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Universal covering maps and radial variation

by

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1 Introduction and Statement of Results

We let $E \subseteq \mathbb{C}$ be a closed set with two or more points. By the uniformization theorem there exists a Fuchsian group of Moebius transformations such that $\mathbb{C} \setminus E$ is conformally equivalent to the quotient manifold $\mathbb{D}/G$. The universal covering map $P : \mathbb{D} \to \mathbb{C} \setminus E$ is then given by $P = \tau \circ \pi$, where π is the natural quotient map onto $\mathbb{D}/G$ and τ is the conformal bijection between $\mathbb{C} \setminus E$ and $\mathbb{D}/G$. In this paper we will show that there exists $e^{i\beta} \in \mathbb{T}$ such that

$$\int_0^1 |P''(re^{i\beta})| dr < \infty. \tag{1.1}$$

Considering $u = \log |P'|$, one obtains (1.1) from variational estimates.

Theorem 1 *There exists $e^{i\beta} \in \mathbb{T}$ and $M > 0$ such that for $r < 1$,*

$$u(re^{i\beta}) < -\frac{1}{M} \int_0^r |\nabla u(\rho e^{i\beta})| d\rho + M.$$

The class of universal covering maps contains two extremal cases. The case where $\mathbb{C} \setminus E$ is simply connected and the case where E consists of two points. We considered the simply connected case in [J-M] where we proved that Anderson's conjecture is true. The second case is easier; well known estimates for the Poincarè metric on the triply punctured sphere give (1.1) when P is the universal covering of $\mathbb{C} \setminus \{0, 1\}$.

In the course of the proof of Theorem 1 we measure the thicknes of E at all scales, and we are guided by the following philosophy. If, at some scale, the boundary E appears to be thick then, locally, the universal covering map behaves like a Riemann map. On the other hand, if E appears to be thin, then, locally, the Poincarè metric of $\mathbb{C} \setminus E$ behaves like the corresponding Poincarè metric of $\mathbb{C} \setminus \{0, 1\}$. With the right estimates for the *transition* from the thick case to the thin case, this philosophy leads to a rigorous proof. Our proof also shows the existence of a very large set of angles β for which Theorem 1 holds.

The following propositions present the main technical results of this paper. Each proposition gives estimates on the radial variation of $u = \log |P'|$. The hypothesis of Proposition 1 covers the case when to an observer at $w = P(\zeta)$ the boundary E looks like a connected set. The hypothesis of Proposition 2 covers the case when the boundary E looks like an isolated point. To express these alternatives analytically, we use the function

$$M(\zeta) = \sup_{z \in T(\zeta)} |\nabla u(z)|(1 - |z|)$$

where

$$T(\zeta) = \{w \in \mathbb{D} : |w-\zeta| \leq 1-|\zeta|, (1-|\zeta|)/2 \leq 1-|w| \leq 1-|\zeta|\}.$$

The first alternative corresponds to the case where $u = \log|P'|$ satisfies a Bloch condition near ζ. The second alternative causes the *failur* of Bloch estimates near ζ. Correspondingly the proof of Proposition 1 uses the condition

$$M(\zeta) \leq \text{ some constant,}$$

whereas Proposition 2 *requires* that

$$M(\zeta) \geq \text{ a very large constant.}$$

Further combinatorial considerations provide the tools for an iterative solution of Theorem 1 based on repeated applications of Propositions 1 and 2.

In both Proposition 1 and 2 the following family of curves plays an important role. We let $L \geq 1$ be a positive integer, and we let $z_1, z_2 \in \mathbb{D}$, $|z_1| < |z_2|$. Then $\Gamma(z_1, z_2, L)$ is the collection of all radial line segments

$$\gamma = \{s \in \mathbb{D} : |z_1| < |s| \leq |z_2|\} \cap (0,t),$$

where $t \in \mathbb{D}$ satisfies $|t| = |z_2|$, $|t - z_2| \leq 2^L(1-|z_2|)$ and where $(0,t)$ denotes the ray connecting $0 \in \mathbb{D}$ to $t \in \mathbb{D}$.

We let M_1, L be positive integers and we fix a point $\zeta \in \mathbb{D}$. Under the hypothesis that $M(\zeta) < C$, the universal covering map P behaves locally like a Riemann map. Hence in the proof of Propositon 1 we work with stopping time arguments and J. Bourgain's estimate for the radial variation of positive harmonic functions.

Proposition 1 *There exist $C_1 \geq 1$ so that the following holds. If $L \geq C_1$ and if $M(\zeta) \leq M_1/2^{C_0 L}$ then there exists $q \in \bar{\mathbb{D}}$ such that*

a)
$$\int_\gamma |\nabla u(w)||dw| \leq C_1 M_1 L, \text{ for } \gamma \in \Gamma(\zeta, q, L).$$

b)
$$\text{If } |q| < 1 \text{ then } u(q) - u(\zeta) \leq -M_1/LC_1.$$

c)
$$1 - |q| \leq (1-|\zeta|)/2 \text{ and } |q-\zeta| \leq (1-|\zeta|)2^L.$$

The constant $C_0 \geq 1$ appearing in the formulation of Proposition 1 is specified in Section 2. Let us assume temporarily that the point q obtained by Proposition 1 also satisfies the condition that $M(q) \leq M_1/2^{C_0 L}$. Then we could apply Proposition 1 again with ζ replaced by q. Doing this would start an iteration leading to the desired variational estimates for u — until a point is reached for which $M(q) > M_1/2^{C_0 L}$. Proposition 2 explains what we do when $M(q) > M_1/2^{C_0 L}$: First using group invariance of P we replace q by a (specially chosen) point w such that

$$|u(w) - u(q)| \leq \ C_2 + d_{\mathbb{D}}(w, q).$$

Then using geometric estimates for the hyperbolic metric in $\mathbb{C} \backslash E$ we prove variational estimates for u along the radius that connects w to the boundary of $\mathbb{D}$. Note that as stated Proposition 2 does not give any information about how *close* w is to q. Only later, when we exploit that the machinery underlying the proof of Proposition 1 is composed of stopping time arguments, are we able to show that $M(q) > M_1/2^{C_0 L}$ implies a bound like

$$d_{\mathbb{D}}(w, q) \leq C_4 L.$$

(This is done in Lemma 3 of Section 3.)

Proposition 2 *There exists $C_2 \geq 1$ so that for $q \in \mathbb{D}$ and $M(q) \geq M_0$ there exist $w, v \in \bar{\mathbb{D}}$ so that the following holds.*

a) $M(w)/C_2 \leq M(q) \leq M(w) C_2$.

b) $|u(w) - u(q)| \leq C_2 + \ d_{\mathbb{D}}(w, q)$.

c) If $|v| < 1$, *then* $M_0/2 \leq M(v) \leq 2M_0$.

d) $(1 - |v|)/(1 - |w|) \leq 2^{(-M(q)+M_0)/C_2} C_2$ *and w lies on the ray* $(0, v)$.

e) If $\gamma \in \Gamma(w, v, L)$, and if w_1, w_2 are points on γ, with $(1 - |w_1|)/(1 - |w_2|) \geq 4$, *then*

$$u(w_2) - u(w_1) \leq -M(w_1)/C_2 + C_2 L.$$

Now we describe in more detail the relative positions of q, w and v. In the case when $|v| = 1$, the point w is the top of the horocycle that is tangent to $\mathbb{T}$ at v and contains q. If $v < 1$, there exist $\zeta_1, \zeta_2 \in \mathbb{T}$ so that w is the top of the hypercycle $S(q)$ containing ζ_1, ζ_2 and q. The point v is then the top of another hypercycle S_0 underneath $S(q)$ that contains ζ_1, ζ_2 and satisfies

$M_0/2 \leq M(v) \leq 2M_0$. We remark also that w will be hyperbolically very close to $k(q)$, for a suitably chosen $k \in G$. And we will see that therefore the right hand side of part (b) does *not* depend on $M(q)$. This is useful since we apply Proposition 2 when $M(q)$ is a very large constant.

Repeatedly applied, Propositions 1 and 2 give the following result.

Proposition 3 *There exists $C_3 \geq 1$ so that for $L > C_3$ and $M = 4C_1^2 C_2 L^2$, a sequence of points $s_k \in \mathbb{D}$ can be found satisfying the following conditions.*

a) $$u(s_k) - u(s_{k-1}) \leq -\frac{1}{M}\int_\gamma |\nabla u(w)||dw| \text{ for } \gamma \in \Gamma(s_{k-1}, s_k, L),$$

b) $$1 - |s_k| \leq (1 - |s_{k-1}|)/4 \text{ and } |s_k - s_{k-1}| \leq 2^L(1 - |s_{k-1}|).$$

2 Bloch estimates and Stopping time Lipschitz domains

In this section we will recapitulate and extend our arguments from [J–M]. In the first paragraphs of this section we discuss the tools necessary to define and analyze stopping time Lipschitz domains. Then we give the proof of Proposition 4 which implies Proposition 1.

We begin by describing a deep result of J. Bourgain [B]. It plays an important role in the proof of Proposition 1. We fix a positive harmonic function g in $\mathbb{D}$, and an interval $I \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus I) \leq L^{-2}$. For $e^{i\alpha} \in I$ we let $\sum(e^{i\alpha}, L)$ be the collection of curves in $\mathbb{D}$ which remain in a Stolz cone with vertex $e^{i\alpha}$ and opening angle $\pi - 1/L$, and have an $L-$Lipschitz parametrization. More precisely the curves in $\sum(e^{i\alpha}, L)$ admit the following representation,

$$\gamma(r) = re^{i\alpha}e^{i\theta(r)},\ 0 \leq r \leq 1,$$

where $|\theta(r)| < L(1-r)$ and $|\theta'(r)| \leq L$. Then the following holds.

Theorem (J. Bourgain) *There exists $e^{i\alpha} \in I$ such that*

a) $$g(e^{i\alpha}) \leq g(0)\left(1 + \frac{1}{c_0 L^2}\right) - c_0\int_0^1 |\nabla g(e^{i\alpha}r)|dr,$$

where $c_0 > 0$ is universal, and such that

b) $$\int_\gamma |\nabla g(w)||dw| \leq CLg(0),$$

whenever $\gamma \in \sum(e^{i\alpha}, L)$. *The constant* $C \geq 1$ *is universal.*

Next we recall the result that a Bloch function is bounded on a dense set of radii. We fix a C_0-Lipschitz domain $W \subseteq \mathbb{D}$. For $w \in W$ we let $s = \text{dist}(w, \partial W)$ and we choose $w_0 \in \partial W$ such that $|w - w_0| = s$. Let r be contained in the intersection $\{y \in \mathbb{D} : |w_0 - y| = s/10\} \cap \partial W$, and let $I = \{y \in \mathbb{D} : |r - y| \leq s/100\} \cap \partial W$. Let h be harmonic in W, and $M = \sup\{|\nabla h(z)| \text{dist}(z, \partial W) : z \in W\}$. In the construction below we use the following theorem [P, Proposition 4.6].

Theorem (Ch. Pommerenke) *There exists a geodesic* γ *in* W, *connecting* w *to a point in* I *such that for* $z \in \gamma$, $|h(w) - h(z)| \leq MA_0$, *where* $A_0 > 0$ *is universal.*

The constant A_0 appearing in the above theorem is a fixed multiple of $1/\omega(I, W, w)$, where $\omega(I, W, w)$ denotes the harmonic measure of I in W evaluated at w. The upper bound for A_0 comes from Beurling's minorisation of harmonic measure.

Finally we discuss an estimate which controls the growth rate of

$$M(\zeta) = \sup_{z \in T(\zeta)} |\nabla u(z)|(1 - |z|).$$

We let g be a Moebius transform without fixed points in $\mathbb{D}$. Then on $\mathbb{D}$ the function $\log d_{\mathbb{D}}(z, g(z))$ is Lipschitz with respect to the hyperbolic metric $d_{\mathbb{D}}$. Taking into account that $u = \log|P'|$ where P is actually a universal covering map, we obtain our next Lemma from the above remark and (3.1), (3.2) below.

Lemma 1 *There exists a universal* $K > 0$ *such that the function* $\log M(z)$ *is* K*−Lipschitz with respect to* $d_{\mathbb{D}}$.

We have completed the discussion of the preliminaries and will now describe the construction of stopping time Lipschitz domains. For the rest of this paper we fix $u = \log|P'|$. We also fix constants $M_1, L \in \mathbb{N}$ such that $M_1 > L > A_0$. We let $C_0 = K^2$, $\zeta \in \mathbb{D}$, and we assume that

$$M(\zeta) \leq M_1/2^{LC_0}.$$

Around ζ, we wish to construct a large Lipschitz domain on which $u - u(\zeta)$ is bounded below and satisfies a Bloch estimate. This is done in two steps each of which uses stopping time procedures on dyadic intervals.

We define the box around ζ as follows,

$$D(\zeta) = \{w \in \mathbb{D} : |\zeta/|\zeta| - w/|w|| \leq 2^L(1 - |\zeta|), \text{ and } 1 - |w| \leq 2^L(1 - |\zeta|)\}.$$

Note that the four sides and the four angles of the box $D(\zeta)$ are of the same size. For a dyadic interval $I \subset \mathbb{T}$ we let $T(I) = \{w \in \mathbb{D} : w/|w| \in I, |I|/2 \leq 1 - |w| \leq |I|\}$ and $M(I) = \sup\{M(\zeta) : \zeta \in T(I)\}$. Defining the first stopping time we let $\mathcal{E} = \{I_j\}$ be the collection of maximal dyadic intervals $\subseteq \mathbb{T}$ that satisfy $T(I) \cap D(\zeta) \neq \phi$ and

$$M(I) \geq M_1/LA_0.$$

We let $E(I)$ be the Euclidean convex hull of $\{w \in \mathbb{D} : 1 - |w| = |I|, \, w/|w| \in I\}$ and $16I \subseteq \mathbb{T}$, where $16I$ is the interval with the same midpoint as I and $|16I| = 16|I|$. Our first Lipschitz domain is given as

$$\mathcal{L}(\zeta) = D(\zeta) \setminus \bigcup_{I \in \mathcal{E}} E(I).$$

On $\mathcal{L}(\zeta)$, the function $u = \log|P'|$ satisfies a Bloch estimate. Indeed for $z \in \mathcal{L}(\zeta)$ we have by construction $|\nabla u(z)|(1 - |z|) \leq M_1/LA_0$, and therefore $|\nabla u(z)|\text{dist}(z, \partial\mathcal{L}(\zeta)) \leq M_1/LA_0$.

Next we will remove the points $w \in \mathcal{L}(\zeta)$ for which $u(w) - u(\zeta) < -M_1/2$. This will be achieved by the following stopping time procedure. Let $\mathcal{V} = \{J\}$ be the collection of maximal dyadic intervals J for which $T(J) \cap \mathcal{L}(\zeta) \neq \phi$ and there exists $v \in T(J)$ for which

$$u(v) - u(\zeta) < -M_1/2.$$

Using Pommerenke's theorem we will extract the information encoded in the stopping time collection $\mathcal{V}$. This requires some preparation. For $J \in \mathcal{V}$ we denote by w the point in $T(J)$ which satisfies $u(w) - u(\zeta) < -M_1/2$, and which is of smallest possible modulus. Let $s = \text{dist}(w, \partial\mathcal{L}(\zeta))$ and choose $w_0 \in \partial\mathcal{L}(\zeta)$ such that $|w - w_0| = s$. Also let $I_i = \{v \in \mathbb{D} : |v - w_i| \leq s/100\} \cap \partial\mathcal{L}(\zeta)$, where w_1, w_2 are the points in the intersection $\{y : |w_0 - y| = s/10\} \cap \partial\mathcal{L}(\zeta)$. By Pommerenke's theorem there exists $y_i \in I_i$ such that for each z on the $\mathcal{L}(\zeta)-$geodesic connecting w to y_i we have the upper bound

$$|u(w) - u(z)| \leq M_1/L.$$

We call this geodesic γ_i. For $i \in \{1, 2\}$ we let R_i be the straight line segment (w, y_i). Note that our construction gives the straight line segments R_1, R_2 in $\mathcal{L}(\zeta)$. Moreover for any point $v \in R_i$, there exists $z \in \gamma_i$ such that z and v can be connected by a curve in $\mathcal{L}(\zeta)$ and the $d_{\mathbb{D}}-$ length of this curve is $\leq K_1$. The constant $K_1 > 0$ is universal. In particular K_1 does not depend on our choice of L. This gives the following estimate for the deviation of u along R_i,

$$|u(w) - u(v)| \leq M_1/L + M_1K_1/LA_0, \text{ for } v \in R_i. \tag{2.1}$$

We let $R_3 \subset \partial\mathcal{L}(\zeta)$ be the shorter arc in $\partial\mathcal{L}(\zeta)$ that connects y_1 and y_2. Finally we define $V(J) \subseteq \mathcal{L}(\zeta)$ to be the domain in $\mathcal{L}(\zeta)$ that is bounded by R_1, R_2, R_3, and we put,

$$W(\zeta) = \mathcal{L}(\zeta) \setminus \bigcup_{J \in \mathcal{V}} V(J).$$

¿From now on we will only consider $L \geq 4 + 4K_1/A_0$. The following list describes the basic properties of the domain $W(\zeta)$, and contains additional important information about the stopping time intervals in $\mathcal{E}$ and $\mathcal{V}$.

REMARKS.

1. If $z \in W(\zeta)$, then $M(z) < M_1/LA_0$ and $u(z) - u(\zeta) > -M_1/2$.

2. The boundary of $W(\zeta)$ can be canonically decomposed into four very simple pieces: Two vertical line segments in $\partial D(\zeta)$, a horizontal line segment in $\partial D(\zeta)$, and a piece that is contained in the graph of a Lipschitz function defined on $\mathbb{T}$. To see this we only need to recall and compare the definitions of $E(I)$ and $V(J)$.

3. It follows from the stopping rule defining $\mathcal{E}$ that $M(I)/M(\zeta) \geq 2^{LC_0}/LA_0$, whenever $I \in \mathcal{E}$. Comparing this estimate with Lemma 1 we find that the intervals $I \in \mathcal{E}$ satisfy $|I| \leq (1 - |\zeta|)/8$.

4. For $I \in \mathcal{E}$ and $q \in E(I)$, we have $q/|q| \in 16I$.

5. The stopping rule for $\mathcal{V}$ together with Lemma 1 implies that any $J \in \mathcal{V}$ satisfies $|J| \leq (1 - |\zeta|)/8$.

6. Let $I \in \mathcal{V}$ and assume that $q \in \partial V(I)$ and $|q| < 1$. Then by our choice of $L > 4 + 4K_1/A_0$ and by (2.1) we obtain that $u(q) - u(\zeta) < -M_1/4$. We point out that this upper bound for the difference $u - u(\zeta)$ on $\partial V(I) \cap \mathbb{D}$ is comparable to the lower bound of that difference in the entire domain $W(\zeta)$. Indeed by Remark 1, for $z \in W(\zeta)$ we have $u(z) - u(\zeta) > -M_1/4$.

$W(\zeta)$ is the domain we will be working with, in this section. The following subset of $\partial W(\zeta)$ is important for the construction below. It contains the points that play a role in the Future.

$$F(\zeta) = \{w \in \partial W(\zeta) : |\zeta/|\zeta| - w| < 2^L(1 - |\zeta|) \text{ and } 1 - |w| \leq (1 - |\zeta|)/2\} \tag{2.2}$$

It follows from Remark 3) and 5) that $F(\zeta)$ is connected. Moreover, by Beurling, we have the following minorization of harmonic measure

$$\omega(F(\zeta), W(\zeta), \zeta) \geq 1 - L^{-2}.$$

The main result of this section is the following Proposition.

Proposition 4 *There exists $q \in F(\zeta)$ such that,*

1) $$\int_\gamma |\nabla u(w)||dw| \leq CM_1 L, \ \textit{for } \gamma \in \Gamma(\zeta, q, L).$$

2) $$\textit{If } |q| < 1, \ \textit{then } u(q) - u(\zeta) < M_1 \left(\frac{C}{c_0 L^2} - \frac{c_0}{CLA_0} \right),$$

where $C \geq 1$ is universal, and where $c_0, A_0 > 0$ are the constants appearing, in Bourgain's theorem resp. Pommerenke's theorem.

Proof. Let $f : \mathbb{D} \to W(\zeta)$ be the Riemann map normalized such that $f(0) = \zeta$. Recall that $F(\zeta)$ is connected and that $w(F(\zeta), W(\zeta), \zeta) \geq 1 - L^{-2}$. Hence $A = f^{-1}(F(\zeta))$ is an interval such that $m(\mathbb{T} \setminus A) \leq L^{-2}$. By Remark 1 the pullback

$$g(w) = u(f(w)) - u(f(0)) + M_1$$

is a positive harmonic function in $\mathbb{D}$. Applying Bourgain's theorem gives $e^{i\alpha} \in A$ such that

$$g(e^{i\alpha}) \leq g(0) \left(1 + \frac{1}{c_0 L^2}\right) - c_0 \int_0^1 |\nabla g(re^{i\alpha})| dr.$$

As $g(0) = M_1$ this is the same as

$$u(f(e^{i\alpha})) - u(f(0)) \leq \frac{M_1}{c_0 L^2} - c_0 \int_{\gamma_0} |\nabla u(w)||dw|, \tag{2.3}$$

where $\gamma_0 = f((0, e^{i\alpha}))$. The second part of Bourgain's theorem gives

$$\int_\gamma |\nabla g(w)||dw| \leq CM_1 L, \qquad \text{for } \gamma \in \sum(e^{i\alpha}, L).$$

With a change of variables we rewrite this line as follows,

$$\int_{f(\gamma)} |\nabla u(w)||dw| \leq CM_1 L, \ \text{for } \gamma \in \sum(e^{i\alpha}, L). \tag{2.4}$$

The admissible curves in (2.4) are $f(\gamma)$ with $\gamma \in \sum(e^{i\alpha}, L)$. Below we will use estimates on harmonic measure to show that the straight line segments in $\Gamma(\zeta, f(e^{i\alpha}), L)$ are also admissable curves. In fact we will show that (2.4) implies,

$$\int_\sigma |\nabla u(w)||dw| \leq CM_1 L, \text{ for } \sigma \in \Gamma(\zeta, q, L). \tag{2.5}$$

Now we let $q = f(e^{i\alpha})$. Note that we chose the interval A such that $f(e^{i\alpha})$ is contained in $F(\zeta)$. By construction the set $F(\zeta)$ splits canonically into three subsets carrying different pieces of information: The subset that intersects $\mathbb{T}$. The subset where $u - u(\zeta) < -M_1/4$. And the set of points z for which we know that somewhere in the Stolz cone centered at z the Bloch constant was larger than $M_1/A_0 L$. Accordingly we continue by distinguishing between the following three cases:

a) $|q| = 1$.

b) $|q| < 1$ and there exists $I \in \mathcal{V}$ such that $q \in \partial V(I)$.

c) $|q| < 1$ and $q \in \partial \mathcal{L}(\zeta)$.

Note that these cases cover all possibilities for $q \in F(\zeta)$. Treating different cases by different means, we will now verify that $q = f(e^{i\alpha})$ satisfies the conclusion of Proposition 4.

ad a) If $q = f(e^{i\alpha})$ satisfies $|q| = 1$ then we only have to show that

$$\int_\sigma |\nabla u(w)||dw| \leq CM_1 L, \text{ for } \sigma \in \Gamma(\zeta, q, L).$$

This however is just the estimate in (2.5).

ad b) By Remark 6 we have that $u(q) - u(\zeta) < -M_1/4$. When we combine this estimate with the variational estimate in (2.5) we obtain the assertions of Proposition 4. Note that in case b) the resulting decay of u is much better than claimed or needed.

ad c) By Remark 4 there exists an interval $I \in \mathcal{E}$, such that $q/|q| \in 16I$. Hence $T(I)$ is contained in a Stolz cone with vertex q. As $I \in \mathcal{E}$ we have

$$M(I) \geq M_1/LA_0.$$

In $W(\zeta)$, the geodesic $\gamma_0 = f((0, e^{i\alpha}))$ passes through a fixed enlargement of $T(I)$. Moreover $\gamma_0 = f((0, e^{i\alpha}))$ is a C^2 curve with uniform constants in $T(I)$. Hence by a simple normal families argument,

$$\int_{\gamma_0} |\nabla u(w)||dw| \geq \frac{M_1}{CLA_0},$$

where $C > 0$ is universal, and in particular independent of L. We insert the last estimate into (2.3) and obtain

$$u(q) - u(\zeta) < M_1 \left(\frac{C}{c_0 L^2} - \frac{c_0}{CLA_0} \right).$$

We have dealt with all possible cases, and Proposition 4 is proven, provided that (2.4) implies (2.5). To show this implication we use the following lemma which is folklore.

We let I, J be adjacent intervals in $\mathbb{T}$ which have $e^{i\alpha}$ as endpoint and $m(I) = m(J) = m(\mathbb{T})/2$. Their images under the Riemann map f are $A = f(I)$ respectively $B = f(J)$. Let $\gamma \subset W(\zeta)$. Using lower bounds for the harmonic measures of A *and* B we obtain useful information about the location of $f^{-1}(\gamma)$.

Lemma (Folklore) *If for any $z \in \gamma$, $w(A, W(\zeta), z) \geq 1/L$ and $w(B, W(\zeta), z) \geq 1/L$, then $f^{-1}(\gamma)$ is contained in a Stolz cone of vertex $e^{i\alpha}$ and of opening angle $\pi - 1/CL$.*

We can now show that (2.4) implies (2.5). We choose $\gamma \in \Gamma(\zeta, q, L)$, i.e., γ is of the form

$$\{s : |q| < |s| < |\zeta|\} \cap (0, t),$$

where t satisfies $|t| = |q|$, $|t - q| \leq 2^L(1 - |q|)$. By elementary geometry and Beurling's minorization of harmonic measure we find $t_1 \in \gamma$, whose hyperbolic distance to t is $\leq LC$, and so that for each $z \in \gamma_1 = \gamma \cap (t_1, 0)$ we have the estimates $w(A, W(\zeta), z) \geq \eta/L$ and $w(B, W(\zeta), z) \geq \eta/L$, with an universal $\eta > 0$. The above folk lemma and the Koebe distortion theorem imply that $f^{-1}(\gamma_1)$ is a curve in $\sum(e^{i\alpha}, CL)$. Hence by (2.4)

$$\int_{\gamma_1} |\nabla u(w)||dw| \leq CM_1 L.$$

Finally for $\gamma_2 = \gamma \cap (t_1, t)$ we estimate

$$\int_{\gamma_2} |\nabla u(w)||dw| \leq M(t_1) d_{\mathbb{D}}(t, t_1) \leq CM_1.$$

■

Remark. We will use Proposition 4 to deduce Proposition 1. Therefore it is important that the constant appearing in condition 2) of Proposition 4,

$$\left(\frac{C}{c_0 L^2} - \frac{c_0}{CLA_0} \right), \tag{2.6}$$

is negative and independent of M_1. But for L large enough the expression in (2.6) is just a small perturbation of $-c_0/CLA_0$. Here our argument *really needs* the additional freedom gained by introducing the parameter L. It now follows that Proposition 4 implies Proposition 1 when we choose $L > 2C^2A_0/c_0^2$ and $C_1 = 2CA_0/c_0$. Note that such a choice is compatible with our previous lower bound on L.

3 When Bloch estimates fail

In this section we prove Proposition 2. We recall that there exits a Fuchsian group G without elliptic elemets so that $\mathbb{C} \setminus E$ is conformally equivalent to $\mathbb{D}/G$. The universal covering map is $P = \tau \circ \pi$ where π is the natural projection, and τ is the conformal bijection between $\mathbb{D}/G$ and $\mathbb{C} \setminus E$. The density of the hyperbolic metric on $\Omega = \mathbb{C} \setminus E$ is given by

$$\lambda_\Omega(P(z))|P'(z)| = \frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{3.1}$$

By the result of A.F Beardon and Ch. Pommerenke [B-P], the density λ_Ω admits the following geometric estimate,

$$\lambda_\Omega(v_0) \sim \frac{1}{\operatorname{dist}(v_0, E)(\beta(v_0)+1)}, \qquad v_0 \in \mathbb{C} \setminus E, \tag{3.2}$$

where

$$\beta(v_0) = \inf\left\{\left|\log\frac{|v_0 - a|}{|a-b|}\right| : |v_0 - a| = \operatorname{dist}(v_0, E) \text{ and } a, b \in E\right\}.$$

If for a given $v_0 \in \mathbb{C} \setminus E$ the infimum in the definition of $\beta(v_0)$ is attained in $a, b \in E$, then one of the following cases holds. (We let $K(a, r)$ denote the open disk with radius $r > 0$ and center a.)

- P1: There exists $B, \eta \in \mathbb{R}_0^+$ such that $\mathbb{C} \setminus E \supset K(a, B) \setminus \bar{K}(a, \eta)$, $\eta < B^{-1}$, $b \notin K(a, B)$ and $\beta(v_0) \sim \log|\operatorname{dist}(v_0, E)/B|$.

- P2: There exists $\eta > 0$ such that $\Omega \supset K(a, \eta^{-1}) \setminus K(a, \eta)$, $a, b \in K(a, \eta)$ and $\beta(v_0) \sim |\log(\operatorname{dist}(v_0, E)/\eta)|$.

We define these cases as giving rise to pictures; for example we will say that we see picture P1 at v_0 if P1 holds.

The following geometric lemma will be very useful when we study the decay of $\log|P'|$ along radial line segments. We consider the following annuli centered at $a \in E$,

$$A_k = \{v \in \mathbb{C} : \operatorname{dist}(v_0, E)/2^{k+1} \le |a - v| \le \operatorname{dist}(v_0, E)/2^k\}, \text{ for } k \in \mathbb{N}_0.$$

We will only use these A_k when $\beta(v_0)$ is large and in this case the annuli A_k are disjoint from E when $k \leq C\beta(v_0)$. We also remark that these annuli allow us to trace the changes of the hyperbolic metric in $\Omega = \mathbb{C}\backslash E$, as we approach the boundary of Ω. In fact, by (3.2), the density of the hyperbolic metric remains essentially constant on each of the A_k, and the corresponding value can be computed from k and $\beta(v_0)$. The formulas are given in the proof below.

Lemma 2 *Let* $s = \operatorname{dist}(v_0, E)$ *and let* $\gamma : [0,1] \to K(a,s) \cap \mathbb{C} \setminus E$ *be a curve satisfying the following conditions:*

1. $\gamma(0) = v_0$.

2. *The linear measure of* $\gamma \cap A_k$ *is bounded by* $C\operatorname{diam} A_k$, $k \in \mathbb{N}$.

3. *There exists* $c < 1/2$ *so that if* $\gamma(t) \in A_k$ *and* $t_1 > t$ *then* $\gamma(t_1) \notin A_{kc}$.

4. $4 > \int_\gamma \lambda_\Omega(w)|dw| > 1/4$.

Then $|\gamma(1) - a|/|v_0 - a| \leq C2^{-\beta(v_0)/C}$ *and* $\beta(\gamma(1)) \leq C\beta(v_0)$, *where* $C \geq 1$ *is universal.*

Proof. First we consider the case when we see the picture P1 at v_0. There exists a smallest $\eta \geq 0$ so that P1 holds. We denote it by $\epsilon \geq 0$. Now we determine how $\beta(v)$ changes when v moves through the annuli A_k. For $v \in A_k$, we have $\operatorname{dist}(v,E) = |v-a| \sim |v_0 - a|/2^k$. Let $k_0 \in \mathbb{N}$ be the first integer for which $|v_0 - a|/2^{k_0} \leq \sqrt{B\epsilon}$. One observes that $\beta(v)$ increases as v moves through the first k_0 annuli, and after that $\beta(v)$ decreases until it reaches ~ 0. In fact, for $v \in A_k$ and $k \leq k_0$ we have $1 + \beta(v) \sim 1 + \beta(v_0) + k$. For $k \geq k_0$ we have $1 + \beta(v) \sim \max\{1, \beta(v_0) + 2k_0 - k\}$. We let $l \in \mathbb{N}$ be the smallest integer for which

$$\gamma \subset \bigcup_{k=1}^{l} A_k.$$

The rest of the proof is used to show that l is comparable to $C\beta(v_0)$. We let $\gamma_k = \gamma \cap A_k$ and we need to consider only the case when $k_0 < l$. Then using hypothesis 2) we estimate as follows.

$$\begin{aligned}
\int_\gamma \lambda_\Omega(v)|dv| = \sum_{k=1}^{l} \int_{\gamma_k} \lambda_\Omega(v)|dv| &\sim \sum_{k=0}^{k_0} \int_{\gamma_k} \frac{|dv|}{\operatorname{dist}(v,E)(1+\beta(v_0)+k)} \\
&+ \sum_{k=k_0}^{l} \int_{\gamma_k} \frac{|dv|}{\operatorname{dist}(v,E)(1+\beta(v_0)+2k_0-k)} \\
&\sim \sum_{k=0}^{k_0} \frac{1}{\beta(v_0)+k} + \sum_{k=k_0}^{l} \frac{1}{\beta(v_0)+2k_0-k} \\
&\sim \left| \log \frac{(\beta(v_0)+k_0)^2}{\beta(v_0)(\beta(v_0)-l+2k_0)} \right|.
\end{aligned}$$

Next using that $\int_\gamma \lambda_\Omega(v)|dv| \geq 1/4$ we obtain

$$\beta(v_0)(\beta(v_0) - l + 2k_0)e^{1/C} \leq (\beta(v_0) + k_0)^2.$$

A simple calculation, using $k_0 \leq l$, gives $l \geq \beta(v_0)/2$. Hypothesis (3) gives the estimate

$$\frac{|\gamma(1) - a|}{|\gamma(0) - a|} \leq 2^{-l/C}.$$

Combining this with $2^{-l/C} < 2^{-\beta(v_0))/2C}$ gives the first conclusion of the lemma when we "see" P1 at v_0 and $k_0 < l$. Finally we remark that the above line of inequalities can be reversed and we obtain also

$$\int_\gamma \lambda_\Omega(v)dv \geq \left|\log \frac{(\beta(v_0)) + k_0)^2}{\beta(v_0))(\beta(v_0) - l + 2k_0)}\right|.$$

Hence if $\int \lambda_\Omega(v) < 4$ then, by a simple calculation, $l \leq C\beta(v_0)$. This gives the second conclusion of Lemma 2. If we see P2 at v_0 then

$$1 + \beta(v) \sim \max\{1, \beta(v_0) - k\},$$

for all k, and $v \in A_k$. Hence this case corresponds to $k_0 = 0$ in the above consideration, and the above calculation can simply be repeated, setting $k_0 = 0$.

■

Proof of Proposition 2. We are given $q \in \mathbb{D}$. The first part of the proof consists of constructing the points $w \in \mathbb{D}$, $v \in \bar{\mathbb{D}}$. The construction is based on the following estimate which holds when $M(q) \geq 1$,

$$\frac{1}{CM(q)} \leq \inf_{g \in G} d_{\mathbb{D}}(q, g(q)) \leq \frac{C}{M(q)}. \tag{3.3}$$

The right hand side of (3.3) follows from Lemma 1 and Koebe's distortion estimate by rescaling. The left hand side is obtained from univalence criteria by rescaling. See [M, Proposition 1.3] for an elementary univalence criterion that suffices here.

Now we select a group element $g \in G$ such that $d_{\mathbb{D}}(q, g(q)) \leq CM(q)^{-1}$. As G does not contain elliptic elements, there are either one or two fixed points of g on $\mathbb{T}$. Each case requires a different construction to obtain w, v.

We first treat the case where g has two fixed points in $\mathbb{T}$. Let $\zeta_1, \zeta_2 \in \mathbb{T}$ be the fixed points of g, and let A be the hyperbolic geodesic connecting ζ_1 to ζ_2. We let $S(q)$ be the hypercycle in $\bar{\mathbb{D}}$ which contains ζ_1, ζ_2 and q. Now we let $K \subseteq \mathbb{D}$ be the region which is bounded by the axis A of g and the interval $I \subset \mathbb{T}$, $m(I) \leq m(\mathbb{T})/2$, whose endpoints are ζ_1, ζ_2. We consider the hypercycle

$$S_0 = \{s \in K : \sinh(d_{\mathbb{D}}(s, g(s))) = \sinh(d_{\mathbb{D}}(q, g(q)))M(q)/M_0\}$$

and the ray R that connects $0 \in \mathbb{D}$ to the midpoint of I. Note that the hypercycle S_0 is well defined; it lies underneath the axis A, and also underneath $S(q)$. Depending on the position of q relative to A the hypercycle $S(q)$ may be above or underneath the axis A. We point out however that when we apply Proposition 2 the hypercycle $S(q)$ will be above the axis A, and the point q we use will be close to the top of $S(q)$. (See Lemma 3 below.) Now we define

$$w = R \cap S(q), \qquad v = R \cap S_0. \tag{3.4}$$

We turn to the case when $g \in G$ has one fixed point $\zeta_1 \in \mathbb{T}$. The first step is again the construction of $w \in \mathbb{D}$, $v \in \mathbb{T}$. We let $S(q)$ be the horocycle through $q \in \mathbb{D}$ and $\zeta_1 \in \mathbb{T}$. Without loss of generality we may assume that $0 \in \mathbb{D}$ is not contained in the disk bounded by $S(q)$. Then we define

$$w = S(q) \cap (0, \zeta_1), \qquad v = \zeta_1. \tag{3.5}$$

Again we point out that we will only apply this when q is near the top of the horocycle.

The following properties of w, v are easily verified:

$$C^{-1} \leq M(w)/M(q) \leq C, \tag{3.6}$$

$$\text{if } |v| < 1, \text{ then } C^{-1} \leq M(v)/M_0 \leq C, \tag{3.7}$$

$$1 - |v|^2/1 - |w|^2 \leq 2^{-M(q)+M_0}, \tag{3.8}$$

$$|u(q) - u(w)| \leq C + |\log((1 - |w|^2)/(1 - |q|^2))|. \tag{3.9}$$

As $S(q), S_0$ are levelsets for $s \mapsto \sinh d_{\mathbb{D}}(s, g(s))$, (3.6) and (3.7) follow from (3.3). Condition (3.8) is a consequence of elementary circle geometry. To verify (3.9) we exploit group invariance of P. We choose $m \in \mathbb{Z}$ so that for $k = g^m$

$$d_{\mathbb{D}}(k(q), w) \leq CM^{-1}(q). \tag{3.10}$$

This is possible by (3.3). As $P = P \circ k$ we obtain $k'(q)P'(k(q)) = P'(q)$. Consequently

$$\log|P'(q)| - \log|P'(k(q))| = \log|k'(q)|,$$

and $1 - |w|^2/2(1-|q|^2) \leq |k'(q)| \leq 1 - |w|^2/1 - |q|^2$. By (3.10) we have

$$|u(k(q)) - u(w)| \leq M(w)d(w, k(q)) < C.$$

Clearly, the last two estimates give (3.9):

$$|u(w) - u(q)| \leq C + |\log((1-|w|^2)/(1-|q|^2))|.$$

So far we have verified conditions a) – d) of Proposition 2. The remaining condition e) follows from our next proposition.

We let R be the radial line segment connecting w and v, that is, $R = (w, v)$. When a point moves along R towards the boundary of $\mathbb{D}$ we observe the following decrease of $u = \log|P'|$:

Proposition 5 *If $z_1, z_2 \in R$ satisfy $1/32 \leq 1 - |z_2|/1 - |z_1| \leq 1/4$, then $u(z_2) - u(z_1) \leq -M(z_1)/C + C$, where $C > 0$ is universal.*

Proof. By choice of R, the line segment $t \mapsto z_1 + t(z_2 - z_1)$ minimizes the $\lambda_{\mathbb{D}}-$distance between the hypercycles (respectively horocycles) $S(z_1)$ and $S(z_2)$. Therefore among all curves connecting $P(z_1)$ and $P(z_2)$ the following,

$$\gamma : t \mapsto P(z_1 + t(z_2 - z_1)),$$

has minimal length with respect to the hyperbolic metric on $\mathbb{C}\setminus E$. And so γ satisfies conditions 1) – 4) of Lemma 2, with $\gamma(0) = v_0 = P(z_1)$ and $\gamma(1) = P(z_2)$. To verify condition 2 of Lemma 2 we first note that for each A_k and $z, z' \in A_k$,

$$C^{-1}\lambda_\Omega(z) \leq \lambda_\Omega(z') \leq C\lambda_\Omega(z).$$

If condition 2 would fail then we could make a new curve with the same initial point and same last point as γ, and such that the hyperperbolic length of this new curve is less than the hyperbolic length of γ. The same argument proves also that condition 3 holds.

Applying Lemma 2 to our curve γ gives the following estimates.

$$\beta(P(z_2)) \leq C\beta(P(z_1)),$$

and

$$|a - P(z_2)|/|a - P(z_1)| \leq C2^{-\beta(z_1)/C}.$$

Combining these estimates with (3.1) and (3.2) we obtain

$$\begin{aligned}
\frac{|P'(z_2)|}{|P'(z_1)|} &= \frac{\lambda_\Omega(P(z_1))(1-|z_1|^2)}{\lambda_\Omega(P(z_2))(1-|z_2|^2)} \\
&\leq C\frac{|a-P(z_2)|}{|a-P(z_1)|}\frac{(\beta(P(z_2))+1)}{(\beta(P(z_1))+1)} \\
&\leq C2^{-\beta(P(z_1))/C}.
\end{aligned}$$

We remark that by rescaling and normal families $M(z_1) \leq C\beta(P(z_1))$; this completes the proof of Proposition 5.

■

Finally we conclude the proof of Proposition 2: Conditions a) – d) of Proposition 2 follow from (3.5) – (3.8). We will now verify condition e), using Proposition 5, Lemma 1, (3.9) and (3.10).

Let $\Lambda \in \Gamma(w, v, L)$ and choose $w_1, w_2 \in \Lambda$ such that $(1-|w_1|)/(1-|w_2|) > 4$. As above we denote $R = (w, v)$. Let us first treat the case when $|v| = 1$. In that case $\Gamma(w, v, L)$ contains only one element namely R, and applying Proposition 5 to $\Lambda = R = (w, v)$ gives condition e) of Proposition 2.

Next we consider the case when $|v| < 1$. This condition implies that our group element g has two fixed points $\zeta_1, \zeta_2 \in \mathbb{T}$. For $i \in \{1, 2\}$ we let $z_i \in R$ be the top of the hypercycle containing w_i and the fixed points $\zeta_1, \zeta_2 \in \mathbb{T}$. As in (3.6) we have $M(w_i)/C \leq M(z_i) \leq M(w_i)C$. Combining (3.9) and (3.10) we obtain $|u(z_i) - u(w_i)| \leq CL$. Applying Proposition 5 to z_1, z_2 gives $u(z_2) - u(z_1) \leq -M(z_1)/C + C$. Summing up we obtain that

$$u(w_2) - u(w_1) \leq -M(w_1)/C_2 + C_2L.$$

■

We will now link the Lipschitz domains of Section 2 to elements of the above construction. Recall that we have isolated the following connected subset on the boundary of our Lipschitz domain $W(\zeta)$,

$$F(\zeta) = \{w \in \partial W(\zeta) : |\zeta/|\zeta| - w| < 2^L(1-|\zeta|) \text{ and } 1-|w| \leq (1-|\zeta|)/2\}.$$

We recall also that for $q \in \mathbb{D}$ we started the proof of Proposition 2 by selecting a group element $g \in G$ satisfying $d_{\mathbb{D}}(q, g(q)) \leq CM(q)^{-1}$. Then we defined $S(q)$ to be the hypercycle containg q and the fixed points ζ_1, ζ_2 of g, when g was hyperbolic. In the case of a parabolic g, $S(q)$ was the horocycle through q that was tangent to $\mathbb{T}$ at the (sole) fixed point of g. In our next lemma we will utilize again that $W(\zeta)$ is the result of stopping time arguments, and we find that for $q \in F(\zeta)$ the top of $S(q)$ is close to q, whenever $M(q)$ is a large constant.

Lemma 3 *Let $q \in F(\zeta)$, and assume that $M(q) \geq M_1/2^{C_0 L}$. Let $w \in \mathbb{D}$ be the top of $S(q)$. Then in $\mathbb{D}$ the hyperbolic distance between q and w is bounded by $C_4 L$.*

Proof. We assume to the contrary that the lemma is false. Under this assumption we will construct a long sequence of points $w_i \in W(\zeta)$ so that $M(w_0) \geq M_1/C_2 2^{LC_0}$ and $M(w_i) \geq 2^i M(w_0)$. On the other hand the points $w_i \in W(\zeta)$ satisfy the stopping time condition $M(w_i) \leq M_1/LA_0$. This gives a contradiction when the sequence of points is long enough.

Now we assume that $d_{\mathbb{D}}(q, w) > CL$ for arbitrary large C. We let R_0 be the straight line segment $R \cap W(\zeta)$ where R is the straight line connecting w to v. We recall that $0, w$ and v are points on the same radial ray. As $d_{\mathbb{D}}(q, w) > CL$, there exists $\tau > 0$ depending only on the Lipschitz constants of $W(\zeta)$, such that the hyperbolic diameter of R_0 is $\geq \tau CL$. Therefore we find points $w_0 = w, w_1, \ldots, w_{i_0}$ on R_0 with $1 - |w_{i+1}|^2/1 - |w_i|^2 < \eta$ and $i_0 \geq \eta\tau CL$. It follows from [Be, Section 7.35] and an elementary calculation that the displacement function decreases at a geometric rate on R_0. Hence

$$d_{\mathbb{D}}(w_{i+1}, g(w_{i+1})) \leq \eta d_{\mathbb{D}}(w_i, g(w_i)), \qquad i \leq i_0.$$

If moreover $\eta > 0$ is small enough, it follows from (3.3) that

$$M(w_i) \geq 2^i M(w_0), \qquad i < i_0. \tag{3.11}$$

Finally, it follows from our hypothesis on $M(q)$ and condition (a) of Proposition 2, that

$$M(w_0) \geq M_1/C_2 2^{C_0 L}. \tag{3.12}$$

On the other hand, in Section 2 the stopping time Lipschitz domain was constructed such that for $w_i \in W(\zeta)$, we have $M(w_i) \leq M_1/LA_0$. This contradicts (3.11) and (3.12) for i_0 large enough, and the assumption was that we can make i_0 as large as we please.

∎

4 Selecting good rays

In this section we first prove Proposition 3 and then Theorem 1. The inductive construction of the points $\{s_k\}$ in Propoosition 3 is based on repeated application of Proposition 1 and 2. These propositions can interact when the constants M_0, M_1, L are specified as follows. We recall that we have imposed the lower bound $L > 4 + 4K_1/A_0$ in Section 2 during the construction of the domains $W(\zeta)$, and that later, in the remark following the proof of Proposition 4, we have chosen L such that also $L > 2C^2A_0/c_0^2$. Now we let $M_0 > 1$ be such that

$$-M_0/C_2 + C_2 \le -M_0/2C_2 \le -1, \tag{4.1}$$

where $C_2 \ge 1$ is the constant apearing in Proposition 2. Finally we take M_1 large enough so that $M_1/2^{C_0L} \ge 2M_0$ and

$$-M_1/LC_1 + 4C_4L \le -M_1/2LC_1. \tag{4.2}$$

We will verify Proposition 3 with $C_3 = \max\{4+4K_1/A_0, 2C^2A_0/c_0^2\}$ and $M = 4C_1^2C_2L^2$. The proof begins with the inductive construction of the sequence $\{s_k\}$. Assuming, as we may that for $u = \log|P'|$, $u(0) = 0$, and $|\nabla u(0)| = 1$ we take $s_0 = 0$. We assume that $s_0, \dots, s_n$ have been constructed such that the conclusion of Proposition 3 holds, and such that $M(s_n) \le M_1/2^{C_0L}$. Now we determine s_{n+1} as follows.

We start by constructing the stopping time Lipschitz domain $W(s_n)$ and apply Proposition 1, to obtain $q \in F(s_n)$ such that

$$u(q) - u(s_n) \le -M_1/C_1L, \tag{4.3}$$

when $|q| < 1$, and

$$\int_\gamma |\nabla u(z)||dz| \le M_1LC_1, \tag{4.4}$$

for $\gamma \in \Gamma(s_n, q, L)$. Now we consider three cases:

1. If $|q| = 1$ then we put $s_{n+1} = q$ and we stop the construction.

2. If $|q| < 1$ and if $M(q) \le M_1/2^{C_0L}$ then we put $s_{n+1} = q$. By (4.3) and (4.4) the induction step is completed. We may continue with the construction of the next point.

3. If $|q| < 1$ and $M(q) > M_1/2^{C_0 L}$ then we apply Proposition 2 to $q \in \mathbb{D}$ and obtain $w \in \mathbb{D}$, $v \in \bar{\mathbb{D}}$ for which the conclusion of Proposition 2 hold. We define $s_{n+1} = v$. In the next paragraph we will verify that s_{n+1} satisfies the conclusion of Proposition 3.

The assumption in the third case is that $M(q) > M_1/2^{C_0 L}$. By Lemma 3 this implies that $d_{\mathbb{D}}(w, q) \leq C_4 L$. We fix $\gamma \in \Gamma(s_n, s_{n+1}, L)$, and we let $\sigma = \gamma \cap \{s : |z_n| < |s| < |q|\}$ and $\rho = \gamma \cap \{s : |w| < |s| < |s_{n+1}|\}$. Note that $\gamma = \sigma \cup \rho$. We estimate the difference $u(s_{n+1}) - u(s_n)$ by breaking it into three pieces: Recalling that $s_{n+1} = v$ and Proposition 2 (e) give

$$u(s_{n+1}) - u(w) \leq -\frac{1}{M} \int_\rho |\nabla u(z)||dz|.$$

Lemma 3 together with Proposition 2 (b) gives $|u(w) - u(q)| \leq C_2 + C_4 L \leq 2C_4 L$, and (4.1) – (4.4) imply

$$u(q) - u(s_n) + 2C_4 L \leq -\frac{1}{M} \int_\sigma |\nabla u(z)||dz|.$$

Summing up we have,

$$\begin{aligned} u(s_{n+1}) - u(s_n) &\leq u(s_{n+1}) - u(w) + u(w) - u(q) + u(q) - u(s_n) \\ &\leq -\frac{1}{M} \left(\int_\sigma |\nabla u|(z)|dz| + \int_\rho |\nabla u|(z)|dz| \right) \\ &\leq -\frac{1}{M} \int_\gamma |\nabla u(z)||dz|. \end{aligned}$$

Finally we have to distinguish between the cases $|v| = |s_{n+1}| = 1$ and $|v| = |s_{n+1}| < 1$. If $|s_{n+1}| = 1$ then we stop the construction, and Proposition 3 is true in that case. If $|s_{n+1}| < 1$ then by Proposition 2 (c) we have $M(s_{n+1}) \leq M_0 2 \leq M_1/2^{C_0 L}$, and we may continue to construct the next point. This completes the proof of Proposition 3.

■

We turn to the proof of Theorem 1. Let $\{s_k\}$ be the sequence of points given by Proposition 3. This sequence converges to a point in $\mathbb{T}$; we denote its limit by $e^{i\beta}$. Now we let $R = (0, e^{i\beta})$ be the ray connecting 0 to $e^{i\beta}$. We will show that *uniformly* on R the radial variation of u is of the smallest possible order. More precisely we will verify that for any $\xi \in R$,

$$u(\xi) \leq -\frac{1}{M} \int_{(0,\xi)} |\nabla u(z)||dz| + MM_1,$$

where M_1 has been chosen in (4.2) and M is the constant appearing in Proposition 3. We decompose $R = (0, e^{i\beta})$ as

$$R = \bigcup \gamma_k,$$

where $\gamma_k = R \cap \{s \in \mathbb{D} : |s_k| \le |s| \le |s_{k+1}|\}$. Note that by condition b) of Proposition 3 the straight line segment γ_k belongs to $\Gamma(s_k, s_{k+1}, L)$. Next we choose an arbitrary point $\xi \in R$. Let $k_0 \in \mathbb{N}$ be such that $\xi \in \gamma_{k_0}$. We will treat two cases depending on how s_{k_0+1} was obtained during the proof of Proposition 3. In the first case s_{k_0+1} was obtained by an application of Proposition 1. As $\xi \in \gamma_{k_0}$ it follows from condition a) of Proposition 1 that,

$$|u(\xi) - u(s_{k_0})| \le \int_{\gamma_{k_0}} |\nabla u(z)||dz| \le C_1 L M_1.$$

Summing a telescoping series we obtain from Proposition 3,

$$u(s_{k_0}) - u(0) \le -\sum_{l=0}^{k_0-1} \frac{1}{M} \int_{\gamma_l} |\nabla u(z)||dz|.$$

We let $\rho = R \cap \{s : |s| < |s_{k_0}|\}$. Now we estimate the difference $u(\xi) - u(0)$ by adding the last two inequalities.

$$\begin{aligned} u(\xi) - u(0) &= u(\xi) - u(s_k) + u(s_k) - u(0) \\ &\le \int_{\gamma_{k_0}} |\nabla u(z)||dz| - \frac{1}{M} \int_\rho |\nabla u(z)||dz| \\ &\le C_1 L M_1 - \frac{1}{M} \int_{(0,\xi)} |\nabla u(z)||dz|. \end{aligned}$$

In the second case s_{k_0+1} was obtained by an application of Proposition 2. This means the following: Applying Proposition 1 to s_{k_0} gives $q \in F(s_{k_0})$ with $M(q) \ge M_1/2^{C_0 L}$; applying Proposition 2 to q gives $w \in \mathbb{D}$, $v \in \bar{\mathbb{D}}$ and $s_{k_0+1} = v$, $M(s_{k_0+1}) \le 2M_0$.

We distinguish between the cases $(1-|w|)/(1-|\xi|) < 4$ and $(1-|w|)/(1-|\xi|) \ge 4$. In the first case we estimate $u(\xi) - u(w) \le 4M(q) \le M_1$. Combining condition b) of Proposition 2 with Lemma 3 and condition b) of Proposition 1 gives

$$u(w) - u(s_{k_0}) \le -M_1/C_1 + 4LC_4.$$

Now we let $\rho = R \cap \{s : |s| < |s_{k_0}|\}$, and using Proposition 3 we estimate as follows.

$$\begin{aligned} u(\xi) - u(0) &= u(\xi) - u(w) + u(w) - u(s_{k_0}) + u(s_{k_0}) - u(0) \\ &\le -\frac{1}{M} \int_\rho |\nabla u(z)||dz| - M_1/2C_1 + M_1 \\ &\le -\frac{1}{M} \int_{(0,\xi)} |\nabla u(z)||dz| + M_1. \end{aligned}$$

Finally we consider the case where $(1-|w|)/(1-|\xi|) \ge 4$. By Proposition 2 (e),

$$u(\xi) - u(w) \le -\frac{1}{M} \int_\sigma |\nabla u(z)||dz|,$$

where $\sigma = R \cap \{s : |w| < |s| < |\xi|\}$. We let $\rho = R \cap \{s : |s| < |s_{k_0}|\}$, then $(0, \xi) = \sigma \cup \rho$. Hence using Proposition 2 (b), Lemma 3 and Proposition 3 we obtain the following estimate

$$\begin{aligned}
u(\xi) - u(0) &= u(\xi) - u(w) + u(w) - u(s_{k_0}) + u(s_{k_0}) - u(0) \\
&\leq -\frac{1}{M} \int_{\sigma} |\nabla u(z)||dz| + 2C_4 L - \frac{1}{M} \int_{\rho} |\nabla u(z)||dz| \\
&\leq -\frac{1}{M} \int_{(0,\xi)} |\nabla u(z)||dz| + 2C_4 L.
\end{aligned}$$

This completes the proof of Theorem 1.

■

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