Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

On a volume constrained variational problem

by

Luigi Ambrosio, Irene Fonseca, Paolo Marcellini and Luc Tartar

Preprint-Nr.: 31 1998

On a Volume Constrained Variational Problem

Luigi Ambrosio [*], Irene Fonseca [†], Paolo Marcellini [‡], Luc Tartar [§]

July 21, 1998

Abstract Existence of minimizers for a volume constrained energy

$$E(u) := \int_\Omega W(\nabla u)\,dx$$

where $\mathcal{L}^N(\{u = z_i\}) = \alpha_i, i = 1, \dots, P$, is proved in the case where z_i are extremal points of a compact, convex set in $\mathbb{R}^d$ and under suitable assumptions on a class of quasiconvex energy densities W. Optimality properties are studied in the scalar-valued problem where $d = 1$, $P = 2$, $W(\xi) = |\xi|^2$, and the Γ-limit as the sum of the measures of the 2 phases tends to $\mathcal{L}^N(\Omega)$ is identified. Minimizers are fully characterized when $N = 1$, and candidates for solutions are studied for the circle and the square in the plane.

1991 Mathematics subject classification (Amer. Math. Soc.): 35A15, 35J65, 49J45, 49K20
Key Words : volume constraints, free boundary problems

1 Introduction

In recent years there has been a remarkable development of techniques in applied analysis motivated in part by questions arising in the study of materials. Some of the underlying mathematical problems lie at the boundary of classical analytical methods, requiring new ideas and the introduction of innovative tools. In this paper we treat a seemingly simple constrained variational problem which falls outside the usual techniques of the Calculus of Variations for proving existence of minimizers.

In 1992 Morton Gurtin, motivated by a problem related to the interface between immiscible fluids [10], suggested that we study existence of minimizers and possible optimal designs for the energy

$$I(u) := \int_\Omega |\nabla u|^2\,dx$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected Lipschitz domain, and $u : \Omega \to \mathbb{R}$ is subjected to the volume constraints

$$\mathcal{L}^N(\{u = 0\}) = \alpha \quad \text{and} \quad \mathcal{L}^N(\{u = 1\}) = \beta. \tag{1.1}$$

Here $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$ and $\alpha, \beta > 0$ satisfy $\alpha + \beta < \mathcal{L}^N(\Omega)$.

Previous works by Alt and Caffarelli [3] and Aguilera, Alt and Caffarelli [2] address a similar problem where only one volume constraint is present and Dirichlet boundary conditions are imposed on u. They obtain existence of minimizers for I and regularity properties for solutions as well as their free boundaries. In our context, and in the presence of two or more constraints, a priori continuity of minimizers would imply separability of the phases $\{u = 0\}$ and $\{u = 1\}$, thus enabling us to use

*Dipartimento di Matematica “F.Casorati”, Università di Pavia, Via Abbiategrasso 205, 27100 Pavia, Italy
†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
‡Dipartimento di Matematica “U. Dini”, Università di Firenze, Viale Morgagni 67/a, 50131 Firenze, Italy
§Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA

their arguments and obtain additional regularity for u and the free boundaries. However, continuity had not been established during the making of this paper, and this seriously limited the choice of variations and required the introduction of analytical methods specific to the multi-phase framework. We must point out that since this work has been completed Tilli (see [17]) pursued it a step further proving locally Lipschitz continuity of minimizers of I.

In this paper we obtain existence of minimizers for I subjected to (1.1). More generally, in Theorem 2.3 we prove existence of solutions of

$$\min\left\{\int_\Omega W(\nabla u)\,dx : u \in W^{1,p}(\Omega;\mathbb{R}^d),\ \mathcal{L}^N(\{u = z_i\}) = \alpha_i,\ i = 1,\dots,P\right\}$$

where $\{z_1,\dots,z_P\}$ are extremal points of a compact, convex set $K \subset \mathbb{R}^d$, with $P \geq 1$, $\alpha_i > 0$ and $\sum_i \alpha_i < \mathcal{L}^N(\Omega)$, provided $W : \mathbb{R}^{d\times N} \to [0,+\infty)$ is a C^1 quasiconvex function with p-growth, $p > 1$, satisfying the structure condition

$$\sum_{i,j=1}^{d}\sum_{k=1}^{N}\frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{jk}\nu^i\nu^j > 0 \quad \text{whenever } \xi^T\nu \neq 0, \xi \in \mathbb{R}^{d\times N}, \nu \in S^{d-1} \tag{1.2}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$. A characterization of (1.2) in terms of the behavior of W along rank-one lines can be found in Remark 2.4(ii). Certain isotropic energy densities, such as functions of the type $W(\xi) = g(|\xi|, |\mathrm{adj}\,\xi|, |\det\xi|)$ verify (1.2) (see Proposition 2.5).

In Section 4, using some optimality properties of minimizers obtained in Section 3 for $W(\xi) := |\xi|^2$, we characterize the asymptotic behavior of minimizers of I subjected to (1.1) as $\alpha \to \mathcal{L}^N(\Omega) - \gamma$ and $\beta \to \gamma$, with $\gamma \in (0, \mathcal{L}^N(\Omega))$. Precisely, we show that the limiting configurations satisfy the constrained least area problem

$$p_\gamma := \min\left\{P_\Omega(E)\ :\ E \subset \Omega,\ \mathcal{L}^N(E) = \gamma\right\}$$

where $P_\Omega(E)$ denotes, as usual, the perimeter of E in Ω. Similar results have been obtained for phase transitions problems where the formation of phases is driven by a double well potential (see [13], [5], [8]), while here the creation of interfaces is due to the volume constraints.

In Section 5 we characterize fully the solutions of (M) when $W(\xi) = |\xi|^2$, Ω is an interval and $d > 1$ (see Subsection 5.1). Explicit solutions are unknown when $\Omega \subset \mathbb{R}^N$ and $N > 1$. We study the cases where Ω is a circle or a square on the plane. If Ω is a circle, then on Subsection 5.2 we determine the minimum energy among radial configurations, and we construct a family of competing configurations u_{ab} with energy strictly lower than the energy for radial functions if $\alpha + \beta << 1$. However, u_{ab} are not solutions either, as they violate some of the optimality conditions obtained in Section 3. We remark that due to Theorem 4.1, if $\alpha + \beta \to \mathcal{L}^2(\Omega)$ then radial configurations will still not be minimizers for (M). Finally, in Subsection 5.3 we address briefly the case where $\Omega = (0,1)^2$, and we show that although the piecewise affine configurations of the form

$$u(x) = \begin{cases} 0 & \text{if } x_1 \leq \alpha, \\ \frac{1}{1-\alpha-\beta}x_1 - \frac{\alpha}{1-\alpha-\beta} & \text{if } \alpha < x_1 < 1-\beta, \\ 1 & \text{if } x_1 \geq 1-\beta \end{cases}$$

satisfy the optimality conditions, they are not minimizers for (M) if $\alpha + \beta << 1$, and, once again by virtue of Theorem 4.1, they will not have least energy when $\alpha + \beta$ approaches the measure of the unit square.

2 Existence

Let us first fix some notation. In the sequel $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$, H^{N-1} is the $(N-1)$-dimensional Hausdorff measure, $\mathcal{M}$ is the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}^d$ and χ_A stands for the characteristic function of a set A. We denote by $\mathbb{R}^{d\times N}$ the vector space of $d \times N$ matrices ξ (d rows, N columns) with components ξ_{ij}, $1 \le i \le d$, $1 \le j \le N$. Finally, Ω is an open, bounded, connected Lipschitz domain, $C_c^k(\Omega)$ is the space of k-differentiable functions with compact support in Ω, $k \in \mathbb{N} \cup \{+\infty\}$.

Proposition 2.1. *For any sequence $(u_h) \subset \mathcal{M}$ converging a.e. to $u \in \mathcal{M}$ and for any closed set $C \subset \mathbb{R}^d$ we have*

$$\mathcal{L}^N(\{x \in \Omega : u(x) \in C\}) \ge \limsup_{n\to+\infty} \mathcal{L}^N(\{x \in \Omega : u_n(x) \in C\}).$$

PROOF. Since $A := \mathbb{R}^d \setminus C$ is open we have

$$\chi_A(u) \le \liminf_{n\to+\infty} \chi_A(u_n) \qquad \text{a.e. in } \Omega.$$

By Fatou's Lemma we have

$$\begin{aligned} \mathcal{L}^N(\{x \in \Omega \,:\, u(x) \in A\}) &= \int_\Omega \chi_A(u)\,dx \le \int_\Omega \liminf_{n\to+\infty} \chi_A(u_n)\,dx \\ &\le \liminf_{n\to+\infty} \int_\Omega \chi_A(u_n)\,dx = \liminf_{n\to+\infty} \mathcal{L}^N(\{x \in \Omega \,:\, u_n(x) \in A\}). \end{aligned}$$

The statement now follows by passing to the complementary sets. □

Note that since any $L^p(\Omega)$-converging sequence has subsequences which converge almost everywhere, the upper semicontinuity property asserted in Proposition 2.1 is still valid with respect to $L^p(\Omega)$-convergence.

Let us consider a finite collection of points $\{z_1, \dots, z_P\}$ in $\mathbb{R}^d$, with $P \ge 1$. In this section we obtain the existence of solutions for the minimization problem

$$\text{(M)} \qquad \min\left\{\int_\Omega W(\nabla u)\,dx : u \in W^{1,p}(\Omega;\mathbb{R}^d),\ \mathcal{L}^N(\{u = z_i\}) = \alpha_i,\ i = 1,\dots,P\right\}$$

where $p > 1$, $\alpha_i > 0$ and $\sum_i \alpha_i < \mathcal{L}^N(\Omega)$, under some technical assumptions on W.

We first find conditions ensuring that the relaxed problem

$$\text{(M)}^* \qquad \min\left\{\int_\Omega W(\nabla u)\,dx : u \in W^{1,p}(\Omega;\mathbb{R}^d),\ \mathcal{L}^N(\{u = z_i\}) \ge \alpha_i,\ i = 1,\dots,P\right\}$$

has a solution.

Proposition 2.2. *Assume that $W : \mathbb{R}^{d\times N} \to [0,\infty)$ is a quasiconvex function satisfying*

$$c|\xi|^p \le W(\xi) \le C(|\xi|^p + 1) \qquad \forall \xi \in \mathbb{R}^{d\times N} \tag{2.1}$$

for some constants $c, C > 0$ and some $p \in (1,+\infty)$. Then problem (M)* has at least one solution.

PROOF. It is easy to check that the class of competing functions in (M)* is not empty. Let (u_h) be a minimizing sequence for the problem and denote by $\bar{u}_h$ the average of u_h on Ω. By Poincaré's inequality and Rellich's Theorem, without loss of generality we may assume that the functions $v_h = u_h - \bar{u}_h$ converge in $L^p(\Omega)$ to some function v. As

$$\mathcal{L}^N(\{u_h = P_1\})\bar{u}_h = \int_{\{u_h = P_1\}} \bar{u}_h\,dx = \int_{\{u_h = P_1\}} (P_1 - v_h)\,dx,$$

we conclude that $(\bar{u}_h)$ is bounded and hence, extracting if necessary another subsequence, the functions (u_h) are converging in $L^p(\Omega)$ to some function $u \in W^{1,p}(\Omega;\mathbb{R}^d)$. By Proposition 2.1 the function u satisfies the constraints of (M)*, and due to the growth condition (2.1) it follows by the lower semicontinuity theorem of Acerbi and Fusco (see [1]) that

$$\int_\Omega W(\nabla u)\,dx \le \liminf_{h\to\infty} \int_\Omega W(\nabla u_h)\,dx.$$

This proves that u is a solution of (M)*. ❑

Note that the previous argument may be carried through when the upper bound on W in (2.1) is replaced by the weaker assumption that $u \mapsto \int_\Omega W(\nabla u)$ is lower semicontinuous in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^d)$.

Next we find conditions on W which ensure that any solution of (M)* actually solves (M).

Theorem 2.3. *Let u be a solution of (M)* and assume that*

(i) W is differentiable, satisfies

$$\sum_{i=1}^d \sum_{k=1}^N \left| \frac{\partial W}{\partial \xi_{ik}} \right| \le C(1+|\xi|^{p-1}) \tag{2.2}$$

for some $C>0$ and all $\xi \in \mathbb{R}^{d\times N}$, and

$$\sum_{i,j=1}^d \sum_{k=1}^N \frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{jk}\nu^i\nu^j > 0 \quad \textit{whenever } \xi^T\nu \neq 0, \xi \in \mathbb{R}^{d\times N}, \nu \in S^{d-1} \tag{2.3}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$.

(ii) $z_1, \dots, z_P$ are extremal points of a compact convex set K.

Then u is a solution of (M).

Proof. We have to prove that

$$\mathcal{L}^N(\{u = z_i\}) = \gamma_i, \quad \text{for all } i = 1, \dots, P.$$

Suppose that $\mathcal{L}^N(\{u=z_1\}) > \alpha_1$. Let $r>0$ be such that $\mathcal{L}^N(\{u=z_1\}) - r > \alpha_1$, and consider a smooth cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N;[0,1])$ such that $\mathcal{L}^N(\operatorname{supp}\varphi) < r$. Without loss of generality, we may assume that the extremal point z_1 is the origin, and let $\nu \in S^{d-1}$ be such that $K \setminus z_1 \subset \{y \in \mathbb{R}^d \,:\, y\cdot\nu > 0\}$. Let $0<\delta<\min\{z_i\cdot\nu \,:\, i=2,\dots,P\}$, and define $f:\mathbb{R}\to[0,+\infty)$ as

$$f(t) := \begin{cases} -t+\delta & \text{if} \quad t \le \delta \\ 0 & \text{otherwise.} \end{cases}$$

Set $w := u\cdot\nu$, and consider the perturbations $u_\varepsilon := u + \varepsilon\varphi f(w)\nu$. If $i=2,\dots,P$, and if $u(x)=z_i$, then $w(x)>\delta$ and $f(w(x))=0$, so that $u_\varepsilon(x) = u(x)$. Therefore

$$\{u_\varepsilon = z_i\} \supset \{u = z_i\}.$$

On the other hand,

$$\mathcal{L}^N(\{u_\varepsilon = z_1\}) \ge \mathcal{L}^N(\{u=z_1\}) - \mathcal{L}^N(\operatorname{supp}\varphi) > \mathcal{L}^N(\{u=z_1\}) - r > \alpha_1,$$

and we conclude that u_ε is admissible for (M)*. Thus, taking into account the growth assumption (2.2) and by virtue of Lebesgue's Dominated Convergence Theorem, we can differentiate under the integral to find

$$\begin{aligned} 0 &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega W\left(\nabla u + \varepsilon \nabla\left[\varphi f(w)\nu\right]|\right) dx \\ &= \sum_{i=1}^{d}\sum_{k=1}^{N} \int_\Omega \frac{\partial W}{\partial \xi_{ik}}(\nabla u)\left[\frac{\partial \varphi}{\partial x_k} f(w)\nu^i + \varphi\, f'(w)\nu^i \sum_{j=1}^{d} \nu^j \frac{\partial u^j}{\partial x_k}\right] dx. \end{aligned} \tag{2.4}$$

Using a partition of unity, it is easy to see that any smooth function with compact support may be written as a finite sum of cut-off functions φ as above, each one of which with small support, so we may consider $\varphi = 1$ in Ω, and (2.4) reduces to

$$\int_{\{w\in(0,\delta)\}} \sum_{i,j=1}^{d}\sum_{k=1}^{N} \frac{\partial W}{\partial \xi_{ik}}(\nabla u)\frac{\partial u^j}{\partial x_k}\nu^i\nu^j\, dx = 0.$$

By (2.3) we deduce that $\nabla w = \nabla u^T \nu = 0$ a.e. on $\{0 < w < \delta\}$, hence the function $\max\{0, \min\{w, \delta\}\}$ is constant in Ω. On the other hand, $\mathcal{L}^N(\{w = 0\}) \geq \mathcal{L}^N(\{u = z_1\}) > 0$ and $\mathcal{L}^N(\{w > \delta\}) \geq \mathcal{L}^N(\{u = z_2\}) > 0$. We have reached a contradiction, and we may conclude now that $\mathcal{L}^N(\{u = z_1\}) = \alpha_1$. $\square$

Remark 2.4. (i) Any differentiable quasiconvex function satisfying the growth condition (2.1) verifies also (2.2) (see [12]).
(ii) In the scalar-valued case where $d = 1$, quasiconvexity reduces to convexity, and the condition (2.3) may be rewritten as

$$\sum_{i=1}^{N} \frac{\partial W}{\partial \xi_i}(\xi)\xi_i > 0 \qquad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Since W is convex, this is equivalent to saying that W has a strict minimum at the origin. More generally, if $W : \mathbb{R}^{d\times N} \to \mathbb{R}$ is a C^1 rank-one convex function then (2.3) holds if and only if

$$t \mapsto W(A + t\nu \otimes \mu) \quad \text{has a strict minimum at } t = 0 \tag{2.5}$$

whenever $\mu \in \mathbb{R}\backslash\{0\}$, $A \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$ and $A^T\nu = 0$. Note that in Proposition 2.2 W is assumed to be quasiconvex, and, consequently, it is rank-one convex. In order to prove the equivalence between (2.3) and (2.5), assume first that $\mu \in \mathbb{R} \setminus \{0\}$, $A \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$, $A^T\nu = 0$, and set

$$\psi(t) := W(A + t\nu \otimes \mu).$$

Since ψ is convex and C^1, ψ has a strict minimum at the origin if and only if $\operatorname{sign}\psi'(t) = \operatorname{sign} t$ for $t \neq 0$. As

$$\sum_{j=1}^{d}(A + t\nu \otimes \mu)_{jk}\nu^j = t\mu_k, \tag{2.6}$$

we have for $t \neq 0$

$$\begin{aligned} \psi'(t) &= \sum_{k=1}^{N}\sum_{i=1}^{d} \frac{\partial W}{\partial \xi_{ik}}(A + t\nu \otimes \mu)\nu^i\mu_k \\ &= \frac{1}{t}\sum_{k=1}^{N}\sum_{i,j=1}^{d} \frac{\partial W}{\partial \xi_{ik}}(A + t\nu \otimes \mu)(A + t\nu \otimes \mu)_{jk}\nu^i\nu^j. \end{aligned} \tag{2.7}$$

It follows from (2.6) that

$$(A + t\nu \otimes \mu)^T \nu = t\mu \neq 0$$

which, together with (2.3) and (2.7), yields

$$\operatorname{sign} \psi'(t) = \operatorname{sign} t.$$

Conversely, if for some $\xi \in \mathbb{R}^{d\times N}$, $\nu \in S^{d-1}$ such that $\xi^T \nu \neq 0$ (2.3) was violated, setting

$$\psi(t) := W(A + t\nu \otimes \xi^T \nu), \quad A := \xi - \nu \otimes \xi^T \nu,$$

then $A^T \nu = 0$,

$$\psi'(1) = \sum_{i,j=1}^{d} \sum_{k=1}^{N} \frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{jk}\nu^i \nu^j \leq 0,$$

and this is in contradiction with (2.5).
(iii) Note that the function $W(\xi) = |\xi|^2$, corresponding to the Dirichlet integral, satisfies all the hypotheses of Theorem 2.3. In this case a simple truncation argument proves that any solution u of (M) satisfies

$$\min\{z_1, z_2\} \leq u \leq \max\{z_1, z_2\}.$$

More generally, in the isotropic (scalar or vectorial) case where $W(\xi) = \Phi(|\xi|)$, the assumption (2.3) reduces to $\Phi'(t) > 0$ for $t > 0$, and it can be shown that any solution u of (M)* takes its values in the closed convex hull K of $\{z_1, \dots, z_P\}$. This follows by comparing u with $\Pi(u)$, where $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ is the orthogonal projection onto a half-space containing all points z_i. Precisely, set

$$\gamma := \inf \left\{ \int_\Omega \Phi(|\nabla u|)\, dx : u \in W^{1,p}(\Omega; K),\, \mathcal{L}^N(\{u = z_i\}) \geq \alpha_i,\, i = 1, \dots, P \right\}.$$

We claim that if an admissible u for (M)* takes values outside K then we may modify it so as to decrease its energy. In fact, if $\mathcal{L}^N(\{x \in \Omega : u(x) \notin K\}) > 0$ then there exists a hyperplane H with normal $\nu \in S^{d-1}$ such that K is contained in one of the half-spaces determined by H, and the other half-space contains a subset E of the range of u with $\mathcal{L}^N(u^{-1}(E)) > 0$. Without loss of generality, we may assume that

$$H := \{y \in \mathbb{R}^d \, : \, y \cdot \nu = 0\}, \quad K \subset \{y \in \mathbb{R}^d \, : \, y \cdot \nu \leq 0\},$$

and that there exists $\delta > 0$ such that

$$\mathcal{L}^N(\{x \in \Omega \, : \, u(x) \cdot \nu > \delta\}) > 0.$$

Let $\{\eta_1, \dots, \eta_{d-1}, \nu\}$ be an orthonormal basis of $\mathbb{R}^d$, and define

$$\Pi(u)(x) := \sum_{i=1}^{d-1} (u(x) \cdot \eta_i)\, \eta_i + f(x)\, \nu$$

where

$$f(x) := \begin{cases} u(x) \cdot \nu & \text{if } u(x) \cdot \nu \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, if $u(x) \in K$ then $\Pi(u)(x) = u(x)$, so $\Pi(u)$ is still admissible. Also $|\nabla \Pi(u)(x)| \leq |\nabla u(x)|$ for a.e. $x \in \Omega$, and

$$|\nabla \Pi(u)(x)| < |\nabla u(x)| \text{ on a set of positive measure.} \tag{2.8}$$

In fact, if $|\nabla\Pi(u)(x)| = |\nabla u(x)|$ a.e. in Ω then $\nabla u \cdot \nu = 0$ a.e. on $\{u \cdot \nu > 0\}$, and so the Sobolev function $v := \max\{(u \cdot \nu), 0\}$ would be constant, in contrast with the conditions

$$\mathcal{L}^N(\{v = 0\}) \geq \mathcal{L}^N(\{u = z_1\}) \geq \alpha_1 > 0, \;\; \mathcal{L}^N(\{v \geq \delta\}) > 0.$$

Since Φ is strictly increasing, by (2.8) we have

$$\int_\Omega \Phi(|\nabla\Pi(u)(x)|)\, dx < \int_\Omega \Phi(|\nabla u(x)|)\, dx.$$

As K is the intersection of a countable family of half-spaces, an iteration of this argument proves that for any u admissible for (M)* there exists a function $\bar{u}$ still admissible for (M)*, with values in K and with smaller energy. Thus every solution of (M)* takes its values on K.
(iv) We do not know whether solutions of (M)* are solutions of (M) if the points z_i are not extremal, even when $d = 1$, $W(\xi) = |\xi|^2$, and there are three or more phases. However, in this particular case it can be easily proved that any *continuous* solution of (M)* is actually a solution of (M). Indeed, if for instance $\mathcal{L}^N(\{u = z_1\}) > \alpha_1$ then we can make local additive variations to obtain that each component of u is harmonic in the open set

$$\left\{x \in \Omega \,:\, u(x) \in \mathbb{R}^d \setminus \{z_2, z_3, \ldots, z_P\}\right\}.$$

This obviously contradicts the fact that $\mathcal{L}^N(\{u = z_1\}) > 0$.

Next we exhibit a class of isotropic energy densities W which satisfy (2.3). We recall that W is *isotropic* if it can be written as

$$W(\xi) = \varphi(\lambda_1(\xi), \ldots, \lambda_N(\xi))$$

for some function φ of the vector of *principal stretches* $(\lambda_1(\xi), \ldots, \lambda_N(\xi))$, where $0 \leq \lambda_1(\xi) \leq \lambda_2(\xi) \leq \ldots \leq \lambda_N(\xi)$, and $\{\lambda_i^2(\xi) : i = 1, \ldots, N\}$ are the eigenvalues of $\xi^T\xi$.

Proposition 2.5. *Let $W : \mathbb{R}^{N\times N} \to \mathbb{R}$ be given by*

$$W(\xi) = \varphi(\lambda_1(\xi), \ldots, \lambda_N(\xi)), \quad \xi \in \mathbb{R}^{N\times N},$$

where $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a symmetric C^1 function such that for every $i = 1, \ldots, N$

$$\frac{\partial\varphi}{\partial\lambda_i}(\lambda) > 0 \quad \textit{whenever } \lambda_i > 0 \;\; \textit{and } \lambda_j \geq 0 \textit{ for all } j = 1, \ldots, N, \; j \neq i.$$

Then W satisfies (2.3).

PROOF. Consider first a matrix $\xi \in \mathbb{R}^{N\times N}$ such that

$$\xi^T \xi\, e_i = \lambda_i^2(\xi)\, e_i, \; i = 1, \ldots, N, \; 0 < \lambda_1(\xi) < \ldots < \lambda_N(\xi),$$

and $\{e_1, \ldots, e_N\}$ is an orthonormal basis of $\mathbb{R}^N$. Fix $B \in \mathbb{R}^{N\times N}$. If t is small enough then

$$0 < \lambda_1(\xi + tB) < \ldots < \lambda_N(\xi + tB),$$

$$\lambda_i(\xi + tB) \to \lambda_i(\xi) \text{ as } t \to 0,$$

and

$$(\xi + tB)^T(\xi + tB)\, e_i(t) = \lambda_i^2(\xi + tB)\, e_i(t), \tag{2.9}$$

where $e_i(t) \to e_i$ as $t \to 0$, and $|e_i(t)| = 1$. Differentiating (2.9) with respect to t, making the inner product of the resulting equation with $e_i(t)$, and using the fact that $e_i(t) \cdot \frac{d}{dt} e_i(t) = 0$, we obtain at $t = 0$

$$Be_i \cdot \xi e_i = \lambda_i(\xi) \frac{d}{dt} \lambda_i(\xi + tB).$$

In the case where $B := \nu \otimes \xi^T \nu$, since $Be_i \cdot \xi e_i = (\xi^T \nu \cdot e_i)^2$, we conclude that

$$\frac{d}{dt} \lambda_i(\xi + tB) = \frac{1}{\lambda_i(\xi)} (\xi^T \nu \cdot e_i)^2. \tag{2.10}$$

We are now in a position to prove (2.3). Let $\xi \in \mathbb{R}^{N \times N}, \nu \in S^{N-1}$, be such that $\xi^T \nu \neq 0$. Writing

$$\xi^T \xi e_i = \lambda_i^2(\xi) e_i$$

for a suitable orthonormal basis of $\mathbb{R}^N$, $\{e_1, \ldots, e_N\}$, then there is $j \in \{1, \ldots, N\}$ such that $\xi^T \nu \cdot e_j \neq 0$, and so $\lambda_j > 0$. Construct a sequence of matrices ξ^n such that $\xi^n \to \xi$ as $n \to +\infty$,

$$(\xi^n)^T \xi^n e_i = \lambda_i^2(\xi^n) e_i, \; 0 < \lambda_1(\xi^n) < \ldots < \lambda_N(\xi^n).$$

Using (2.10), we conclude that

$$\begin{aligned}
\sum_{i,j=1}^{d} \sum_{k=1}^{N} \frac{\partial W}{\partial \xi_{ik}}(\xi) \xi_{jk} \nu^i \nu^j &= \lim_{n \to \infty} \sum_{i,j=1}^{d} \sum_{k=1}^{N} \frac{\partial W}{\partial \xi_{ik}}(\xi^n) \xi_{jk}^n \nu^i \nu^j \\
&= \lim_{n \to \infty} \frac{d}{dt}\bigg|_{t=0} W(\xi^n + t\nu \otimes (\xi^n)^T \nu) \\
&= \lim_{n \to \infty} \sum_{i=1}^{N} \frac{\partial \varphi}{\partial \lambda_i}(\lambda_1(\xi^n), \ldots, \lambda_N(\xi^n)) \frac{1}{\lambda_i(\xi^n)} ((\xi^n)^T \nu \cdot e_i)^2 \\
&\geq \limsup_{n \to \infty} \frac{\partial \varphi}{\partial \lambda_j}(\lambda_1(\xi^n), \ldots, \lambda_N(\xi^n)) \frac{1}{\lambda_j(\xi^n)} ((\xi^n)^T \nu \cdot e_j)^2 \\
&= \frac{\partial \varphi}{\partial \lambda_j}(\lambda_1(\xi), \ldots, \lambda_N(\xi)) \frac{1}{\lambda_j} (\xi^T \nu \cdot e_i)^2 > 0.
\end{aligned}$$

□

A simple class of polyconvex functions satisfying the hypotheses of Proposition 2.5 is formed by energy densities of the type

$$W(\xi) = g(|\xi|, |\mathrm{adj}\, \xi|, |\det \xi|)$$

where $g(\eta, \mu, \lambda)$ is a C^1 convex function on $[0, +\infty)^3$ such that

$$\frac{\partial g}{\partial \mu}(\eta, \mu, \lambda) \geq 0, \; \frac{\partial g}{\partial \lambda}(\eta, \mu, \lambda) \geq 0, \text{ and } \frac{\partial g}{\partial \eta}(\eta, \mu, \lambda) > 0 \qquad \text{for all } (\eta, \mu, \lambda) \text{ with } \eta > 0.$$

Here $\det \xi$ denotes the determinant of the $N \times N$ matrix ξ, and $\mathrm{adj}\, \xi$ is the adjugate of the matrix ξ, i.e. the matrix of the minors of order $N - 1$ with the property

$$(\mathrm{adj}\, \xi)^T \xi = \xi^T \mathrm{adj}\, \xi = \det \xi \, \mathbb{I}. \tag{2.11}$$

Using the same terminology for the principal stretches as above, we rewrite

$$W(\xi) = \varphi(\lambda_1(\xi), \ldots, \lambda_N(\xi))$$

where

$$\varphi(\lambda_1, \ldots, \lambda_N) := g\left(\sqrt{\sum_{i=1}^{N} \lambda_i^2}, \; \sqrt{\sum_{i=1}^{N} \lambda_1^2 \ldots \lambda_{i-1}^2 \lambda_{i+1}^2, \ldots \lambda_N^2}, \; \lambda_1 \ldots \lambda_N \right)$$

with $\lambda_{-1}, \lambda_{N+1} := 1$.

3 Optimality Properties of the Solutions

As it was shown in the previous section (see Theorem 2.3, Remark 2.4), the problem

$$\text{(M)} \qquad \min\left\{\int_\Omega |\nabla u|^2\,dx : u \in W^{1,1}(\Omega),\ \mathcal{L}^N(\{u=0\}) = \alpha,\ \mathcal{L}^N(\{u=1\}) = \beta\right\}$$

admits solutions, and any solution belongs to $u \in W^{1,2}(\Omega;[0,1])$. Below we study some optimality properties of these solutions.

Theorem 3.1. *Let $u \in W^{1,2}(\Omega;[0,1])$ be a solution of* (M). *Then*
(i)

$$\int_\Omega \varphi f'(u)|\nabla u|^2 + f(u)\nabla\varphi\cdot\nabla u\,dx = 0$$

for all $\varphi \in C^1(\overline{\Omega})$ and all $f \in W^{1,\infty}(\Omega)$ with $f(0) = f(1) = 0$;
(ii)

$$\int_\Omega |\nabla u|^2\, g(u)\,dx = \left(\int_\Omega |\nabla u|^2\,dx\right)\left(\int_0^1 g(s)\,ds\right)$$

for all $g \in L^\infty(\mathbb{R})$;
(iii) *Δu is a signed Radon measure in Ω with support contained in $\overline{\{u=0\}} \cup \overline{\{u=1\}}$, and*

$$|\Delta u|(\Omega) \le 2\int_\Omega |\nabla u|^2\,dx.$$

Moreover,

$$\langle \Delta u, \phi\rangle = \lim_{n\to+\infty} n\int_{\{u<1/n\}} \phi\,|\nabla u|^2\,dx - n\int_{\{u>1-1/n\}} \phi\,|\nabla u|^2\,dx$$

for every $\phi \in C_c(\Omega)$;
(iv) *if $F \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ satisfies* $\operatorname{div} F = 0$ *then*

$$\sum_{i,j=1}^N \int_\Omega \frac{\partial F_i}{\partial x_j}\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,dx = 0. \tag{3.1}$$

Proof. By Theorem 2.3 and Remark 2.4 ii) we know that any solution u of (M) belongs to $W^{1,2}(\Omega;[0,1])$. Taking φ and f under the assumptions of part (i), it is clear that

$$\{u=1\} \subset \{u+\varepsilon\varphi f(u) = 1\} \quad \text{and} \quad \{u=0\} \subset \{u+\varepsilon\varphi f(u) = 0\}.$$

Therefore $u+\varepsilon\varphi f(u)$ is admissible for $(\mathrm{M})^*$, and in light of Remark 2.4 ii),

$$\begin{aligned} 0 &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |\nabla(u+\epsilon\varphi f(u))|^2\,dx \\ &= 2\int_\Omega \varphi f'(u)|\nabla u|^2 + f(u)\nabla\varphi\cdot\nabla u\,dx, \end{aligned}$$

proving (i). Part (ii) follows immediately from (i) setting

$$\varphi \equiv 1, \quad f(t) := \int_0^t g(s)\,ds - t\int_0^1 g(s)\,ds.$$

To obtain (iii), consider the piecewise affine functions

$$f_n(t) := \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le t \le 1-\frac{1}{n} \\ -nt+n & \text{if } 1-\frac{1}{n} \le t \le 1. \end{cases}$$

By (i) we have for all $\varphi \in C_c^1(\Omega)$

$$\begin{aligned}\langle \Delta u, \varphi\rangle &= -\int_\Omega \nabla u \cdot \nabla\varphi \, dx = -\lim_{n\to+\infty}\int_\Omega \nabla u\cdot\nabla\varphi\, f_n(u)\,dx \\ &= -\lim_{n\to+\infty}\int_\Omega |\nabla u|^2\varphi\, f_n'(u)\,dx = -\lim_{n\to+\infty}\langle \mu_n,\varphi\rangle,\end{aligned}$$

where the finite Radon measures μ_n are defined as

$$\mu_n := |\nabla u|^2\, f_n'(u)\,\mathcal{L}^N\lfloor\Omega.$$

By (ii) we have

$$|\mu_n|(\Omega) = \int_\Omega |\nabla u|^2\,|f_n'(u)|\,dx = \left(\int_\Omega |\nabla u|^2\,dx\right)\left(\int_0^1 |f_n'(s)|\,ds\right) \le 2\int_\Omega |\nabla u|^2\,dx,$$

thus there exists a Radon measure μ such that, up to a subsequence,

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |\mu(\Omega)| \le 2\int_\Omega |\nabla u|^2\,dx.$$

We conclude that

$$\Delta u = -\mu = \text{weak-*}\lim_{n\to+\infty} -n|\nabla u|^2\mathcal{L}^N\lfloor\{u<1/n\} + n|\nabla u|^2\mathcal{L}^N\lfloor\{u>1-1/n\}.$$

Finally, let F be a Lipschitz mapping on Ω, with $F=0$ on $\partial\Omega$, and such that div $F=0$. Consider the flow

$$\begin{cases} \dfrac{dw}{dt}(t,x) &= F\left(w(t,x)\right) \\ w(0,x) &= x \end{cases} \qquad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

It is well known that

$$\det \nabla_x w(t,x) = 1. \tag{3.2}$$

Indeed, using (2.11) we have

$$\begin{aligned} N\frac{d}{dt}\det\nabla_x w(t,x) &= \operatorname{adj}\nabla_x w(t,x)\cdot\frac{d}{dt}\nabla_x w(t,x) = \operatorname{adj}\nabla_x w(t,x)\cdot\nabla_x(F(w(t,x))) \\ &= \operatorname{adj}\nabla_x w(t,x)\nabla_x w^T(t,x)\cdot\nabla F(w(t,x)) = \det\nabla_x w(t,x)\mathbb{I}\cdot\nabla F(w(t,x)) \\ &= \det\nabla_x w(t,x)\,\operatorname{div}F(w(t,x)) = 0.\end{aligned}$$

Therefore

$$\det\nabla_x w(t,x) = \det\nabla_x w(0,x) = 1.$$

Define

$$u_\varepsilon(x) := u(w_\varepsilon(x)),\ \text{ where } w_\varepsilon(x) := w(\varepsilon,x).$$

These functions satisfy the volume constraints of (M) because by (3.2)

$$\mathcal{L}^N(\{u_\varepsilon=0\}) = \int_{\{u=0\}} \det\nabla w_\varepsilon(x)\,dx = \mathcal{L}^N(\{u=0\}),$$

and, similarly, $\mathcal{L}^N(\{u_\varepsilon = 1\}) = \mathcal{L}^N(\{u = 1\})$. If $u \in C^2$ then

$$\begin{aligned} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega |\nabla(u \circ w_\varepsilon)|^2\, dx &= 2\int_\Omega \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{dw_{\varepsilon\, j}}{d\varepsilon} \frac{\partial u}{\partial x_i}\, dx = 2\int_\Omega \sum_{i,j=1}^N F_j \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i}\, dx \\ &= -2\int_\Omega \sum_{i,j=1}^N \frac{\partial F_j}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i}\, dx. \end{aligned}$$

By a simple approximation argument, the formula above is still valid if $u \in W^{1,2}(\Omega; \mathbb{R})$ is a solution of (M), and we conclude that

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega |\nabla(u \circ w_\varepsilon)|^2\, dx = -2\int_\Omega \nabla F \nabla u \cdot \nabla u\, dx.$$

□

Remark 3.2. If u is locally a Lipschitz function in Ω, statement (ii) can be reformulated as

$$\int_{\{u=t\}} |\nabla u|\, dH^{N-1} = \int_\Omega |\nabla u|^2\, dx$$

for $\mathcal{L}^1$-a.e. $t \in (0,1)$. In order to prove this assertion we will use the *co-area* formula (see [7], Chapter 3)

$$\int_\Omega h(x)|\nabla v(x)|\, dx = \int_{-\infty}^{+\infty} \left(\int_{\{v=t\}} h(x)\, dH^{N-1}(x) \right) dt \tag{3.3}$$

valid for any Borel function $h : \Omega \to [0, +\infty]$ and $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R})$. By part (i), with $\varphi \equiv 1$, $f \in C^1_c(\mathbb{R})$, and setting $h(x) := f'(u(x))|\nabla u|$, we obtain

$$0 = \int_\Omega |\nabla u|^2 f'(u)\, dx = \int_0^1 f'(t) \left(\int_{\{u=t\}} |\nabla u|\, dH^{N-1} \right) dt = 0.$$

This proves the existence of a constant C such that $\int_{\{u=t\}} |\nabla u|\, dH^{N-1} = C$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$. Using the co-area formula once again, we conclude that $C = \int_0^1 |\nabla u|^2\, dx$. As mentioned in the introduction, following the present work Tilli ([17]) has obtained the locally Lipschitz property of u.

In Proposition 3.4 we will need to exploit certain divergence-free fields with a given trace on a Lipschitz domain, and the result below can be used to ensure their existence.

Proposition 3.3. *If $\theta \in L^2(\Omega)$ satisfies*

$$\int_\Omega \theta\, dx = 0$$

then there exists $f \in W^{1,2}_0(\Omega; \mathbb{R}^N)$ such that $\operatorname{div} f = \theta$.

PROOF. We first recall a consequence of Tartar's equivalence lemma ([16]), which, in turn, generalizes a result of Peetre (see [15]): let E_1 be a Banach space, and let E_2, E_3, be normed spaces. If $A : E_1 \to E_2$ is a linear bounded operator and $B : E_1 \to E_3$ is a compact linear operator, then Range(A) is closed provided

$$\|u\|_{E_1} \le C\, [\|Au\|_{E_2} + \|Bu\|_{E_3}] \tag{3.4}$$

for some constant $C > 0$. We apply the equivalence lemma with $E_1 := L^2(\Omega)$, $E_2 := [H^{-1}(\Omega)]^N$, $E_3 := H^{-1}(\Omega)$, $Au := \nabla u$, and $Bu := u$. With these choices, the estimate (3.4) reduces to

$$\|u\|_{L^2} \leq C\,[\|\nabla u\|_{H^{-1}} + \|u\|_{H^{-1}}]$$

and it has been proved by Nečas in [14].

Since Range(A) is closed, then so is Range(A^T), where $A^T : [H_0^1(\Omega)]^N \to L^2(\Omega)$ is the divergence operator. We conclude that

$$\left\{\theta \in L^2(\Omega) : \int_\Omega \theta\, dx = 0\right\} = [\mathrm{Ker}(A)]^\perp = \overline{\mathrm{Range}(A^T)} = \mathrm{Range}(A^T)$$

and the statement follows. □

An immediate consequence of Proposition 3.3 is that if $\tau \in H^{1/2}(\partial\Omega, \mathbb{R}^N)$ satisfies

$$\int_{\partial\Omega} \tau \cdot n_\Omega \, dH^{N-1} = 0,$$

where n_Ω is the unit outer normal to $\partial\Omega$, then the problem

$$\begin{cases} \operatorname{div} g = 0 & \text{in } \Omega \\ \\ g = \tau & \text{on } \partial\Omega \end{cases}$$

admits a solution $g \in W^{1,2}(\Omega; \mathbb{R}^N)$. Indeed, it suffices to apply Proposition 3.3 to the function $\theta := -\operatorname{div} h$, where $h \in W^{1,2}(\Omega; \mathbb{R}^N)$ is such that $h = \tau$ on $\partial\Omega$, to obtain a function $f \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ satisfying $\operatorname{div} f = \theta$, and set $g := f + h$.

In the following proposition we exploit (3.1), to show that the normal derivative on the boundary of the level sets $\{u = 0\}$, $\{u = 1\}$, is locally constant. As it turns out, the normal derivative for minimizers is globally constant ([17], and also see [2] in the case of one volume constraint).

Proposition 3.4. *If $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ satisfies (3.1), if $\Delta u = 0$ in $\{0 < u < 1\}$, and if the free boundary $S := \{u = 0\} \cup \{u = 1\}$ is C^1, then then $\partial u/\partial n$ is locally constant on S.*

Proof. Let $g \in C_c^\infty(B_r)$, where B_r is an open ball of radius r in Ω such that $B_r \cap \{u = 1\} = \emptyset$. Suppose in addition that

$$\int_{\partial\{u=0\}} g \cdot \nu \, dH^{N-1} = 0$$

where ν is the outer normal to $\{u > 0\}$. In view of Proposition 3.3 and the remark thereafter, we consider the fields F^+ and F^- such that

$$\begin{cases} \operatorname{div} F^+ = 0 & \text{in } B_r^+ := B_r \setminus \{u = 0\} \\ F^+ = g & \text{on } S^+ := \partial B_r^+ \cap \{u = 0\} \\ F^+ = 0 & \text{on } \partial B_r^+ \setminus \{u = 0\}, \end{cases}$$

$$\begin{cases} \operatorname{div} F^- = 0 & \text{in } B_r^- := B_r \cap (\mathrm{int}\{\mathrm{u} = 0\}) \\ F^- = g & \text{on } \partial B_r^- \cap \{u = 0\} \\ F^- = 0 & \text{on } \partial B_r^- \setminus \{u = 0\}, \end{cases}$$

and define

$$F := \begin{cases} F^+ & \text{in } B_r^+ \\ F^- & \text{in } B_r^- \\ 0 & \text{otherwise.} \end{cases}$$

A smoothing argument shows that (3.1) holds with F (which, a priori, is only in $W_0^{1,2}(B_r, \mathbb{R}^N)$ and not necessarily Lipschitz) because $|\nabla u|$ is bounded in B_r by assumption. Hence, we have

$$\begin{aligned} 0 &= \sum_{i,j=1}^N \int_{B_r} \frac{\partial F_i}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx = \sum_{i,j=1}^N \int_{B_r^+} \frac{\partial F_i^+}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx \\ &= \sum_{i,j=1}^N \int_{S^+} g_i \nu_j \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dH^{N-1} - \int_{B_r^+} F_i^+ \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx. \end{aligned}$$

Note that

$$\sum_{j=1}^N \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) = \frac{\partial u}{\partial x_i} \Delta u + \frac{\partial}{\partial x_i} \left[\frac{1}{2} |\nabla u|^2 \right] = \frac{\partial}{\partial x_i} \left[\frac{1}{2} |\nabla u|^2 \right]$$

hence

$$\begin{aligned} 0 &= \sum_{i=1}^N \int_{S^+} g_i \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial n}\, dH^{N-1} - \int_{B_r^+} F_i^+ \frac{\partial}{\partial x_i} \left[\frac{1}{2} |\nabla u|^2 \right] dy \\ &= \int_{S^+} \sum_{i=1}^N g_i \nu_i \left[\left| \frac{\partial u}{\partial n} \right|^2 - \frac{1}{2} |\nabla u|^2 \right] dH^{N-1} = \frac{1}{2} \int_{S^+} g \cdot \nu \left| \frac{\partial u}{\partial n} \right|^2 dH^{N-1} \end{aligned}$$

by the identity $\partial u / \partial x_i = \nu_i \partial u / \partial n$ on the boundary. We have proved the implication

$$\int_{S^+} g \cdot \nu \, dH^{N-1} = 0 \qquad \Longrightarrow \qquad \int_{S^+} g \cdot \nu \left| \frac{\partial u}{\partial n} \right|^2 dH^{N-1} = 0 \tag{3.5}$$

for $g \in C_c^\infty(B_r, \mathbb{R}^N)$. for $g \in C_c^\infty(B_r, \mathbb{R}^N)$, and this ensures the existence of a constant λ such that

$$\int_{S^+} g \cdot \nu \left| \frac{\partial u}{\partial n} \right|^2 dH^{N-1} = \lambda \int_{S^+} g \cdot \nu \, dH^{N-1} \qquad \text{for all } g \in C_0^\infty(B_r, \mathbb{R}^N).$$

We conclude that the normal derivative of u is locally constant on S^+. □

4 Asymptotic Behavior of the Solutions

In this section we investigate the asymptotic behavior of solutions $u_{\alpha\beta}$ of

$$(\mathrm{M})_{\alpha\beta} \qquad \min \left\{ \int_\Omega |\nabla u|^2 \, dx : \ u \in W^{1,2}(\Omega), \mathcal{L}^N(\{u=0\}) = \alpha, \, \mathcal{L}^N(\{u=1\}) = \beta \right\}$$

as $(\alpha + \beta) \nearrow \mathcal{L}^N(\Omega)$. We denote by $m_{\alpha\beta}$ the Dirichlet integral of $u_{\alpha\beta}$ and, for any constant $\gamma \in (0, |\Omega|)$, we set

$$p_\gamma := \min \left\{ P_\Omega(E) \, : \, E \subset \Omega, \ \mathcal{L}^N(E) = \gamma \right\} \tag{4.1}$$

where $P_\Omega(E)$ denotes, as usual, the perimeter of E in Ω. The main result of this section is the following:

Theorem 4.1. *For any $\gamma \in (0, \mathcal{L}^N(\Omega))$ we have*

$$\lim_{\substack{\alpha \to \mathcal{L}^N(\Omega)-\gamma \\ \beta \to \gamma \\ \alpha+\beta<\mathcal{L}^N(\Omega)}} \big(\mathcal{L}^N(\Omega) - (\alpha+\beta)\big) m_{\alpha\beta} = p_\gamma^2. \tag{4.2}$$

Moreover, any limit point in the $L^2(\Omega)$ topology of $u_{\alpha\beta}$ is the characteristic function of a minimizing set for (4.1).

Theorem 4.1 will be deduced easily from Theorem 4.2 below, recalling that the theory of Γ-convergence ensures that minimizers of $(\mathrm{M})_{\alpha\beta}$ converge to minimizers for (4.1), and that minima for $(\mathrm{M})_{\alpha\beta}$ tend to the minimum for the limiting problem, i.e. (4.2) holds.

Theorem 4.2. *For any $u \in L^2(\Omega)$ and $\alpha, \beta > 0$ with $\alpha + \beta < \mathcal{L}^N(\Omega)$, we define*

$$F_{\alpha\beta}(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 & \textit{if } u \in H^1(\Omega), \mathcal{L}^N(\{u \le 0\}) \ge \alpha, \mathcal{L}^N(\{u \ge 1\}) \ge \beta \\ +\infty & \textit{otherwise,} \end{cases}$$

and

$$G_\gamma(u) := \begin{cases} [P_\Omega(E)]^2 & \textit{if } u = \chi_E \textit{ and } \mathcal{L}^N(E) = \gamma \\ +\infty & \textit{otherwise.} \end{cases}$$

Then,

$$\Gamma\big(L^2(\Omega)\big) - \lim_{\substack{\alpha \to \mathcal{L}^N(\Omega)-\gamma \\ \beta \to \gamma \\ \alpha+\beta<\mathcal{L}^N(\Omega)}} \big(\mathcal{L}^N(\Omega) - (\alpha+\beta)\big) F_{\alpha\beta}(u) = G_\gamma(u) \qquad \textit{for all } u \in L^2(\Omega).$$

Proof. Without loss of generality, we can assume that $\mathcal{L}^N(\Omega) = 1$. We fix sequences $\{\alpha_n\}$, $\{\beta_n\}$, converging to $(1-\gamma)$, γ, respectively, and we denote by $F^+(u)$, $F^-(u)$, the upper and lower Γ-limits, precisely:

$$F^+(u) := \inf_{\{u_n\}} \left\{ \limsup_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) F_{\alpha_n\beta_n}(u_n) : u_n \to u \text{ in } L^2(\Omega) \right\}$$

and

$$F^-(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) F_{\alpha_n\beta_n}(u_n) : u_n \to u \text{ in } L^2(\Omega) \right\}.$$

We have to prove that $F^- \ge G_\gamma \ge F^+$.

Step 1. We first establish the inequality $F^- \ge G_\gamma$, namely

$$\liminf_{h\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) F_{\alpha_n\beta_n}(u) \ge G_\gamma(u) \tag{4.3}$$

for any sequence $\{u_n\}$ converging to u in $L^2(\Omega)$. It is not restrictive to assume that the $\liminf$ in (4.3) is a finite limit, and to assume, by a truncation argument, that $0 \le u_n \le 1$.

We first prove that $u = \chi_E$ is a characteristic function and that $\mathcal{L}^N(E) = \gamma$. Indeed, by Proposition 2.1 with $C = \{0\}$ and $C = \{1\}$ we infer

$$\mathcal{L}^N(\{u = 0\}) \ge \limsup_{n\to+\infty} \mathcal{L}^N(\{u_n = 0\}) = (1-\gamma), \qquad \mathcal{L}^N(\{u = 1\}) \ge \limsup_{n\to+\infty} \mathcal{L}^N(\{u_n = 1\}) = \gamma.$$

In particular, there exists a Borel set $E \subset \Omega$ such that $u = \chi_E$. Since

$$\int_\Omega u_n \, dx \ge \beta_n + \int_{L_n} u_n \, dx,$$

with $L_n = \{0 < u_n < 1\}$, passing to the limit as $n \to +\infty$ we obtain

$$\mathcal{L}^N(E) = \int_\Omega u\,dx \geq \gamma$$

as claimed.

Denoting by $\int_\Omega |Du|$ the total variation of a $L^1_{\rm loc}(\Omega)$ function u (see for instance [9]), we notice that

$$\begin{aligned} \int_\Omega |Du_n| &= \int_{L_n} |\nabla u_n|\,dx \leq \left(\mathcal{L}^N(L_n)\right)^{1/2} \left(\int_\Omega |\nabla u_n|^2\,dx\right)^{1/2} \\ &\leq \left[\left(1-(\alpha_n+\beta_n)\right)\int_\Omega |\nabla u_n|^2\,dx\right]^{1/2} \end{aligned}$$

$F_{\alpha_n\beta_n}(u_n) < +\infty$. Therefore, as$P_\Omega(E) = \int_\Omega |D\chi_E|$ and $u \mapsto \int_\Omega |Du|$ is $L^1_{\rm loc}(\Omega)$ lower semicontinuous we get

$$\begin{aligned} G_\gamma(u) &= [P_\Omega(E)]^2 = \left(\int_\Omega |Du|\right)^2 \leq \liminf_{n\to+\infty}\left(\int_\Omega |Du_n|\right)^2 \\ &\leq \liminf_{n\to+\infty}\left(1-(\alpha_n+\beta_n)\right)\int_\Omega |\nabla u_n|^2\,dx, \end{aligned} \tag{4.4}$$

and this proves (4.3).

Step 2. Now we prove the inequality $F^+(u) \leq G_\gamma(u)$. It is not restrictive to assume that $u = \chi_E$ is a characteristic function, $\mathcal{L}^N(E) = \gamma$ and $P_\Omega(E) < +\infty$.

We first assume that $E = D \cap \Omega$ for some bounded open set D with smooth boundary in $\mathbb{R}^N$, and we prove that

$$F^+(u) \leq \left[H^{N-1}(\partial D \cap \overline{\Omega})\right]^2. \tag{4.5}$$

Let

$$d(x) := \begin{cases} \operatorname{dist}(x,\partial D) & \text{if } x \notin D \\ -\operatorname{dist}(x,\partial D) & \text{if } x \in D \end{cases}$$

be the signed distance function from D. Due to the smoothness of D, for $\sigma > 0$ sufficiently small we have that

$$\left\{x \in \mathbb{R}^N \,:\, d(x) = t\right\} = \left\{\Phi_t(x) \,:\, x \in \partial D\right\} \tag{4.6}$$

for $t \in (-\sigma,\sigma)$, where $\Phi_t(x) := x + t\nu(x)$ and ν is the unit outer normal to D. For n large enough it holds

$$\mathcal{L}^N(\{x \in \Omega \,:\, |d(x)| < \sigma\}) > 1 - (\alpha_n + \beta_n)$$

and hence we can find λ_n, $\mu_n \in (-\sigma,\sigma)$ such that $\lambda_n < \mu_n$ and

$$\mathcal{L}^N(\{x \in \Omega \,:\, d(x) \leq \lambda_n\}) = \alpha_n, \qquad \mathcal{L}^N(\{x \in \Omega \,:\, d(x) \geq \mu_n\}) = \beta_n.$$

By construction, the functions

$$u_n(x) = \frac{[\min\{d(x),\mu_n\} - \lambda_n]^+}{\mu_n - \lambda_n}$$

satisfy the constraint $\mathcal{L}^N(\{u_n = 0\}) = \alpha_n$, $\mathcal{L}^N(\{u_n = 1\}) = \beta_n$, and clearly $u_n \to u$ in $L^2(\Omega)$. Using the identity $|\nabla d| = 1$ and the co-area formula (3.3) with $h \equiv 1$ we can estimate

$$\begin{aligned}
\big(1-(\alpha_n+\beta_n)\big)\int_\Omega |\nabla u_n|^2\,dx &= \big(1-(\alpha_n+\beta_n)\big)\int_{\{x\in\Omega\,:\,\lambda_n<d(x)<\mu_n\}} |\nabla u_n|^2\,dx \\
&= \frac{1-(\alpha_n+\beta_n)}{(\mu_n-\lambda_n)^2}\mathcal{L}^N(\{x\in\Omega\,:\,\lambda_n<d(x)<\mu_n\}) \\
&= \left[\frac{\mathcal{L}^N(\{x\in\Omega\,:\,\lambda_n<d(x)<\mu_n\})}{\mu_n-\lambda_n}\right]^2 \\
&= \left[\frac{1}{\mu_n-\lambda_n}\int_{\lambda_n}^{\mu_n} H^{N-1}(\{x\in\Omega\,:\,d(x)=t\})\,dt\right]^2 .
\end{aligned}$$

Hence, to get (4.5) we need only to prove the inequality

$$\limsup_{t\to 0} H^{N-1}(\{x\in\Omega\,:\,d(x)=t\}) \le H^{N-1}(\partial D\cap\overline{\Omega}). \tag{4.7}$$

Indeed, let us fix an open set $A \supset \overline{\Omega}$. By (4.6), for $|t| < \min\{\sigma, \mathrm{dist}(\overline{\Omega}, \partial A)\}$ we have

$$\{x\in\Omega\,:\,d(x)=t\} \subset \Phi_t(A\cap\partial D),$$

hence

$$\limsup_{t\to 0} H^{N-1}(\{x\in\Omega\,:\,d(x)=t\}) \le \limsup_{t\to 0} H^{N-1}(\Phi_t(A\cap\partial D)) = H^{N-1}(A\cap\partial D).$$

Clearly (4.7) follows by letting $A \downarrow \overline{\Omega}$.

Finally, by Lemma 4.3 below we can find a sequence of bounded open sets D_n with smooth boundary in $\mathbb{R}^N$ such that $u_n := \chi_{D_n\cap\Omega}$ converge to $u = \chi_E$ in $L^2(\Omega)$, $\mathcal{L}^N(D_n\cap\Omega) = \gamma$, and

$$\lim_{n\to+\infty} H^{n-1}(\partial D_n\cap\overline{\Omega}) = P_\Omega(E).$$

Applying (4.4) to u_n and using the lower semicontinuity of $u \mapsto F^+(u)$ (see [6]), we obtain

$$F^+(u) \le \liminf_{n\to+\infty} F^+(u_n) \le \liminf_{n\to+\infty} H^{N-1}(\partial D_n\cap\overline{\Omega}) = P_\Omega(E).$$

□

Lemma 4.3. *Let $E \subset \Omega$ be a set with finite perimeter such that $0 < \mathcal{L}^N(E) < \mathcal{L}^N(\Omega)$. There exists a sequence of bounded open sets $D_n \subset \mathbb{R}^N$ with smooth boundary in $\mathbb{R}^N$ such that $\mathcal{L}^N(E) = \mathcal{L}^N(D_n\cap\Omega)$, χ_{D_n} converges to χ_E in $L^2(\Omega)$, and*

$$\lim_{n\to+\infty} H^{N-1}(\partial D_n\cap\overline{\Omega}) = P_\Omega(E).$$

PROOF. Let us first assume the existence of nonempty balls B, B' such that $B \subset E$ and $B' \subset \Omega\setminus E$. By a local reflection argument (see for instance [4]) we can extend E to a bounded set with finite perimeter E' in $\mathbb{R}^N$ such that $|D\chi_{E'}|(\partial\Omega) = 0$. It is possible to find bounded open sets E_n with smooth boundary, converging to E' and such that (see [9])

$$\lim_{n\to+\infty} P_{\mathbb{R}^N}(E_n) = P_{\mathbb{R}^N}(E').$$

By the lower semicontinuity of perimeter in open sets we infer

$$
\begin{aligned}
P_\Omega(E) = P_\Omega(E') &\leq \liminf_{n\to+\infty} P_\Omega(E_n) \leq \limsup_{n\to+\infty} H^{N-1}(\partial E_n \cap \overline{\Omega}) \\
&= \limsup_{n\to+\infty} P_{\mathbb{R}^N}(E_n) - P_\Omega(E_n) \\
&\leq \limsup_{n\to+\infty} P_{\mathbb{R}^N}(E_n) - \liminf_{n\to+\infty} P_{\mathbb{R}^N\setminus\overline{\Omega}}(E_n) \\
&\leq P_{\mathbb{R}^N}(E') - P_{\mathbb{R}^N\setminus\overline{\Omega}}(E') = P_\Omega(E') = P_\Omega(E)
\end{aligned}
$$

whence $H^{N-1}(\partial E_n \cap \overline{\Omega})$ converges tp $P_\Omega(E)$.

Since $\mathcal{L}^N(E_n \cap \Omega)$ converges to $\mathcal{L}^N(E)$, possibly adding to E_n small balls contained in B' and possibly removing from E_n small balls contained in B we obtain sets D_n with the same properties such that $\mathcal{L}^N(D_n \cap \Omega) = \gamma$.

To prove the general case, we notice that any set $E \subset \Omega$ such that $0 < P_\Omega(E) < +\infty$ can be approximated, in area and perimeter, by sets E_h such that $\mathcal{L}^N(E_h) = \mathcal{L}^N(E)$ and such that both E_h and $\Omega \setminus E_h$ have nonempty interior: the approximation can for instance be achieved choosing a point $x \in \Omega$ where the density of E is $1/2$ and defining

$$
E_h := E \cup B_{1/h}(x) \setminus B_{\rho_h}(x) \qquad h \geq 1
$$

with $\rho_h = \left(\mathcal{L}^N(B_{1/h}(x) \setminus E)/\omega_N\right)^{1/N}$, chosen in order to satisfy the volume constraint. Hence, since the approximation property is true for E_h, a diagonal argument leads to the existence of D_n also in the general case. □

Proof of Theorem 4.1. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences converging to $(1-\gamma)$, γ, respectively, and let $u_n \in W^{1,2}(\Omega;[0,1])$ be the corresponding solutions to $(\mathrm{M})_{\alpha_n,\beta_n}$ (see Remark 2.4(ii)) . By the general properties of Γ-convergence (see [6]) we need only to know that the sequence $\{u_n\}$ is relatively compact in $L^2(\Omega)$.

Let $E \subset \Omega$ be a set of finite perimeter such that $\mathcal{L}^N(E) = \gamma$, and in view of Theorem 4.2 let $\{v_n\}$ be a sequence converging to χ_E in $L^2(\Omega)$ and such that

$$
\lim_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) \int_\Omega |\nabla v_n|^2 \, dx = [P_\Omega(E)]^2.
$$

Since u_n are minimizing, we have

$$
\limsup_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) \int_\Omega |\nabla u_n|^2 \, dx \leq [P_\Omega(E)]^2,
$$

and (4.4) yields

$$
\limsup_{n\to+\infty} \int_\Omega |\nabla u_n| \, dx \leq P_\Omega(E) < +\infty.
$$

Since the embedding $BV(\Omega) \subset L^1(\Omega)$ is compact, and as the functions are equibounded, we conclude that $\{u_n\}$ is relatively compact in the $L^2(\Omega)$ topology. □

5 Competing Configurations

5.1 One dimensional solutions

In the scalar case where Ω is an interval, $d = 1$ and $\{z_1, z_2\} = \{0, 1\}$, the solutions of (M) can be easily characterized. Assuming with no loss of generality that $\Omega = (0, 1)$, we claim that any minimizer u in Ω is affine in $\{0 < u < 1\}$. In fact, denoting by $\{A_i\}_{i\in I}$ the connected components of $\{0 < u < 1\}$, we have

$$\int_0^1 |u'|^2\,dx = \sum_i \int_{A_i} |u'|^2\,dx = \sum_i \frac{1}{\mathcal{L}^1(A_i)}.$$

The inequality between arithmetic mean an harmonic mean gives

$$\int_0^1 |u'|^2\,dx \geq \frac{[\mathrm{card}(I)]^2}{\sum_i \mathcal{L}^1(A_i)} = \frac{[\mathrm{card}(I)]^2}{1-\alpha-\beta}.$$

This proves that $\mathrm{card}(I) = 1$, i.e. $\{0 < u < 1\}$ has only one connected component, and that the least energy is $1/(1-\alpha-\beta)$.

This argument can basically be repeated in the vector-valued case with $\Omega = (0, 1)$ and $d > 1$. Recalling that in this case we have existence for any finite set of constrained points $K = \{z_1, \ldots, z_P\}$ (not necessarily extremal points of a convex set, see Remark 2.4(iv)), it can be shown that the problem is equivalent to finding the shortest connection between these points. In fact, setting $\gamma := 1 - \sum_i \alpha_i$ the remaining length fraction, we denote by *path* any finite sequence $w := \{w_1, \ldots, w_r\}$ such that

$$\{w_1, \ldots, w_r\} = K,$$

and we claim that the infimum of (M) is given by

$$\gamma^{-1} \inf\left\{\sum_{j=1}^{r-1} |w_{j+1} - w_j| \;:\; w \text{ is a path}\right\}^2.$$

In fact, using the Lagrange multiplier rule, the above infimum can be represented by

$$\text{(P)} \qquad \inf\left\{\sum_{j=1}^{r-1} \frac{|w_{j+1} - w_j|^2}{a_j} \;:\; w \text{ is a path},\ \sum_{j=1}^{r} a_j = \gamma\right\}.$$

If $u \in H^1(0, 1)$ is any admissible function for (M) and I is any connected component of $A_u := \{u \notin K\}$, then the condition $u(\partial I) \subset K$ implies that

$$\int_I |u'|^2\,dt \geq \frac{[\mathrm{osc}(u, I)]^2}{\mathcal{L}^1(I)} \geq \delta^2$$

where $\delta > 0$ is the least distance between two points in K. Hence, A_u has only finitely many connected components. It is now easy to establish a one to one correspondence, with equivalence of the energies, between admissible functions $u \in H^1(0, 1)$ for (M) and admissible pairs (w_i, a_i) for (P). In fact, given $u \in H^1(0, 1)$ admissible for (M), the length of each connected component $I = (s, t)$ of A_u determines a certain positive number a_j, and we set

$$w_j := \lim_{x\downarrow s} u(x), \qquad w_{j+1} := \lim_{x\uparrow t} u(x).$$

If u is a solution for (M) then

$$\int_I |u'|^2\,dx = \frac{|w_{j+1} - w_j|^2}{a_j},$$

and clearly (w_j, a_j) is admissible for (P). Conversely, any $\{(w_j, a_j)\}$ admissible for (P) corresponds to a function u admissible for (M) which is piecewise affine, with slope $|w_{j+1} - w_j|/a_j$ in intervals with length a_j, and whose whose level sets $\{u = z_i\}$ are formed by n_i intervals (possibly reducing to a single point) with total length α_i, where

$$n_i := \mathrm{card}\left(\{j \,:\, w_j = z_i\}\right).$$

5.2 The circle : radial and comparison configurations

Here we study candidates for solutions of the problem

$$\text{(Mr)} \qquad \min\left\{\int_B |\nabla u|^2\,dx : u \in W^{1,2}_r(B),\, \mathcal{L}^N(\{u=0\}) = \alpha, \mathcal{L}^N(\{u=1\}) = \beta\right\}$$

where B is the unit ball of $\mathbb{R}^2$ and $W^{1,2}_r(B)$ denotes the space of radial functions in $W^{1,2}(B)$.

Let $u(x) := g(|x|) \in W^{1,2}_r(B)$, where g is continuous in $(0,1]$. We define

$$\bar{r} := \sup\{r \in (0,1] \,:\, g(r) \in \{0,1\}\}.$$

If $g(\bar{r}) = 0$ we can make a nonincreasing rearrangement of g, preserving the measure of level sets of u (see for instance [11], Lemma 7.17) in order to obtain a new function $\tilde{u}(x) = \tilde{g}(|x|)$ whose Dirichlet integral does not exceed the one of u, and still admissible for (M). If $g(\bar{r}) = 1$ the same argument can be applied to $1-u$. This proves that to determine the minimum energy we may restrict ourselves to nonincreasing or nondecreasing functions g.

An elementary computation shows that $g(r) = a + b\ln r$ in $\{0 < g < 1\}$ for suitable constants a, b. In the nonincreasing case these constants can be computed using the volume constraints to find $b = 1/\ln(r_0/r_1)$ and $a = -b\ln r_1$, where

$$r_0 := \sqrt{\frac{\beta}{\pi}}, \qquad r_1 := \sqrt{\frac{\pi - \alpha}{\pi}}.$$

With these choices of a, b, the Dirichlet integral reduces to $4\pi/\ln((\pi-\alpha)/\beta)$. Taking into account also the nondecreasing case, we find that the minimal energy of (M) is

$$\min\left\{\frac{4\pi}{\ln\frac{\pi-\alpha}{\beta}}, \frac{4\pi}{\ln\frac{\pi-\beta}{\alpha}}\right\}.$$

We claim that, in general, the solutions of (M) are not radial. Consider the family of functions

$$u_{ab}(z) := a + b\ln\left|\frac{z+1}{z-1}\right|^2$$

in the complex variable $z = x + iy$, and defined in the unit disk $\Omega = \{|z| < 1\}$. These functions are harmonic, and their level sets are circles orthogonal to $\partial\Omega$, i.e., the solutions of the constrained least area problem (4.1). This might suggest that the functions $\min\{\max\{0, u_{ab}\}, 1\}$ are solutions of (M), for suitable a, b depending on α, β, at least when $\alpha+\beta$ is close to π. However, this is not true because the normal derivative is not constant on level sets and the necessary condition for minimality stated in Proposition 3.4 is violated. This can be seen either by direct computation or by the conformal change of variables $w = \log[(z+1)/(z-1)]$, mapping the circles on vertical segments in the w plane and Ω onto a strip; in the new configuration the functions have constant normal derivative, hence in the original one this property is not true.

However, these functions can be used to show that for α, $\beta << 1$ the solutions of (M) are not radial; in fact, using the equations $\Delta u_{ab} = 0$, $\partial u_{ab}/\partial n = 0$, it can be proved that

$$\int_\Omega |\nabla u_{ab}|^2 \, dz = \int_{\{x=0\}} \frac{\partial u_{ab}}{\partial x} \, dH^1 = 4b \int_{-1}^{1} \frac{1}{y^2+1} \, dy = 2b\pi.$$

Denoting by r_0 and r_1 the radii of the circles $\{u_{ab} = 0\}$, $\{u_{ab} = 1\}$, respectively, for α, $\beta << 1$ we have

$$1 \sim a + b \ln \frac{4}{r_1^2}, \qquad 0 \sim a + b \ln \frac{r_0^2}{4}, \qquad \alpha \sim \frac{\pi r_0^2}{2}, \qquad \beta \sim \frac{\pi r_1^2}{2}$$

and we find that the least energy of (M) cannot exceed

$$\frac{2\pi}{\ln \frac{4\pi^2}{\alpha\beta}}.$$

For $\alpha = \beta$, this quantity is asymptotically 4 times smaller than the least energy of radial solutions.

5.3 The square: piecewise affine and comparison configurations

Let $\Omega = (0,1)^2$, $d = 1$, fix $\alpha, \beta \in (0,1)$, with $\alpha + \beta < 1$, and consider the piecewise linear function $u : (0,1)^2 \to \mathbb{R}^2$ such that

$$u(x) = \begin{cases} 0 & \text{if } x_1 \leq \alpha, \\ \frac{1}{1-\alpha-\beta} x_1 - \frac{\alpha}{1-\alpha-\beta} & \text{if } \alpha < x_1 < 1-\beta, \\ 1 & \text{if } x_1 \geq 1-\beta \end{cases}$$

We claim that, in spite the fact that u satisfies the optimality conditions found on Section 3, u will not solve (M) neither when $\alpha + \beta << 1$ nor when $\alpha + \beta$ is close to 1. Indeed

$$\int_\Omega |\nabla u|^2 \, dx = \frac{1}{1-\alpha-\beta},$$

and if we consider a competing configuration v such that $v = 0$ on a right triangle with right angle at $(0,0)$, $v = 1$ on a right triangle with right angle at the vertex $(1,1)$, and v is linear in between, then it can be shown that

$$\int_\Omega |\nabla v|^2 \, dx = \frac{1-\alpha-\beta}{(\sqrt{2} - \sqrt{\alpha} - \sqrt{\beta})^2}.$$

In particular,

$$\int_\Omega |\nabla v|^2 \, dx < \int_\Omega |\nabla u|^2 \, dx$$

for $\alpha + \beta$ sufficiently small, for instance if $\alpha + \beta < 3 - 2\sqrt{2}$. Finally, considering the limiting configuration which is equal to one on a quarter of a circle centered at $(0,0)$ with radius r, and it is constantly equal to zero elsewhere on the square, then $r = 2\sqrt{\beta/\pi}$, and the perimeter of the interface is $\sqrt{\pi\beta}$. By Theorem 4.1 we conclude that if $\sqrt{\pi\beta} < 1$ then u cannot be a solution for (M).

Acknowledgments The authors would like to thank Morton Gurtin for having suggested the problem of minimization of the Dirichlet integral under volume constraints, and for many stimulating discussions that followed the progress of this paper. The authors are indebted to the Center for Nonlinear Analysis (CNA, Carnegie Mellon University) and to the Max Planck Institute for Mathematics in the Sciences in Leipzig (MPI), for its hospitality and for providing an exceptional environment for the conclusion of this work. The research of I. Fonseca and L. Tartar was partially supported by the National Science Foundation through the Center for Nonlinear Analysis, by the National Science Foundation under grants No. DMS–9500531 and DMS–9704762 respectively, and by the MPI. The research of L. Ambrosio and P. Marcellini was partially supported by the Italian Ministry of Education (MURST).

References

[1] Acerbi, E. and N. Fusco, Semicontinuity problems in the calculus of variations, *Arch. Rat. Mech. Anal.* **86** (1984), 125–145.

[2] Aguilera, N., H.W. Alt and L.A. Caffarelli, An optimization problem with volume constraint, *SIAM J. Control and Optimization* **24** (1986), 191–198.

[3] Alt, H.W. and L.A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.* **325** (1981), 105–144.

[4] Ambrosio, L., N. Fusco and D. Pallara, *Free discontinuity problems and special functions with bounded variation,* Oxford U.P., forthcoming.

[5] Bouchitté, G., Singular perturbations of variational problems arising from a two phases transition model, *Appl. Math. Optim.* **21** (1990), 289–314.

[6] Dal Maso, G., *An introduction to* Γ*-Convergence*, Birkhäuser, 1993.

[7] Evans, L. C. and R.F. Gariepy, *Lecture Notes on Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, 1992.

[8] Fonseca, I. and L. Tartar, The gradient theory of phase transitions for systems with two potential wells, *Proc. Royal Soc. Edin.* **111 A** (1989), 89–102.

[9] Giusti, E., *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, 1994.

[10] Gurtin, M. E., D. Polignone and J. Vinals, Two-phase binary fluids and immisible fluids described by an order parameter, *Math. Models and Methods in Applied Science* **6** (1996), 815–831.

[11] Lieb, E. H. and M. Loss, *Analysis*, Graduate Studies in Mathematics, **14**, American Mathematical Society, 1997.

[12] Marcellini, P., Approximation of quasiconvex functions and lower semicontinuity of multiple integrals, *Manuscripta Math.* **51** (1995), 1–28.

[13] Modica, L., Gradient theory of phase transitions and minimal interface criterion, *Arch. Rat. Mech. Anal.* **98** (1987), 123–142.

[14] Nečas, J., Sur les normes équivalentes dans $W^{k,p}(\Omega)$ et sur la coercivité des formes formellement positives, in *Équations aux Dérivées Partielles*, Les Presses de l'Université de Montréal, 1996.

[15] Peetre, J., Another approach to elliptic boundary value problems, *Comm. Pure Appl. Math.* **14** (1961), 711–731.

[16] Tartar, L., Sur un lemme d'équivalence utilisé en analyse numerique, *Calcolo* **24**(2) (1987), 129–140.

[17] Tilli, P., On the regularity of the solutions to a variational problem with two free boundaries, to appear.