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by

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Regularity and Optimal Design Results for Elastic Membranes

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Abstract The effective energy of a mixture of two elastic materials in a thin film is characterized using Gamma-limit techniques. For cylindrical shaped inclusions it is shown that 3D-2D asymptotics and optimal design commute from a variational viewpoint. Regularity of local minimizers for the resulting design is addressed.

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1 Introduction

There has been an increasing involvement of the mathematics community in the study of thin structures due, in part, to the remarkable technological, industrial and medical applications of thin films. Taking the viewpoint that their mechanical, elastic, and electromagnetic properties are inherited from the corresponding bulk properties as the material domain is scaled thinner and thinner, in order to study thin film behavior we undertake a 3D-2D asymptotic analysis for a flat domain as its thickness tends to 0.

Although the linear and semilinear frameworks were relatively well understood several years ago (see [20], [21], [43], [44]), it was only recently that some progress has been made in a truly nonlinear setting (see [14], [15], [18], [27], [30], [38]; see also [1], [2], [35]).

When we consider the optimization of the mechanical properties of a thin film through the distribution of stiffness subject to a fixed volume fraction, we are naturally led to a problem in optimal design involving dimension reduction, i.e., a 3D-2D asymptotic analysis. Optimal design of two-phase mixtures is a contemporary topic which goes back to the pioneering works of F. Murat and L. Tartar (see [42], [45]; see also [6], [7], [16], [33], [36], [39]).

We consider a thin 3D domain $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ and $\omega \subset \mathbb{R}^2$ is a bounded, open, Lipschitz domain. We assume that the body is composed by two elastic materials with energy densities W_1 and W_2, W_i $(i = 1, 2)$ are continuous nonnegative functions on $\mathbb{R}^{3\times 3}$, so that at each material point $x \in \Omega_\varepsilon$ the bulk energy density corresponding to a given strain $\xi \in \mathbb{R}^{3\times 3}$ is given by

$$W(x, \xi) := \chi(x) W_1(\cdot) + (1 - \chi(x)) W_2(\cdot)$$

where $\chi(\cdot)$ denotes the characteristic function of the first phase.

In search for the best design, the one which minimizes the *compliance*, we need to characterize

$$\lim_{\varepsilon \to 0^+} \sup_{\chi} \inf_{u} \frac{1}{\varepsilon} \left\{ \int_{\Omega_\varepsilon} (\chi W_1 + (1 - \chi) W_2)(Du)\, dx - \int_{\Omega_\varepsilon} f \cdot u \, dx : \right.$$
$$\left. u = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon),\ \chi \in L^\infty(\Omega_\varepsilon; \{0, 1\}),\ \frac{1}{\mathcal{L}^3(\Omega_\varepsilon)} \int_{\Omega_\varepsilon} \chi(x)\, dx = \theta \right\},$$

where $f(x)$ is a given load on Ω_ε and $\theta \in [0, 1]$ is a fixed volume fraction. Here, and in what follows, $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure in $\mathbb{R}^N$. This problem, involving a sup-inf, is still out of our reach mathematically, and so we turn to the characterization of the above limit when $\sup_\chi \inf_u$ is replaced by $\inf_{\chi, u}$. We may think of it not only as finding the "worst possible design", but also in terms of damage. Elastic materials are prone to defects that impair their elastic stiffness, and we may want to predict the evolution of the damaged areas and how they will be affected by the scaling in the thickness. In this setting, $\{\chi = 1\}$ and $\{\chi = 0\}$ represent the damaged and healthy regions, respectively. We take the viewpoint that damage evolution is governed by the principle of global energy minimization (see [5], [31]), precisely, for fixed $\varepsilon > 0$ we look for

$$\inf_{\chi, u} d_\varepsilon(\chi, u)$$

where

$$d_\varepsilon(\chi, u) := \frac{1}{\varepsilon} \left\{ \int_{\Omega_\varepsilon} (\chi W_1 + (1 - \chi) W_2)(Du) dx - \int_{\Omega_\varepsilon} f \cdot u dx + \kappa \int_{\Omega_\varepsilon} \chi dx : \right.$$
$$\left. u = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon),\ \chi \in L^\infty(\Omega(\varepsilon); \{0, 1\}),\ \frac{1}{\mathcal{L}^3(\Omega_\varepsilon)} \int_{\Omega_\varepsilon} \chi(x)\, dx = \theta \right\},$$

and κ is the *critical energy release rate*. The characterization of the resulting energy density has been obtained in a fairly general context in [18], but here we restrict ourselves to the case treated in [28] where the inclusions are of cylindrical type, i.e., $\chi = \chi(x_\alpha)$, with x_a denoting the pair of variables x_1, x_2. Precisely, if $\theta_0 \in [0, 1]$ and if $v \in W^{1,p}(\omega; \mathbb{R}^3)$ for some $p > 1$, with W_i growing at infinity not faster than a polynomial of degree p, we show that there is an energy density $\overline{W}$ on the middle plate ω, intrinsically related to that found by H. Le Dret and A. Raoult (see [38]), such that

$$G(\theta_0, v) = \inf \left\{ 2 \int_\omega \overline{W}(\theta, D_a v)\, dx_a : \frac{1}{\mathcal{L}^2(\omega)} \int_\omega \theta \, dx_a = \theta_0 \right\},$$

where

$$\begin{aligned} G(\theta_0, v) &:= \\ &\inf_{\{\chi_\varepsilon\},\{v_\varepsilon\}} \Bigg\{ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_\Omega (\chi_\varepsilon(x_\alpha) W_1 + (1 - \chi_\varepsilon(x_\alpha)) W_2) \left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon \right) dx_\alpha dx_3 \\ &\qquad v_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^3), \chi_\varepsilon \in L^\infty(\omega; \{0,1\}), \\ &\qquad v_\varepsilon \to v \quad \text{in } L^p(\Omega; \mathbb{R}^3), \frac{1}{\mathcal{L}^2(\omega)} \int_\omega \chi_\varepsilon(x_\alpha)\, dx_\alpha = \theta \Bigg\}. \end{aligned} \tag{1.1}$$

Here $\Omega := \omega \times (-1, 1)$, and we used the change of variables $(y_\alpha, y_3) \in \Omega_\varepsilon \mapsto (x_\alpha, x_3) \in \Omega$, with $y_\alpha = x_\alpha$, $y_3 = \varepsilon x_3$. We write x'_α when x_a varies over the unit square Q' of $\mathbb{R}^2$, the derivative D_α stands for the pair of partial derivatives $(D_1 := \partial/\partial x_1, D_2 := \partial/\partial x_2)$, $|x_\alpha| := \sqrt{x_1^2 + x_2^2}$, and dx_α (resp. dx'_α) will denote $dx_1 dx_2$ (resp. $dx'_1 dx'_2$).

Next, we address the regularity of equilibrium configurations for the thin structure, i.e. minimizers of the mapping $v \in W^{1,p}(\omega; \mathbb{R}^3) \mapsto I(\theta_0, v)$ for fixed constant volume fraction θ_0, where

$$I(\theta, v) := \int_\omega \overline{W}(\theta, D_\alpha v)\, dx_\alpha.$$

It is well known that regularity properties of (local) minimizers for vectorial variational problems are extremely hard to get, and there is not a systematic theory that will provide insight into this issue. Recently some progress on this direction has been made under certain constitutive hypotheses on $\overline{W}$, and initially this analysis was undertaken mostly within the framework of image segmentation in computer vision (see [17], [19], [23], [25], [41]). As it turns out, although the physical or technological motivations are not related, some mathematical models for image segmentation and for the interplay between fracture and damage for (non linear) elastic materials are, essentially, the same, involving similar mathematical challenges and difficulties. As an example, in the image segmentation models proposed by D. Mumford and J. Shah (see [41]) and in the fracture and damage set up used by I. Fonseca and G. Francfort (see [27]), we consider the free discontinuity problem

$$\mathcal{G}(u; K) := \int_{\omega \setminus K} \overline{W}(\theta_0, D_\alpha v)\, dx_\alpha + \beta \int_{\omega \setminus K} |u - g|^q \, dx_\alpha + \eta H^1(K \cap \omega)$$

where H^{N-1} stands for the $N-1$-Hausdorff measure, K is a closed set, $\beta, \eta > 0$, $p, q > 1$, and $v \in C^1(\omega \setminus K)$. This model was studied at length by E. De Giorgi and his collaborators (see [23], [25]; see also [10], [11], [12]) in the case of scalar-valued v, and the quest for existence of minimizers for $\mathcal{G}$ led to the relaxation of this functional in a functional space where compactness is guaranteed. Precisely, L. Ambrosio and E. De Giorgi introduced in [24] the space of *functions of special bounded variation SBV* (see also [8], [9], [10]), and they re-wrote $\mathcal{G}$ as

$$\mathcal{G}_{\text{relax}}(v) := \int_\omega \overline{W}(\theta_0, \nabla_\alpha v)\, dx_\alpha + \beta \int_\omega |v - g|^q \, dx_\alpha + \eta H^1(S(v) \cap \omega) \tag{1.2}$$

where $v \in SBV(\omega; \mathbb{R}^d)$. The distributional derivative $D_\alpha v$ of v in ω can be written as

$$D_\alpha v = \nabla_\alpha v \lfloor \omega + (v^+ - v^-) \otimes \nu H^1 \lfloor S(v),$$

the jump set of v, $S(v)$, is 1- rectifiable, ν is the normal to $S(v)$, and v^+, v^-, are the traces of v on $S(v)$. As it is usual, we use the symbol $\lfloor$ to indicate the restriction of a measure, i.e., if μ is a measure on a set A and if $B \subset A$ is μ-measurable, then $\mu \lfloor B(X) := \mu(B \cap X)$ for all $X \subset A$. The model (1.2) was used in [27] in the vectorial setting to treat the interplay between fracture and damage in an elastic material.

Once existence of a minimizer v for $\mathcal{G}_{\text{relax}}$ is established, the next task is the search for regularity properties of local minimizers of $I(\theta_0, \cdot)$ which will guarantee that $(v; \overline{S(v)})$ is now a minimizer for the initial energy $\mathcal{G}$. We recall that v is a *local minimizer* for $\mathcal{G}$ if

$$\int_{B_r} \overline{W}(\theta_0, D_\alpha v)\, dx_a \le \int_{B_r} \overline{W}(\theta_0, D_\alpha u)\, dx_\alpha$$

for all ball $B_r \subset \omega$ and for all u such that $u - v \in W_0^{1,2}(B_r; \mathbb{R}^3)$. In other words, we would like to be able to assert that $S(v)$ is "essentially closed" and that outside $S(v)$ the function v is smooth. Precisely, we want to prove that

$$H^1((\overline{S(v)} \setminus S(v)) \cap \omega) = 0$$

and

$$v \in W^{1,2}(\omega \setminus \overline{S(v)}; \mathbb{R}^3).$$

In the scalar case, where $v : \omega \to \mathbb{R}^d, d = 1$, this result was obtained by De Giorgi, Carriero and Leaci in [25] for the Dirichlet integral, and later extended to the vectorial case, where $d > 1$, by Carriero and Leaci (see [19]) when $\overline{W}(\xi) := |\xi|^p$, $p > 1$. More recently, in collaboration with Acerbi and with Fusco (see [3], [4], [29]) we were able to prove that local minimizers $v \in W^{1,2}(\omega; \mathbb{R}^3)$ of $I(\theta, \cdot)$ are in $C^{0,\gamma}(\Omega; \mathbb{R}^3)$ for all $\gamma \in (0, 1)$, whenever the density energy $\overline{W}$ is of the form

$$\overline{W}(\theta, \xi) = \frac{1}{2}|\xi|^2 + h(|M(\xi)|), \tag{1.3}$$

the function h grows linearly at infinity, it is not necessarily convex, and $M(\xi)$ denotes the vector of all 2×2 minors of ξ. In turn, this regularity entails that minimizers $v \in SBV(\omega;\mathbb{R}^3)$ of

$$u \mapsto \int_\omega \left(\frac{1}{2}|\nabla u|^2 + h(|M(\nabla u)|)\right) dx_\alpha + \beta \int_\omega |u-g|^q \, dx_\alpha + \eta H^1(S(u)\cap\omega)$$

are "classical" minimizers in that $v \in W^{1,2}(\omega \setminus \overline{S(v)};\mathbb{R}^3)$ and $H^1((\overline{S(v)}\setminus S(v))\cap \omega) = 0$. We use an argument similar to the one introduced by P. Bauman, N. Owen and D. Phillips in [13] (see also [26]) to exploit the higher integrability of two auxiliary functions of the derivatives of u, $A := (|D_1 u|^2 - |D_2 u|^2)/2$, $B := D_1 u \wedge D_2 u$. As it turns out, when $h(|M(\xi)|) = |\xi|$ then A and B are harmonic functions. For related regularity results, see also ([22], [37]).

In Section 2 we will give a brief overview of the optimal design problem for thin films with cylindrical inclusions, and in Section 3 we will outline the proof of Hölder regularity for local minimizers of $I(\theta,\cdot)$ when (1.3) holds.

2 The 3D-2D Optimal Design Problem

We consider two energy density functions $W_i : \mathbb{R}^{3\times 3} \to [0,+\infty)$, $i = 1,2$, such that

$$\alpha|\xi|^p \le W_i(\xi) \le \beta(1+|\xi|^p), \quad \xi \in \mathbb{R}^{3\times 3},$$

$\alpha, \beta > 0$ and $1 < p < +\infty$. As in [38], for $i = 1,2$, we define

$$\overline{W}_i(\overline{\xi}) := \inf_{z\in\mathbb{R}^3} W_i(\overline{\xi}|z), \quad \overline{\xi} \in \mathbb{R}^{3\times 2},$$

where $(\overline{\xi}|z)$ denotes the 3×3 matrix which first two columns are the ones of $\overline{\xi}$ and third column is the vector z.

For a fixed volume fraction $\theta_0 \in [0,1]$ and for $\overline{\xi} \in \mathbb{R}^{3\times 2}$ we set

$$\overline{W}(\theta,\overline{\xi}) := \inf_{\chi,\varphi} \left\{ \int_{Q'} \left(\chi(x'_\alpha)\overline{W}_1 + (1-\chi(x'_\alpha))\overline{W}_2\right)(\overline{\xi} + D_\alpha \varphi)\, dx'_\alpha : \varphi \in W_0^{1,p}(Q';\mathbb{R}^3), \int_{Q'} \chi(x'_\alpha)\, dx'_\alpha = \theta_0 \right\}.$$

Also, if $\theta \in L^\infty(\omega;[0,1])$ we define the energy

$$J(\theta,v) := \inf_{\{\chi_\varepsilon\},\{v_\varepsilon\}} \left\{ \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_\Omega (\chi_\varepsilon(x_\alpha)W_1 + (1-\chi_\varepsilon(x_\alpha))W_2)\left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon\right) dx_\alpha dx_3 : v_\varepsilon \in W^{1,p}(\Omega;\mathbb{R}^3), \chi_\varepsilon \in L^\infty(\omega;\{0,1\}), v_\varepsilon \to v \quad \text{in } L^p(\Omega;\mathbb{R}^3), \chi_\varepsilon \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\omega;[0,1]) \right\}.$$

Recalling the effective energy G as introduced in (1.1), it can be shown that

Theorem 2.1

$$J(\theta, v) = 2\int_\omega \overline{W}(\theta(x_\alpha), D_\alpha v(x_\alpha))\, dx_\alpha \tag{2.1}$$

and

$$G(\theta_0, v) = \inf\left\{ J(\theta; v) : \frac{1}{\mathcal{L}^2(\omega)}\int_\omega \theta(x_\alpha)\, dx_\alpha = \theta_0 \right\}. \tag{2.2}$$

Although we will not present here a complete, detailed proof of this theorem, we indicate the main ideas of each one of the following five steps of the proof.

Step 1: We claim that

$$J(\theta, v) \geq 2\int_\omega \overline{W}(\theta(x_\alpha, D_\alpha v(x_\alpha))\, dx_\alpha. \tag{2.3}$$

Consider any admissible sequence pair $\{(\chi_n, v_n)\}$ with corresponding $\varepsilon_n \to 0^+$. We localize the energy by constructing the sequence of finite, Radon measures with traces on the open sets A of ω given by

$$\begin{aligned}\mu_n(A) := \quad & \Big[\chi_n W_1 \left(D_\alpha v_n \Big| \tfrac{1}{\varepsilon_n} D_3 v_n\right) \\ & \quad +(1-\chi_n(x_\alpha))W_1\left(D_\alpha v_n \Big| \tfrac{1}{\varepsilon_n} D_3 v_n\right)\Big]\, \mathcal{L}^3 \lfloor A \times (-1,1).\end{aligned}$$

To establish (2.3) we must show that

$$\frac{d\mu}{d\mathcal{L}^2}(x_0) \geq 2\overline{W}(\theta(x_0), D_\alpha v(x_0)) \quad \mathcal{L}^2 \text{ a.e. } x_0 \in \omega.$$

Let x_0 be a Lebesgue point for θ and a point of approximate differentiability for v. Choosing a suitable sequence of radii $\delta_j \to 0^+$ such that $\mu(\partial Q'(x_0, \delta_j)) = 0$, and changing variables, we have

$$\begin{aligned}
&\frac{d\mu}{d\mathcal{L}^2}(x_0) = \lim_{j\to+\infty} \frac{1}{\delta_j^2}\, \mu(Q'(x_0, \delta_j)) \\
&= \lim_{j\to+\infty} \frac{1}{\delta_j^2} \lim_{n\to+\infty} \int_{Q'(x_0,\delta_j)\times(-1,1)} (\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n\right) dx_\alpha dx_3 \\
&= \lim_{j,n\to+\infty} \int_Q (\chi_{j,n}(x_\alpha)W_1 + (1-\chi_{j,n}(x_\alpha))W_2)\left(D_\alpha v_{j,n} \Big| \frac{\delta_j}{\varepsilon_n} D_3 v_{j,n}\right) dx_\alpha dx_3 \\
&\geq \lim_{j,n\to+\infty} \int_Q (\chi_{j,n}(x_\alpha)\overline{W}_1 + (1-\chi_{j,n}(x_\alpha))\overline{W}_2)\,(D_\alpha v_{j,n})\, dx_\alpha dx_3,
\end{aligned}$$

where

$$\begin{aligned}
\chi_{j,n}(x'_\alpha) \quad &:= \chi_n((x_0)_\alpha + \delta_j x'_\alpha), \\
v_{j,n}(x'_\alpha, x_3) \quad &:= \tfrac{v_n((x_0)_\alpha + \delta_j x'_\alpha, x_3) - v(x_0)}{\delta_j}.
\end{aligned}$$

Since $\lim_{j\to+\infty}\lim_{n\to+\infty}||v_{j,n}-Dv(x_0)x'_\alpha||_{L^p(Q)}=0$, a standard slicing method to connect $v_{j,n}$ to $x\mapsto Dv(x_0)x'_\alpha$ near the lateral boundary $\partial Q'\times(-1,1)$, together with Fubini's Theorem, entails

$$\frac{d\mu}{d\mathcal{L}^2}(x_0)\geq \limsup_{j,n\to+\infty}\int_Q(\chi_{j,n}(x_\alpha)\overline{W}_1+(1-\chi_{j,n}(x_\alpha))\overline{W}_2)(D_\alpha w_{j,n})\,dx_\alpha dx_3, \quad (2.4)$$

where $D_\alpha w_{j,n}=D_\alpha v(x_0)$ near $\partial Q'\times(-1,1)$. By definition of $\overline{W}$, the proof of (2.3) would be concluded if

$$\int_{Q'}\chi_{j,n}(x'_\alpha)\,dx'_\alpha=\theta(x_0).$$

If this is not the case, then we must modify slightly the support of the characteristic function $\chi_{j,n}$ so as to satisfy this volume constraint without increasing too much the resulting total energy. We illustrate how to resolve this problem when

$$\int_{Q'}\chi_{j,n}(x'_\alpha)\,dx'_\alpha<\theta(x_0)<1.$$

The limiting cases $\theta(x_0)\in\{0,1\}$ must be treated with a different argument. Since

$$\begin{aligned}\lim_{j,n\to+\infty}\int_{Q'}\chi_{j,n}(x'_\alpha)\,dx'_\alpha &= \lim_{j\to+\infty}\frac{1}{\delta_j^2}\lim_{n\to+\infty}\int_{Q'(x_0,\delta_j)}\chi_n(x_\alpha)\,dx_\alpha\\ &= \lim_{j\to+\infty}\frac{1}{\delta_j^2}\int_{Q'(x_0,\delta_j)}\theta(x_\alpha)\,dx_\alpha\\ &=\theta(x_0),\end{aligned}$$

we have that

$$\lim_{j,n\to+\infty}\mathcal{L}^2(A_{j,n})=\theta(x_0),$$

where

$$A_{j,n}:=\{x_\alpha\in Q':\chi_n((x_0)_\alpha+\delta_j x'_\alpha)=1\}.$$

As, by assumption, $\mathcal{L}^2(A_{j,n})<\theta(x_0)$, we set

$$K_{j,n}:=\left[\left[\frac{1}{\sqrt{\theta(x_0)-\mathcal{L}^2(A_{j,n})}}\right]\right],$$

where $[[x]]$ stands for the integer part of x. Then, for j,n, large enough and since $\theta(x_0)<1$, we have that

$$K_{j,n}(\theta(x_0)-\mathcal{L}^2(A_{j,n}))\leq\sqrt{\theta(x_0)-\mathcal{L}^2(A_{j,n})}\leq 1-\theta(x_0),$$

so that it is possible to decompose $Q' \setminus A_{j,n}$ (a set of measure at least $1-\theta(x_0)$) as

$$Q' \setminus A_{j,n} = \cup_{i=1}^{K_{j,n}} \hat{A}_i \cup B,$$

where $\hat{A}_i$ are mutually disjoint and

$$\mathcal{L}^2(\hat{A}_i) = \theta(x_0) - \mathcal{L}^2(A_{i,n}), \quad i = 1, \ldots, K_{j,n}.$$

Due to the coercivity of W_i and by (2.4), $\{D_\alpha w_{j,n}\}$ is uniformly bounded in $L^p(Q)$, and so there exists an index $i(j,n) \in \{1, \ldots, K_{j,n}\}$ such that

$$\int_{\hat{A}_{i(j,n)}} (1 + |D_\alpha w_{j,n}|^p) \, dx_\alpha \leq \frac{C}{K_{j,n}}.$$

With $1_{\hat{A}_{i(j,n)}}$ denoting the characteristic function of $\hat{A}_{i(j,n)}$, define

$$\hat{\chi}_{j,n} := \chi_{j,n} + 1_{\hat{A}_{i(j,n)}},$$

so that

$$\int_{Q'} \hat{\chi}_{j,n} \, dx_\alpha = \theta(x_0).$$

Clearly

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^2}(x_0) &\geq \liminf_{j,n\to+\infty} \int_{Q'\times(-1,1)} (\hat{\chi}_{j,n}(x_\alpha)\overline{W}_1 + (1-\hat{\chi}_{j,n}(x_\alpha))\overline{W}_2)(D_\alpha w_{j,n}) \, dx_\alpha dx_3 \\ &\geq \overline{W}(\theta(x_0), D_\alpha v(x_0)).\end{aligned}$$

To conclude (2.3) we must assert now that

$$J(\theta, v) \leq 2 \int_\omega \overline{W}(\theta, D_\alpha v) \, dx_\alpha. \tag{2.5}$$

We proceed by approximation in Steps 2–4, where we localize J with respect to the domain of integration. Precisely, if A is an open subset of ω, we define

$$\begin{aligned}&J(\theta, v; A) := \\ &\inf_{\{\chi_\varepsilon\},\{v_\varepsilon\}} \Bigg\{ \lim_{\varepsilon\to0^+} \frac{1}{\varepsilon} \int_{A\times(-1,1)} (\chi_\varepsilon(x_\alpha)W_1 + (1-\chi_\varepsilon(x_\alpha))W_2) \left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon \right) dx : \\ &\qquad v_\varepsilon \in W^{1,p}(A\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \in L^\infty(\omega;\{0,1\}), \\ &\qquad v_\varepsilon \to v \quad \text{in } L^p(A\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\omega;[0,1]) \Bigg\}.\end{aligned}$$

<u>*Step 2*</u>: (2.5) holds for $A = Q'(a,r) \subset \omega$, θ, constant and v affine. With no loss of generality, and upon a suitable translation and rescaling, we may assume that $A = Q'$, i.e., $a = 0, r = 1$.

By definition of $\overline{W}$, and using a measurability selection criterium, we may find $\chi_\eta \in L^\infty(Q';[0,1])$, with $\int_{Q'} \chi_\eta(x'_\alpha)\, dx'_\alpha = \theta$, $\varphi_\eta \in W_0^{1,\infty}(Q';\mathbb{R}^3)$, and $\xi_\eta \in L^p(Q';\mathbb{R}^3)$, such that

$$\overline{W}(\theta, D_\alpha v) = \lim_{\eta\to 0^+} \int_{Q'} (\chi_\eta(x'_\alpha)\overline{W}_1 + (1-\chi_\eta(x'_\alpha))\overline{W}_2)(D_\alpha v + D_\alpha \varphi_\eta \,|\, \xi_\eta(x'_\alpha))\, dx'_\alpha.$$

In addition, the continuity and growth properties of $\overline{W}_i$, and the density of $W_0^{1,\infty}(Q';\mathbb{R}^3)$ in $L^p(Q';\mathbb{R}^3)$, allow us to assume that $\xi_\eta \in W_0^{1,\infty}(Q';\mathbb{R}^3)$.

Extend $\chi_\eta, \varphi_\eta, \xi_\eta$, Q'-periodically to $\mathbb{R}^2$ and set

$$v_{\eta,n}(x_\alpha, x_3) := v(x_\alpha) + \frac{1}{n}\varphi_\eta(nx_\alpha) + \frac{1}{n^2}x_3\xi_\eta(nx_\alpha),$$

$$\chi_{\eta,n}(x_\alpha) := \chi_\eta(nx_\alpha).$$

Note that

$$\lim_{\eta\to 0^+} \lim_{n\to+\infty} ||v_{\eta,n} - v||_{L^p(Q;\mathbb{R}^3} = 0,$$

$$\lim_{\eta\to 0^+} \lim_{n\to+\infty} \left| \int_{Q'} \chi_{\eta,n}\psi\, dx'_\alpha - \int_{Q'} \theta\psi\, dx'_\alpha \right| = 0$$

for all ψ on a countable, dense subset of $L^1(Q)$, and also

$$\begin{aligned}
&\lim_{\eta\to 0^+} \lim_{n\to+\infty} \int_Q (\chi_{\eta,n}(x_\alpha)W_1 + (1-\chi_{\eta,n}(x_\alpha))W_2) \\
&\qquad\qquad \left(D_\alpha v_n + D_\alpha\varphi_{\eta,n}(x_\alpha)\middle| n^2 D_3 v_{\eta,n}\right)\, dx_\alpha dx_3 \\
&= \lim_{\eta\to 0^+} \lim_{n\to+\infty} \int_Q (\chi_\eta(nx_\alpha)W_1 + (1-\chi_\eta(nx_\alpha))W_2) \\
&\qquad \left(D_\alpha v + D_\alpha\varphi_\eta(nx_\alpha) + \frac{1}{n}x_3 D_\alpha\xi_\eta(nx_\alpha)\middle|\xi_\eta(nx_\alpha)\right)\, dx_\alpha\, dx_3 \\
&= 2\overline{W}(\theta, D_\alpha v).
\end{aligned}$$

Therefore, we may find a decreasing sequence $\eta(n) \searrow 0^+$ such that, setting

$$v_n := v_{\eta(n),n},\ \chi_n := \chi_{\eta(n),n},$$

we have

$$v_n = v \text{ on } \partial Q' \times (-1,1), \tag{2.6}$$

and

$$v_n \to v \text{ in } L^p(Q;\mathbb{R}^3),\ \chi_n \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(A),$$

with

$$\begin{aligned}
J(v,\theta;A) &\le \liminf_{n\to+\infty} \int_Q (\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n\middle| n^2 D_3 v_n\right) dx_\alpha dx_3 \\
&= 2\overline{W}(\theta, D_\alpha v).
\end{aligned}$$

Step 3: Given the matching boundary conditions of the sequence $\{v_n\}$ obtained in the previous step (see (2.6)), and since any triangle T on the plane may be covered with squares of the type $Q'(a,r), r > 0, a \in \mathbb{R}^2$, up to a set of arbitrarily small measure, it is now clear that a simple covering argument will ensure that (2.5) still holds for a piecewise constant function θ, a piecewise affine function $v \in W^{1,p}(\omega;\mathbb{R}^3)$, and a triangular domain T, where the sequence $\{\varepsilon_n\}$ for the upper bound approximating pair $\{(\chi_n, v_n)\}$ is taken always to be $\{1/n\}$.

Step 4: In order to conclude the proof of (2.5) for v an arbitrary element of $W^{1,p}(\omega;\mathbb{R}^3)$ and θ an arbitrary element of $L^\infty(\omega;[0,1])$, we let $\{v_k, \theta_k\}$ be a sequence of piecewise affine and continuous v's and piecewise constant θ's on a triangulation of ω such that, as $k \to +\infty$,

$$v_k \to v \quad \text{in } W^{1,p}(\omega;\mathbb{R}^3)$$

and

$$\theta_k \to \theta \quad \text{in } L^p(\omega;[0,1]), 1 \leq q < +\infty.$$

The proof now follows from Step 3, via a standard diagonalization argument, and using the fact that $(\theta, \overline{\xi}) \in [0,1] \times \mathbb{R}^{3\times 2} \mapsto \overline{W}(\theta, \overline{\xi})$ is an upper-semicontinuous function.

Step 5: Finally, in order to establish (2.2), we start by remarking that, trivially,

$$G(\theta_0, v) \geq \inf \left\{ J(\theta; v) : \frac{1}{\mathcal{L}^2(\omega)} \int_\omega \theta(x_\alpha)\, dx_\alpha = \theta_0 \right\}.$$

Indeed, if $\{\chi_\varepsilon\}$ is such that

$$\frac{1}{\mathcal{L}^2(\omega)} \int_\omega \chi_\varepsilon(x_\alpha)\, dx_\alpha = \theta_0,$$

then a subsequence of $\{\chi_\varepsilon\}$, still indexed by ε, is such that

$$\chi_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(\omega;[0,1]),$$

with

$$\frac{1}{\mathcal{L}^2(\omega)} \int_\omega \theta(x_\alpha)\, dx_\alpha = \theta_0.$$

Conversely, let $\theta \in L^\infty(\omega;[0,1])$ be such that

$$\frac{1}{\mathcal{L}^2(\omega)} \int_\omega \theta(x_\alpha)\, dx_\alpha = \theta_0$$

and consider a sequence $\{\chi_n, v_n, \varepsilon_n\}$ such that

$$\chi_n \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\omega;[0,1]), \quad v_n \to v \ \text{ in } L^p(\Omega;\mathbb{R}^3), \quad \varepsilon_n \to 0^+,$$

with

$$J(v,\theta) = \lim_{n\to+\infty} \int_\Omega (\chi_n(x_\alpha) W_1 + (1-\chi_n(x_\alpha)) W_2) \left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n \right) dx_\alpha dx_3.$$

As in the proof of Step 1, we may construct another sequence of characteristic functions $\hat{\chi}_n$ such that

$$\int_\omega \hat{\chi}_n(x_\alpha)\, dx_a = \theta_0 \quad \text{for all } n$$

and

$$\begin{aligned} J(v,\theta) &\geq \limsup_{n\to+\infty} \int_\Omega (\hat{\chi}_n(x_\alpha) W_1 + (1-\hat{\chi}_n(x_\alpha)) W_2) \left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n \right) dx_\alpha dx_3 \\ &\geq G(\theta_0, v). \end{aligned}$$

□

3 Regularity of Minimizers for a Class of Membrane Energies

In this section we summarize the argument used in [4] (see also [3]) to obtain Hölder regularity for local minimizers of a certain class of energies which appear naturally in the 3D-2D asymptotic analysis for thin films (see Section 2).

As before, ω is an open, bounded subset of $\mathbb{R}^2$, and let $h : [0,+\infty) \to [0,\infty)$ be a C^1 function such that

(H1) $h(t) \leq C(1+t)$ for some $C > 0$;

(H2) there exist $\Lambda \in [0,+\infty)$ such that

$$\lim_{t\to+\infty} h'(t) = \Lambda;$$

(H3) there exist $\alpha, C > 0$ such that for all $t \geq 1$

$$\left| h'(t) - \frac{h(t)}{t} \right| \leq \frac{C}{t^\alpha}.$$

Without loss of generality, we may assume that $0 < \alpha < 1$.

Given $u \in W^{1,2}(\omega;\mathbb{R}^d)$ we define

$$M(\nabla u) := \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2},$$

i.e., the 2-covector whose components are the 2×2 subdeterminants of ∇u. Let (see (1.3))

$$F(u) := \int_\omega \left[\frac{1}{2} |\nabla u|^2 + h(|M(\nabla u)|) \right] dx.$$

where, for simplicity of notation, we write x in place of x_α.

We claim that

Theorem 3.1 *If $u \in W^{1,2}(\omega;\mathbb{R}^d)$ is a $W^{1,2}$-local minimizer of F then $u \in C^{0,\gamma}_{\mathrm{loc}}$ for all $\gamma \in (0,1)$.*

As mentioned in the Introduction, under the constitutive relation (1.3) and following the argument of [25], this regularity result will entail the existence of minimizers $v \in SBV(\omega;\mathbb{R}^d)$ for $\mathcal{G}_{\mathrm{relax}}$ satisfying

$$H^1((\overline{S(v)} \setminus S(v)) \cap \omega) = 0.$$

The method we will pursue here is well known in regularity theory and it relies heavily on the properties of *Morrey spaces* $L^{p,\lambda}$ (see [32], [34]).

Definition 3.2 *Given $\lambda \geq 0$ we say that $f \in L^{p,\lambda}(\omega;\mathbb{R})$ if there exists a constant $C > 0$ such that*

$$\int_{B_\rho(x)\cap\omega} |f|^p \, dx \leq C\rho^\lambda$$

for all $x \in \omega$ and $0 < \rho < \operatorname{diam}\omega$. The function f is said to be in $L^{p,\lambda}_{\mathrm{loc}}(\omega)$ if $f \in L^{p,\lambda}(\omega')$ for all $\omega' \subset\subset \Omega$.

It can be shown that

$$L^{p,0}(\omega) = L^p(\omega),\ L^{p,2}(\omega) = L^\infty(\omega),\ L^{p,\lambda}(\omega) = \{0\} \quad \text{if } \lambda > 2,$$

and that $L^{p,\lambda}(\omega)$ is a Banach space endowed with the norm

$$||f||_{L^{p,\lambda}(\omega)} := \left\{ \sup_{x\in\omega,\ 0<\rho<\operatorname{diam}\omega} \rho^{-\lambda} \int_{B_\rho(x)\cap\omega} |f|^p \, dx \right\}^{\frac{1}{p}}.$$

Morrey proved that (see Theorem 3.5.2, [40])

Lemma 3.3 *If $u \in W^{1,2}_{\mathrm{loc}}(\omega)$ and $\nabla u \in L^{2,\lambda}_{\mathrm{loc}}(\omega)$ for some $0 < \lambda < 2$ then $u \in C^{0,\lambda/2}_{\mathrm{loc}}(\omega)$.*

In light of this result, to prove Theorem 3.1 it suffices to show that if u is a $W^{1,2}$-local minimizer of F then for all $0 \leq \lambda < 2$, with B_ρ, B_R, denoting balls in ω of radii ρ, R, respectively,

$$\int_{B_\rho} |\nabla u|^2 \, dx \leq C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |\nabla u|^2 \, dx + C\rho^\lambda \tag{3.1}$$

for all $0 < \rho < R$ with $B_R \subset\subset \omega$. In turn, this inequality will follow from the proposition below.

Proposition 3.4 *If* $\nabla u \in L^{2,\lambda}_{\mathrm{loc}}(\omega;\mathbb{R}^d)$ *for some* $0 \le \lambda < 2$ *then*

$$\nabla u \in L^{2,q_0(\lambda)}_{\mathrm{loc}}(\omega;\mathbb{R}^d),$$

where $q_0(\lambda) := \alpha + \lambda(1-\alpha/2)$.

In fact, using an iterative scheme where

$$\lambda_0 := 0, \quad \lambda_{k+1} := q_0(\lambda_k),$$

then

$$\lim_{k\to+\infty} \lambda_k = \lim_{k\to+\infty} \alpha \sum_{i=0}^{k} \left(1-\frac{\alpha}{2}\right)^i = 2,$$

and we conclude that (3.1) holds for all $0 \le \lambda < 2$. We remark, however, that the proof of Proposition 3.4 hinges heavily on the higher integrability properties of the functions

$$A := \frac{|D_1 u|^2 - |D_2 u|^2}{2}, \quad B := (D_1 u)\cdot(D_2 u),$$

where $D_1 u$ and $D_2 u$ stand for the column vectors in $\mathbb{R}^d$ of the derivatives of u with respect to x_1 and to x_2, respectively. Precisely,

Lemma 3.5 *The functions* A *and* B *solve the system*

$$\begin{cases} \Delta A &= D^2_{11} g - D^2_{22} g \\ \\ \Delta B &= 2D^2_{12} g, \end{cases}$$

where

$$g := h(|M(\nabla u)|) - |M(\nabla u)|\, h'(|M(\nabla u)|).$$

In addition, if $\nabla u \in L^{2,\lambda}_{\mathrm{loc}}(\omega;\mathbb{R}^{2d})$ *for some* $0 \le \lambda < 2$ *then* $\sqrt{|A|+|B|} \in L^{2,2\alpha+\lambda(1-\alpha)}_{\mathrm{loc}}(\omega;\mathbb{R}^d)$.

We were unable to extend this two-dimensional argument to, say, $N = 3$, i.e., we could not find A, B, C, solving an appropriate system of PDEs, and entailing regularity for local minimizers.

Proof of Proposition 3.4. Fix $\phi \in W^{1,2}_0(\omega;\mathbb{R}^d)$, $k \in \mathbb{N}$, and assume that $\nabla u \in L^{2,\lambda}_{\mathrm{loc}}(\omega;\mathbb{R}^{2d})$ for some $0 \le \lambda < 2$. For $\varepsilon \in \mathbb{R}$ set $u_\varepsilon(x) := u(x) + \varepsilon\phi(x)$.

Local minimality of u entails

$$\limsup_{\varepsilon\to 0^-} \frac{F(u) - F(u_\varepsilon}{\varepsilon} \le 0,$$

which, in turn, yields

$$
\begin{aligned}
(M+1)\int_\omega \nabla u\cdot\nabla\phi\,dx \quad &+\int_\omega G\cdot\nabla\phi\,dx\\
&\le C\int_\omega\sqrt{|A|+|B|}|\nabla\phi|\,dx+\theta_k\int_\omega|\nabla u||\nabla\phi|\,dx,
\end{aligned}
$$

$\theta_k := \sup_{t\ge k}|\Lambda-h'(t)|$, $G=(G_1,G_2)$ and

$$
G_1 := \chi_{\{0<|M(\nabla u)|<k\}}\,(\Lambda-h'(|M(\nabla u)|))\,\frac{M(\nabla u)}{|M(\nabla u)|}\wedge D_2u,
$$

$$
G_2 := \chi_{\{0<|M(\nabla u)|<k\}}\,(h'(|M(\nabla u)|)-\Lambda)\,\frac{M(\nabla u)}{|M(\nabla u)|}\wedge D_1u.
$$

It can be shown that

$$
|G|\le C(k)(1+\sqrt{|A|+|B|})\quad\text{a.e. in }\omega.
$$

Next, for a fixed ball $B_R\subset\subset\omega$ we solve the Dirichlet problem

$$
\begin{cases}
(M+1)\Delta v=\operatorname{div}G \quad \text{in } B_R\\[2ex]
v-u\in W_0^{1,2}(B_R;\mathbb{R}^d).
\end{cases}
$$

We have

$$
(M+1)\int_{B_R}(\nabla u-\nabla v)\cdot\nabla\phi dx\le C\int_{\omega\cap B_R}\sqrt{|A|+|B|}|\nabla\phi|dx+\theta_k\int_{B_R}|\nabla u||\nabla\phi|dx,
$$

so setting $\Phi:=u-v$ and using the fact that

$$
|G|\le C|\nabla u|,\qquad \int_{B_R}|\nabla v|^2\le C\int_{B_R}|\nabla u|^2,
$$

we deduce that

$$
\int_{B_R}|\nabla u-\nabla v|^2\,dx\le C\int_{B_R}(|A|+|B|)\,dx+C\theta_k\int_{B_R}|\nabla u|^2\,dx.
$$

Finally, $\nabla v\in L^{2,q(\lambda)}_{\rm loc}(B_R;\mathbb{R}^{2d})$ and for all $0<\rho\le R$ (see [32], Theorem 3, page 87)

$$
\int_{B_\rho}|\nabla v|^2\,dx\le C\left(\frac{\rho}{R}\right)^{q(\lambda)}\int_{B_R}|\nabla v|^2\,dx+C(k)\rho^{q(\lambda)},
$$

from which we conclude that

$$
\int_{B_\rho}|\nabla u|^2\,dx\le C\left[\left(\frac{\rho}{R}\right)^{q(\lambda)}+\theta_k\right]\int_{B_R}|\nabla u|^2\,dx+C(k)R^{q(\lambda)}.
$$

By (H2) if k is large then θ_k is small, and a simple variant of Lemma 2.1 in Chapter 3 in [32] will now yield

$$\int_{B_\rho} |\nabla u|^2 \, dx \leq C \left(\frac{\rho}{R}\right)^{\lambda'} \int_{B_R} |\nabla u|^2 \, dx + C\rho^{\lambda'},$$

for all $0 < \lambda' < q(\lambda)$, and thus for $\lambda' = q_0(\lambda)$. $\square$

The rest of this section is dedicated to the proof of Lemma 3.5, where we will use the following auxiliary result (see [4]).

Lemma 3.6 *Let $p > 1$ and $0 \leq \lambda < 2$. If $f_{ij} \in L^{p,\lambda}_{\mathrm{loc}}(\omega)$ for $i, j \in \{1, 2\}$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ is a distributional solution of*

$$\Delta u = \sum D_{ij} f_{ij}$$

then $u \in L^{p,\lambda}_{\mathrm{loc}}(\omega)$.

Proof of Lemma 3.5. We consider a variation of the domain. Precisely, let $\Phi := (\varphi, \psi) \in C^1_0(\Omega; \mathbb{R}^2)$, and let $\varepsilon > 0$ be small enough so that, with $\Phi_\varepsilon(x) := x + \varepsilon\Phi(x)$, then $\Phi_\varepsilon : \Omega \to \Omega$ is a smooth diffeomorphism satisfying

$$\det D\Phi_\varepsilon(x) = 1 + \varepsilon \operatorname{div} \Phi(x) + \omega_1(x, \varepsilon),$$

$$\det D\Phi_\varepsilon^{-1}(y) = 1 - \varepsilon \operatorname{div} \Phi(\Phi_\varepsilon^{-1}(y)) + \omega_2(y, \varepsilon),$$

where $\omega_i(\cdot, \varepsilon)/\varepsilon \to 0$, as $\varepsilon \to 0$, uniformly in ω. From the local minimality assumption we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(u_\varepsilon) = 0,$$

i.e.

$$\int_\omega \left[\frac{1}{2}|\nabla u|^2 \operatorname{div} \Phi - \nabla u \nabla\Phi \cdot \nabla u\right] dx$$

$$= \int_\omega [\, |M(\nabla u)| h'(|M(\nabla u)|) - h(|M(\nabla u)|) \,] \operatorname{div} \Phi \, dx.$$

This equation may be rewritten as

$$\int_\omega [A(D_2\psi - D_1\varphi) - B(D_1\psi + D_2\varphi)] \, dx = \int_\omega -g(D_1\varphi + D_2\psi) \, dx,$$

that is,

$$\begin{cases} D_1 A + D_2 B &= D_1 g \\ D_2 A - D_1 B &= -D_2 g \end{cases}$$

and the first assertion follows. By (H3)

$$|g| \leq C(1 + |M(\nabla u)|^{1-\alpha})$$

and so, assuming that $\nabla u \in L^{2,\lambda}_{\mathrm{loc}}(\omega; \mathbb{R}^{2d})$, we have that $|M(\nabla u)| \in L^{1,\lambda}_{\mathrm{loc}}(\Omega; \mathbb{R})$ and

$$g \in L^{\frac{1}{1-\alpha},\lambda}_{\mathrm{loc}}(\omega).$$

We may now use Lemma 3.6 to obtain that

$$A, B \in L^{\frac{1}{1-\alpha},\lambda}_{\mathrm{loc}}(\omega),$$

and by Hölder inequality we conclude that

$$\sqrt{|A| + |B|} \in L^{2,2\alpha+\lambda(1-\alpha)}_{\mathrm{loc}}(\omega).$$

$\square$

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