let $\varphi \in \mathcal{D}^{1,n}(\Omega)$. Then there exists a sequence of functions $\hat{\varphi}_k \in \mathcal{D}^{1,n}(U)$ such that $D\hat{\varphi}_k \to D\hat{\varphi}$ in $L^n(U)$. We extend $\hat{\varphi}_k$ by zero to S^n. Since Ω^C has property A we conclude that either $\mathcal{H}^n(U^C) > 0$ or U^C consists of at most two points. In the first case we conclude by Poincaré's inequality that $\hat{\varphi}_k \to \hat{\varphi}$ in $L^n(S^n)$ and hence $\varphi_k = \hat{\varphi}_k \circ \pi^{-1} \to \varphi$ in $L^n_{\text{loc}}(\Omega)$. In the second case there exists a sequence of functions $\hat{\omega}_k \in C_0^\infty(U)$ such that $\hat{\omega}_k \to 1$ in $L^n(S^n)$ and $\|D\hat{\omega}_k\|_{L^n(S^n)} \le (k|(\varphi - \varphi_k)_{S^n}|)^{-1}$. Define $\hat{\psi}_k = \varphi_k + (\varphi - \varphi_k)_{S^n}\hat{\omega}_k$. It is easy to check that $\hat{\psi}_k \to \hat{\varphi}$ in $W^{1,n}(U)$. If $\varphi \in L^\infty(\Omega)$ we define $\tilde{\psi}_k = T_M(\psi_k)$ where T_M is a smooth cut–off function in the range with $T_M(z) = z$ for all $|z| \le M$ and $M \ge \|\varphi\|_{L^\infty(\Omega)}$. ∎

Remark. It is easy to see that the lemma holds true for arbitrary open domains Ω. The proof uses a suitable definition of n–capacity on S^n; the first case in the proof of the lemma corresponds to $\text{cap}_n(U^C) > 0$ while the second case corresponds to $\text{cap}_n(U^C) = 0$

Lemma A.2 *Let $\Omega \subset \mathbb{R}^n$ be an open domain such that Ω^C has property A. Assume that $u \in \mathcal{D}^{1,n}(\Omega)$.*

1) *If $g \in C_0^\infty(\mathbb{R})$, then $g \circ u \in \mathcal{D}^{1,n}(\Omega)$.*
2) *If $\eta \in C_0^\infty(\mathbb{R}^n)$, then $\eta u \in \mathcal{D}^{1,n}(\Omega)$.*
3) *Assume that $\Omega^C \subset Q(0,r)$ and $\eta \in C^\infty(\mathbb{R}^n)$ satisfies $\eta \equiv 0$ on $Q(0,r)$ and $\eta \equiv 1$ on $\mathbb{R}^n \setminus Q(0,2r)$. Then $\eta \in \mathcal{D}^{1,n}(\Omega)$.*

Proof. In view of Lemma A.1 we may choose a sequence $u_k \in C_0^\infty(\Omega)$ such that $u_k \to u$ in $L^n_{\text{loc}}(\Omega)$, $u_k \to u$ almost everywhere, and $Du_k \to Du$ in $L^n(\Omega)$. It follows from the triangle inequality that

$$\begin{aligned} &\int_\Omega |D(g \circ u) - D(g \circ u_k)|^n dx \\ &\qquad \le c \int_\Omega |(Dg) \circ u_k - (Dg) \circ u|^n |Du|^n dx + c \int_\Omega |(Dg) \circ u_k|^n |Du_k - Du|^n dx. \end{aligned}$$

The first integral converges to zero by the dominated convergence theorem since $|Du|^n \in L^1(\Omega)$. The second integral converges to zero since $|Dg| \le C$ and $Du_k \to Du$ in $L^n(\Omega)$ by assumption. This proves the first assertion. Moreover

$$\int_\Omega |D(\eta u_k) - D(\eta u)|^n dx \le c \int_\Omega |D\eta(u_k - u)|^n dx + c \int_\Omega |\eta(Du_k - Du)|^n dx.$$

Therefore the second assertion follows from the convergence of u_k to u in $L^n_{\text{loc}}(\Omega)$ since η has compact support. The last assertion is an immediate consequence of the fact that constant functions are in $\mathcal{D}^{1,n}(\mathbb{R}^n)$ for $n \ge 2$. ∎

Lemma A.3 *Let $u \in \mathcal{D}^{1,n}(\Omega; \mathbb{R}^m)$ be a solution of system (1.1) with $f \in L^1(\Omega; \mathbb{R}^m) \cap C^\infty(\Omega; \mathbb{R}^m)$ in place of μ. Then the equality*

$$\int_\Omega \sigma(x, u, Du) D\varphi dx = \int_\Omega f \varphi dx$$

holds for all $\varphi \in \mathcal{D}^{1,n}(\Omega) \cap L^\infty(\Omega)$.

Proof. This follows easily from Lemma A.1. □

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