Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint-Nr.: 40 1998

BERNSTEIN TYPE THEOREMS FOR HIGHER CODIMENSION

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Abstract. We show a Bernstein theorem for minimal graphs of arbitrary dimension and codimension under a bound on the slope that improve previous results and is independent of the dimension and codimension. The proof depends on the regularity theory for the harmonic Gauss map and the geometry of Grassmann manifolds.

1. Introduction

The celebrated theorem of Bernstein says that the only entire minimal graphs in Euclidean 3−space are planes. More precisely, let $z = f(x, y)$ be a smooth function defined on all of $\mathbb{R}^2$, whose graph in $\mathbb{R}^3$ is a minimal surface. Then f is a linear function, and the graph is a plane.

The efforts to generalize Bernstein's theorem to higher dimensions led to profound developments in analysis and geometric measure theory. The final result in a successive series of achievements by several mathematicians was the theorem of J. Simons [S] that an entire mimimal graph has to be planar for dimension≤ 7, while Bombieri, de Giorgi and Giusti [B-G-G] shortly afterwards produced a counterexample to such an assertion in dimension 8 and higher. By way of contrast, J. Moser [M] had earlier proved a Bernstein type result in arbitrary dimension under the additional assumption that the slope of the graph is uniformly bounded.

All these results hold for hypersurfaces, i.e. minimal graphs of codimension 1.

For higher codimension, the situation becomes more complicated, and Lawson-Osserman [L-O] had given explicit counterexamples to Bernstein type results in higher codimension: for example they showed that the cone over a Hopf map is an entire Lipschitz solution to the minimal surface system.

On the other hand, Hildebrandt-Jost-Widman [H-J-W] had obtained a Bernstein type result in arbitrary codimension under the assumption of a certain quantitative bound for the slope. Their method was to show that the Gauss map of

*Thanks the Max Planck Institute for Mathematics in Sciences in Leipzig for providing good conditions during the preparation of this paper , and also NNSFC and SFECC for support

such a minimal submanifold which was known to be a harmonic map into a Grassmannian has to be constant, if its range is contained in a sufficiently restricted set of the Grassmannian. This restriction required the bound on the slope.

In fact, that result applies not only to minimal graphs but also to ones of parallel mean curvature as the Gauss map continues to stay harmonic under that condition. Actually, Chern [C] had shown that a hypersurface in Euclidean space which is an entire graph of constant mean curvature necessarily is a minimal hypersurface. Thus, by Simons' theorem, it is a hyperplane for dimension≤ 7. Chern's result was generalized by Chen-Xin [C-X].

The proof of Hildebrandt-Jost-Widman derived Hölder estimates for harmonic maps with values in Riemannian manifolds with an upper bound for sectional curvature and by a scaling argument then concluded a Liouville type theorem for harmonic maps under assumptions including the above mentioned harmonic Gauss maps. The Hölder estimates needed a bound on the radius of the image, and examples show that [H-J-W] had achieved the optimal bound in the general framework for that paper. Nevertheless, when applied to the special case of interest here, namely harmonic maps with values in Grassmannians, it turns out that the bound for the slope reqired in [H-J-W] for their Bernstein type result depends on the dimension and codimension of the minimal graphs. In the present paper, we obtain such a Bernstein type result under a bound for the slope which is better than the one in [H-J-W] and independent of dimension and codimension.

Theorem 1. *Let $z^i = f^i(x_1, \cdots, x_n),\ i = 1, \cdots, m$ be smooth functions defined everywhere in $\mathbb{R}^n$. Suppose their graph $M = (x, f(x))$ is a submanifold with parallel mean curvature in $\mathbb{R}^{m+n}$. Suppose that there exists a number β_0 with*

$$\beta_0 < \begin{cases} 2, & \text{when} \quad m \geq 2, \\ \infty & \text{when} \quad m = 1; \end{cases} \tag{1.1}$$

such that

$$\Delta_f \leq \beta_0 \qquad \text{for all} \quad x \in \mathbb{R}^n, \tag{1.2}$$

where

$$\Delta_f(x) = \{det(\delta_{\alpha\beta} + \sum_i f^i_{x_\alpha}(x) f^i_{x_\beta}(x))\}^{\frac{1}{2}}. \tag{1.3}$$

Then $f^1, \cdots, f^m$ are linear functions on $\mathbb{R}^n$ representing an affine $n-$plane in $\mathbb{R}^{m+n}$.

The case $m = 1$ of course is simply Moser's result.

In the framework of geometric measure theory the Bernstein problem is transfered to the rigidity problem of $m-$dimensional minimal submanifolds M in the sphere S^{m+n}. Namely, it follows from the general work of Allard [A] on varifolds that the tangent cone at infinity of such an $m-$dimensional submanifold of $\mathbb{R}^{m+n+1}$ is the cone over a minimal submanifold of S^{m+n}. J. Simons [S] proved that a compact minimal submanifold of the sphere whose normal planes lie in a sufficiently small neighborhood is a totally geodesic subsphere. Reilly [R] and Fischer-Colbrie [FC] improved the previous work successively. Like [H-J-W], however, these results require bounds that depend on dimension and codimension, and we are able here to get a similar generalizations as above.

Theorem 2. *Let M be an $m-$dimensional compact or simple*[1] *Riemannian manifold which is a minimal submanifold in S^{m+n}. Suppose that there is a fixed oriented $n-$plane P_0 and a number γ_0,*

$$\gamma_0 > \begin{cases} \frac{1}{2} & if \quad min(m,n) \geq 2, \\ 0 & if \quad min(m,n) = 1; \end{cases}$$

such that

$$\langle P, P_0 \rangle \geq \gamma_0$$

holds for all normal $n-$planes P of M in S^{m+n}. Then M is contained in a totally geodesic subsphere of S^{m+n}.

As mensioned, the bounds in our results (2 in Thm. 1 and $\frac{1}{2}$ in Thm. 2) are sharper than the ones previously known. As the examples of Lawson-Osserman show some such bound is necessary. However, these counterexamples are not sharp for our bounds, and so one may ask whether our bounds are optimal. This is probably not the case because the essential point in our argument is to construct a strictly convex function on a sufficiently large region in a Grassmannian. The question of finding the optimal bound - at least for the strategy pursued here - then amounts to finding the largest such convex supporting region in a Grassmannian. Geometric intuition suggests that such an optimal region may on one hand be larger than the one constructed in our paper, but on the other hand by no means as explicitely presentable as ours and possibly also nonunique in a general way. In other words, the optimal bound will probably turn out to be some number that may virtually be impossible to write down explicitely, and thus perhaps also be of comparatively little significance.

Our general strategy is the same as in [H-J-W]. By using the Ruh-Vilms theorem [R-V], we prove the harmonic Gauss map is constant under certain conditions. The main refinement in this paper is a detailed study of geodesic convex sets in the Grassmannian manifold $G_{n,m}$. We define a new specific geodesic convex set B_G in $G_{n,m}$ which is larger than the usual geodesic convex ball and interesting in its own right. We expect further applications.

2. Preliminaries

In this section we describe some basic notions and results on harmonic maps and submanifols in the Euclidean space and the sphere which will be used in later sections.

Let (M, g) and (N, h) be Riemannian manifolds with metric tensors g and h, respectively. Harmonic maps are described as critical points of the following energy functional

$$E(f) = \frac{1}{2} \int_M e(f) * 1, \tag{2.1}$$

[1] as defined below

where $e(f)$ stands for the energy density. The Euler-Lagrange equation of the energy functional is

$$\tau(f) = 0, \tag{2.2}$$

where $\tau(f)$ is the tension field. In local coordinates

$$e(f) = g^{ij}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}h_{\beta\gamma}, \tag{2.3}$$

$$\tau(f) = (\Delta_M f^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j})\frac{\partial}{\partial y^\alpha}, \tag{2.4}$$

where $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbols of the target manifold N. Here and in the sequel we use the summation convention. For more details on harmonic maps consult [E-L]

A Riemannian manifold M is said to be simple, if it can be described by coordinates x on $\mathbb{R}^n$ with a metric

$$ds^2 = g_{ij}dx^i dx^j, \tag{2.5}$$

for which there exist positive numbers λ and μ such that

$$\lambda|\xi|^2 \le g_{ij}\xi^i\xi^j \le \mu|\xi|^2 \tag{2.6}$$

for all x and ξ in $\mathbb{R}^n$. In other words, M is topologically $\mathbb{R}^n$ with a metric for which the associated Laplace operator is uniformly elliptic on $\mathbb{R}^n$.

Hildebrandt-Jost-Widman proved a Liouville-type theorem for harmonic maps in [H-J-W]:

Theorem 2.1. *Let f be a harmonic map from a simple or compact Riemannian manifold M into a complete Riemannian manifold N, the sectional curvature of which is bounded above by a constant $\kappa \ge 0$. Denote by $B_R(Q)$ a geodesic ball in N with radius $R < \frac{\pi}{2\sqrt{\kappa}}$ which does not meet the cut locus of its center Q. Assume also that the range $f(M)$ of the map f is contained in $B_R(Q)$. Then f is a constant map.*

By using the composition formula for the tension field, one easily verifies that the composition of a harmonic map $f : M \to N$ with a convex function $\phi : f(M) \to \mathbb{R}$ is a subharmonic function on M. The maximum principle then implies

Proposition 2.2. *Let M be a compact manifold without boundary, $f : M \to N$ a harmonic map with $f(M) \subset V \subset N$. Assume that there exists a strictly convex function on V. Then f is a constant map.*

Let $M \to \mathbb{R}^{m+n}$ be an n-dimensional oriented submanifold in Euclidean space. For any point $x \in M$, by parallel translation in the ambient Euclidean space, the tangent space T_xM is moved to the origin of $\mathbb{R}^{m+n}$ to obtain an n-subspace in $\mathbb{R}^{m+n}$, namely, a point of the Grassmannian manifold $\gamma(x) \in G_{n,m}$. Thus, we define a generalized Gauss map $\gamma : M \to G_{n,m}$. E. Ruh and J. Vilms discovered the relation between the property of the submanifold and the harmonicity of its Gauss map in [R-V].

Theorem 2.3. *Let M be a submanifold in $\mathbb{R}^{m+n}$. Then the mean curvature vector of M is parallel if and only if its Gauss map is a harmonic map.*

Let $M \to S^{m+n} \hookrightarrow \mathbb{R}^{m+n+1}$ be an m-dimensional submanifold in the sphere. For any $x \in M$, by parallel translation in $\mathbb{R}^{m+n+1}$, the normal space N_xM of M in S^{m+n} is moved to the origin of $\mathbb{R}^{m+n+1}$. We then obtain an n-subspace in $\mathbb{R}^{m+n+1}$. Thus, the so-called normal Gauss map $\gamma : M \to G_{n,m+1}$ has been defined. There is a natural isometry η between $G_{n,m+1}$ and $G_{m+1,n}$ which maps any n-subspace into its orthogonal complementary $(m+1)$-subspace. The map $\eta^* = \eta \circ \gamma$ maps any point $x \in M$ into an $(m+1)$-subspace consisting of T_xM and the position vector of x.

On the other hand, to study properties of the submanifold M in the sphere we may investigate the cone CM generated by M. CM is the image under the map from $M \times [\,0,\infty\,)$ into $\mathbb{R}^{m+n+1}$ defined by $(x,t) \to tx$, where $x \in M$, $t \in [\,0,\infty\,)$. CM has a singularity at $t = 0$. To avoid the singularity we consider the truncated cone CM_ε, which is the image of $M \times [\,\varepsilon,\infty\,)$ under the same map, where ε is any positive number.

J. Simons showed in [S] that M is a minimal m-submanifold if and only if CM_ε is an $(m+1)$-dimensional minimal cone. In fact, we have (see [X], Prop 3.2):

Proposition 2.4. *CM_ε has parallel mean curvature in $\mathbb{R}^{m+n+1}$ if and only if M is a minimal submanifold in S^{m+n}.*

From Theorem 2.3 and Proposition 2.4 it follows that

Proposition 2.5. *M is a minimal $m-$dimensional submanifold in the sphere S^{m+n} if and only if its normal Gauss map $\gamma : M \to G_{n,m+1}$ is a harmonic map.*

3. Geometry of Grassmannian Manifolds

Let N be a Riemannian manifold with curvature tensor $R(\cdot,\cdot)$.

Let γ be a geodesic issueing from x_0 with $\gamma(0) = x_0$ and $\gamma(t) = x$, where t is the arc length parameter. Define a self-adjoint map

$$R_{\dot\gamma} : w \to R(\dot\gamma, w)\dot\gamma. \tag{3.1}$$

Let v be a unit eigenvector of $R_{\dot\gamma(0)}$ with eigenvalue μ and $\langle v, \dot\gamma(0)\rangle = 0$. Let $v(t)$ be the vector field obtained by parallel translation of v along γ. In the case of N being a locally symmetric space with nonnegative sectional curvature, $v(t)$ is an eigenvector of $R_{\dot\gamma(t)}$ with eigenvalue $\mu \geq 0$, namely

$$R(\dot\gamma(t), v(t))\dot\gamma = \mu v(t).$$

Thus,

$$J(t) = \begin{cases} \frac{1}{\sqrt{\mu}}\sin(\sqrt{\mu}t)v(t), & \text{when} \quad \mu > 0 \\ t\,v(t), & \text{when} \quad \mu = 0 \end{cases}$$

is a Jacobi field along $\gamma(t)$ with $J(0) = 0$. On the other hand, the Hessian of the distance function r from x_0 can be computed by those Jacobi fields. Now, we assume γ is a geodesic without a conjugate point up to distance r from x_0. For orthonormal vectors $X, Y \in T_{\gamma(r)}B_r(x_0)$ there exist unique Jacobi fields J_1 and J_2 such that

$$J_1(0) = J_2(0) = 0,\ J_1(r) = X,\ J_2(r) = Y,$$

since there is no conjugate point of x_0 along γ. We then have

$$\begin{aligned}
\mathrm{Hess}(r)(X,Y) &= \langle \nabla_X \dot\gamma, Y\rangle \\
&= \langle \nabla_{J_1}\dot\gamma, J_2\rangle|_{\gamma(0)}^{\gamma(r)} \\
&= \int_0^r \frac{d}{dt}\langle \nabla_{J_1}\dot\gamma, J_2\rangle\, dt \\
&= \int_0^r (\langle \nabla_{\dot\gamma}\nabla_{J_1}\dot\gamma, J_2\rangle + \langle \nabla_{J_1}\dot\gamma, \nabla_{\dot\gamma}J_2\rangle)dt \\
&= \int_0^r (\langle R(J_1,\dot\gamma)\dot\gamma, J_2\rangle + \langle \nabla_{\dot\gamma}J_1, \nabla_{\dot\gamma}J_2\rangle)dt \\
&= \int_0^r \left(\frac{d}{dt}\langle \nabla_{\dot\gamma}J_1, J_2\rangle - \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}J_1 + R(\dot\gamma, J_1)\dot\gamma, J_2\rangle\right) dt \\
&= \langle \nabla_{\dot\gamma}J_1, J_2\rangle
\end{aligned}$$

Assume that μ_i and $v_i(t)$ are eigenvalues and orthonormal eigenvectors of $R_{\dot\gamma(t)}$. Then

$$J_i(t) = \frac{1}{\sqrt{\mu_i}}\sin(\sqrt{\mu_i}t)v_i(t)$$

are $n-1$ orthogonal Jacobi fields, where $\mu_i > 0$.

$$\begin{aligned}
\mathrm{Hess}(r)(J_i, J_j) &= \langle \nabla_{\dot\gamma}J_i, J_j\rangle \\
&= \left\langle \cos(\sqrt{\mu_i}r)v_i(r), \frac{1}{\sqrt{\mu_j}}\sin(\sqrt{\mu_j}r)v_j(r)\right\rangle, \\
&= \frac{1}{\sqrt{\mu_j}}\sin(\sqrt{\mu_j}r)\cos(\sqrt{\mu_i}r)\delta_{ij}
\end{aligned}$$

and

$$\mathrm{Hess}(r)(v_i(r), v_j(r)) = \sqrt{\mu_i}\cot(\sqrt{\mu_i}r)\delta_{ij} \tag{3.2}$$

(In the case $\mu_i = 0, \mathrm{Hess}(r)(v_i(r), v_i(r)) = \frac{1}{r}$). On the other hand

$$\mathrm{Ric}(\dot\gamma,\dot\gamma) = \sum_i \langle R(\dot\gamma, v_i)\dot\gamma, v_i\rangle = \sum_i \mu_i.$$

Let $\mathbb{R}^{m+n}$ be an $(m+n)-$dimensional Euclidean space. The set of all oriented $n-$subspaces (called $n-$planes) constitutes the Grassmannian manifold $G_{n,m}$, which is the irreducible symmetric space $SO(m+n)/SO(m)\times SO(n)$. Our discussion of the geometry of $G_{n,m}$ will be based on the work of Wong [W]. References for other contributions on the geometry of Grassmannian are listed in [H-J-W].

Let P and Q be two points in $G_{n,m}$. Wong defined the angles between P and Q as the critical values of the angle between a nonzero vector x in P and its orthogonal projection x^* in Q as x runs through P. Assume that $e_1, \cdots, e_n$ are orthonormal vectors which span P, and $f_1, \cdots, f_n$ for Q. For a nonzero vector

$$x = \sum_\alpha x_\alpha e_\alpha,$$

its orthogonal projection in Q is

$$x^* = \sum_\alpha x^*_\alpha f_\alpha.$$

Thus, for any y in Q we have

$$\langle x - x^*, y\rangle = 0.$$

Assume that

$$a_{\alpha\beta} = \langle e_\alpha, f_\beta\rangle .$$

We then have

$$x^*_\beta = \sum_\alpha a_{\alpha\beta} x_\alpha,$$

and

$$\begin{aligned}\cos\theta &= \frac{\left\langle \sum_\alpha x_\alpha e_\alpha, \sum_\beta x^*_\beta f_\beta\right\rangle}{\sqrt{\sum_\alpha x_\alpha^2}\sqrt{\sum_\alpha {x^*_\alpha}^2}} = \frac{\sum_{\alpha,\,\beta} a_{\alpha\beta} x_\alpha x^*_\beta}{\sqrt{\sum_\alpha x_\alpha^2}\sqrt{\sum_\alpha {x^*_\alpha}^2}}\\ &= \frac{\sum_{\alpha,\,\beta,\,\gamma} a_{\alpha\beta} a_{\gamma\beta} x_\alpha x_\gamma}{\sqrt{\sum_\alpha x_\alpha^2}\sqrt{\sum_{\alpha,\,\beta,\,\gamma} a_{\alpha\beta} a_{\gamma\beta} x_\alpha x_\gamma}} \\ &= \frac{\sqrt{\sum_{\alpha,\,\beta} A_{\alpha\beta} x_\alpha x_\beta}}{\sqrt{\sum_\alpha x_\alpha^2}},\end{aligned} \tag{3.3}$$

where $A_{\alpha\beta} = \sum_\gamma a_{\alpha\gamma} a_{\beta\gamma}$ is symmetric in α and β. It follows that the angles θ_α between P and Q are

$$\theta_\alpha = \cos^{-1}(\lambda_a), \qquad 0 \le \theta_\alpha \le \frac{\pi}{2},$$

where λ_α^2 are the eigenvalues of the symmetric matrix $(A_{\alpha\beta})$.

Let $\{e_\alpha, e_{n+i}\}$ be a local orthonormal frame field in $\mathbb{R}^{m+n}$, where $i, j, \cdots = 1, \cdots, m$; $\alpha, \beta, \cdots = 1, \cdots, n$; $a, b, \cdots = 1, \cdots, m+n$ (say, $n \le m$). Let $\{\omega_\alpha, \omega_{n+i}\}$ be its dual frame field so that the Euclidean metric is

$$g = \sum_\alpha \omega_\alpha^2 + \sum_i \omega_{n+i}^2.$$

The Levi-Civita connection forms ω_{ab} of $\mathbb{R}^{m+n}$ are uniquely determined by the equations

$$
\begin{aligned}
d\omega_a &= \omega_{ab} \wedge \omega_b, \\
\omega_{ab} + \omega_{ba} &= 0.
\end{aligned}
\tag{3.4}
$$

Now, we consider any point $P \in G_{n,m}$ and a sufficiently close point Q. The canonical Riemannian metric on $G_{n,m}$ can be defined by the sum of the squares of the n angles between P and Q, namely

$$
ds^2 = \sum_{\alpha,\, i} \omega^2_{\alpha\, n+i} \tag{3.5}
$$

¿From (3.4) and (3.5) it is easily seen that the curvature tensor of $G_{n,m}$ is

$$
\begin{aligned}
\mathrm{R}_{\alpha i\, \beta j\, \gamma k\, \delta l} =& \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ik}\delta_{jl} + \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{kl} \\
&- \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{il}\delta_{kj} - \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{ij}\delta_{kl}
\end{aligned}
$$

in a local orthonormal frame field $\{e_{\alpha\, i}\}$, which is dual to $\{\omega_{\alpha\, n+i}\}$

Let $\dot\gamma = x_{\alpha i} e_{\alpha\, i}$ and $v = v_{\alpha i} e_{\alpha\, i}$. Then

$$
\begin{aligned}
\langle \mathrm{R}(\dot\gamma, e_{\beta j})\dot\gamma, v\rangle &= x_{\alpha i} x_{\gamma k} v_{\delta l} \, \langle \mathrm{R}(e_{\alpha i}, e_{\beta j}) e_{\gamma k}, e_{\delta l}\rangle \\
&= x_{\alpha i} x_{\gamma k} v_{\delta l} (\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ik}\delta_{jl} + \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{kl} \\
&\qquad\qquad - \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{il}\delta_{kj} - \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{ij}\delta_{kl}) \\
&= x_{\beta i} x_{\alpha i} v_{\alpha j} + x_{\alpha j} x_{\alpha l} v_{\beta l} - 2\, x_{\beta l} x_{\alpha j} v_{\alpha l}
\end{aligned}
\tag{3.6}
$$

By an action of $SO(m) \times SO(n)$

$$
x_{\alpha i} = \lambda_\alpha \delta_{\alpha i},
$$

where $\sum_\alpha \lambda_\alpha^2 = 1$. In fact, there exist an $n \times n$ orthogonal matrix U and an $m \times m$ matrix U', such that

$$
UXU' = \begin{pmatrix} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_n & \end{pmatrix},
$$

where $X = (x_{ai})$ is an $n \times m$ matrix. From (3.1) and (3.6) we have

$$
\begin{aligned}
\mathrm{R}_{\dot\gamma} v &= (\lambda_\alpha \lambda_\beta \delta_{\alpha i}\delta_{\beta i} v_{\alpha j} + \lambda_\alpha^2 \delta_{\alpha j}\delta_{\alpha l} v_{\beta l} - 2\, \lambda_\alpha\lambda_\beta \delta_{\alpha j}\delta_{\beta l} v_{\alpha l}) e_{\beta j} \\
&= (\lambda_\beta^2 v_{\beta j} + \lambda_\alpha^2 \delta_{\alpha j} v_{\beta\alpha} - 2\, \lambda_\alpha\lambda_\beta \delta_{\alpha j} v_{\alpha\beta}) e_{\beta j} \\
&= \begin{cases} (\lambda_\beta^2 v_{\beta\alpha} + \lambda_\alpha^2 v_{\beta\alpha} - 2\, \lambda_\alpha\lambda_\beta v_{\alpha\beta}) e_{\beta\alpha}, & \text{when} \quad j = \alpha = 1, \cdots, n; \\ \lambda_\beta^2 v_{\beta s} e_{\beta s}, & \text{when} \quad j = s = n+1, \cdots, m. \end{cases}
\end{aligned}
$$

For any $n \times m$ matrix V, there is an orthogonal decomposition

$$
V = (V_1, 0) + (V_2, 0) + (0, V_3),
$$

where V_1 is an $n\times n$ symmetric matrix, V_2 is an $n\times n$ skew-symmetric matrix and V_3 is an $n\times(m-n)$ matrix. For $v_{\beta\alpha}=v_{\alpha\beta}$

$$\mathrm{R}_{\dot\gamma}v_{\beta\alpha}e_{\beta\alpha}=(\lambda_\alpha-\lambda_\beta)^2v_{\beta\alpha}e_{\beta\alpha}.$$

For $v_{\beta\alpha}=-v_{\alpha\beta}$

$$\mathrm{R}_{\dot\gamma}v_{\beta\alpha}e_{\beta\alpha}=(\lambda_\alpha+\lambda_\beta)^2v_{\beta\alpha}e_{\beta\alpha}.$$

In summary, $\mathrm{R}_{\dot\gamma}$ has eigenvalues:

$$\begin{array}{ll}
\lambda_1^2 & \text{with multiplicity } m-n\\
\vdots & \vdots\\
\lambda_n^2 & \text{with multiplicity } m-n\\
(\lambda_\alpha+\lambda_\beta)^2 & \text{with multiplicity } 1\\
(\lambda_\alpha-\lambda_\beta)^2 & \text{with multiplicity } 1\\
0 & \text{with multiplicity } n-1
\end{array}$$

for each pair α and β with $\alpha\neq\beta$. From (3.2) it follows that the eigenvalues of the Hessian of the distance function r from a fixed point at the direction $X=(x_{\alpha i})=(\lambda_\alpha\delta_{\alpha i})$ are the same as the ones at $X_1=(|\lambda_\alpha|\delta_{\alpha i})$. They are as follows.

$$\begin{array}{ll}
\lambda_1\cot(\lambda_1 r) & \text{with multiplicity } m-n\\
\vdots & \vdots\\
\lambda_n\cot(\lambda_n r) & \text{with multiplicity } m-n\\
(\lambda_\alpha+\lambda_\beta)\cot(\lambda_\alpha+\lambda_\beta)r & \text{with multiplicity } 1\\
(\lambda_\alpha-\lambda_\beta)\cot(\lambda_\alpha-\lambda_\beta)r & \text{with multiplicity } 1\\
\dfrac{1}{r} & \text{with multiplicity } n-1
\end{array}\tag{3.7}$$

where $\lambda_\alpha>0$ without loss of generality.

Let P_0 be an oriented n-space in $\mathbb{R}^{m+n}$. We represent it by n vectors e_α, which are complemented by m vectors e_{n+i}, such that $\{e_\alpha,e_{n+i}\}$ form an orthonormal base of $\mathbb{R}^{m+n}$. Then we can span the n-planes P in a neighborhood $\mathbb{U}$ of P_0 by n vectors f_α:

$$f_\alpha=e_\alpha+z_{\alpha i}e_{n+i},$$

where $(z_{\alpha i})$ are the local coordinates of P in $\mathbb{U}$.

Let $(x_{\alpha i})=(\lambda_\alpha\delta_{\alpha i})$ be a unit tangent vector at P_0. The geodesic from P_0 at the direction $(x_{\alpha i})$ in $\mathbb{U}$ is (see [W])

$$(z_{\alpha i}(t))=\begin{pmatrix}\tan(\lambda_1 t) & & 0 & \\ & \ddots & & 0\\ 0 & & \tan(\lambda_n t) & \end{pmatrix}\tag{3.8}$$

where t is the arc length parameter and $0 \le t < \frac{\pi}{2|\lambda_\alpha|}$ with $|\lambda_\alpha| = \max(|\lambda_1|, \cdots, |\lambda_n|)$.

Thus, a geodesic in $G_{n,m}$ between two $n-$spaces is simply obtained by rotating one into the other in Euclidean space, by rotating corresponding basis vectors. We also see that in the case of codimension > 1, the above zero eigenvalues of the curvature tensor are obtained for a pair of tangent vectors in the Grassmannian that correspond to rotating two given orthogonal basis vectors of an $n-$plane separately into two different mutually orthogonal directions orthogonal to that $n-$plane. Take as an example the $2-$plane spanned by e_1, e_2 in $\mathbb{R}^4$. One tangent direction in $G_{2,4}$ would be to move e_1 into e_3 and keep e_2 fixed, and the other tangent direction would move e_2 into e_4 and keep e_1 fixed. The largest possible eigenvalue, namely 2 in case of codimension > 1 (e. g. for $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}, \lambda_3 = \cdots = \lambda_n = 0$), is realized if one takes two orthogonal directions, say e_1, e_2, in a given $n-$plane and two other such directions, say f_1, f_2 orthogonal to that $n-$plane, and the two tangent directions corresponding to rotating e_1 to f_1, e_2 to f_2 and e_1 to f_2, e_2 to $-f_1$, respectively. This geometric picture is useful for visualizing our subsequent constructions.

Now, let us define an open set $B_G(P_0)$ in $\mathbb{U} \subset G_{n,m}$. In $\mathbb{U}$ we have the normal coordinates around P_0, and then the normal polar coordinates around P_0. Define $B_G(P_0)$ in normal polar coordinates around P_0 as follows:

$$B_G(P_0) = \left\{ (X, t); \qquad X = (\lambda_\alpha \delta_{\alpha i}), \quad 0 \le t < t_X = \frac{\pi}{2(|\lambda_{\alpha'}| + |\lambda_{\beta'}|)} \right\}, \tag{3.9}$$

where $\lambda_{\alpha'}$ and $\lambda_{\beta'}$ are two eigenvalues with largest absolute values. ¿From (3.8) we see that $B_G(P_0)$ lies inside the cut locus of P_0. We also know from (3.7) that the square of the distance function r^2 from P_0 is a strictly convex smooth function in $B_G(P_0)$.

Remark. The above definition of $B_G(P_0)$ is for the case of $m \ge n > 1$. If $n = 1$, $G_{1,m}$ is the usual sphere S^m and the defined set is the open hemisphere as usual.

Let $P = ((\lambda_\alpha \delta_{\alpha i}), t)$ and $Q = ((\lambda'_\alpha \delta_{\alpha i}), t')$ be two points in $B_G(P_0)$. Then the local expression of P in $\mathbb{U}$ is the $n \times m$ matrix $(\tan(\lambda_\alpha t)\delta_{\alpha i})$, similarly that of Q is $(\tan(\lambda'_\alpha t')\delta_{\alpha i})$. Consider a curve Γ between P and Q defined by

$$(\tan(\lambda_\alpha t(1-h) + \lambda'_\alpha t' h)\delta_{\alpha i})$$

in $\mathbb{U}$, there $0 \le h \le 1$ is the parameter for Γ. We claim that Γ is a geodesic.

Let P' be the middle point defined by

$$\left(\tan\left(\frac{\lambda_\alpha t + \lambda'_\alpha t'}{2}\right)\delta_{\alpha i}\right)$$

in $\mathbb{U}$. In $\mathbb{R}^{m+n}$ the $n-$plane P is spanned by orthogonal vectors

$$f_\alpha = e_\alpha + \tan(\lambda_\alpha t) e_{n+\alpha}$$

whose unit ones are

$$\tilde{f}_\alpha = \cos(\lambda_\alpha t)e_\alpha + \sin(\lambda_\alpha t)e_{n+\alpha}.$$

Similarly, P' is spanned by n vectors

$$\tilde{f}'_\alpha = \cos\frac{\lambda_\alpha t + \lambda'_\alpha t'}{2}e_\alpha + \sin\frac{\lambda_\alpha t + \lambda'_\alpha t'}{2}e_{n+\alpha}.$$

By using (3.3) we obtain the angles between P and P' is $\frac{1}{2}|\lambda_\alpha t - \lambda'_\alpha t'|$ and their distance is

$$d(P, P') = \frac{1}{2}\sqrt{\sum_\alpha (\lambda_\alpha t - \lambda'_\alpha t')^2}. \tag{3.10}$$

Take any point P'' between P and P' on Γ

$$P'' = \left(\tan(\alpha_\alpha t(1-h) + \lambda'_\alpha t' h)\delta_{\alpha i}\right),$$

where $0 < h < \frac{1}{2}$. By computation we know that

$$d(P, P'') = h\sqrt{\sum_\alpha (\lambda_\alpha t - \lambda'_\alpha t')^2} \tag{3.11}$$

and

$$d(P'', P') = (\frac{1}{2} - h)\sqrt{\sum_\alpha (\lambda_\alpha t - \lambda'_\alpha t')^2}. \tag{3.12}$$

(3.10), (3.11) and (3.12) show that the restriction of Γ to the segment between P and P' is the minimal geodesic. The same holds for the other segment of Γ. Hence, the entire curve Γ is a geodesic between P and Q.

We claim that geodesic Γ has the following properties:

(1) $\Gamma \subset B_G(P_0)$;
(2) $\Gamma \subset B_G(P')$.

For any $h \in [0, 1]$, $\Gamma(h)$ lies on a geodesic starting from P_0 in the direction $\left(\frac{1}{A}(\lambda_\alpha t(1-h) + \lambda'_\alpha t' h)\delta_{\alpha i}\right)$, where $A > 0$ is the normalizer. By our construction of $B_G(P_0)$ the radius in this direction is

$$s = \frac{\pi A}{2\left(|\lambda_1 t(1-h) + \lambda'_1 t' h| + |\lambda_2 t(1-h) + \lambda'_2 t' h|\right)},$$

where for notational simplicity we assume that the first and the second components are the two with largest absolute values. As P and Q are in $B_G(P_0)$ and by the conditions for t and t',

$$\begin{aligned}
&|\lambda_1 t(1-h) + \lambda'_1 t' h| + |\lambda_2 t(1-h) + \lambda'_2 t' h| \\
&\quad \le (|\lambda_1| + |\lambda_2|)t(1-h) + (|\lambda'_1| + |\lambda'_2|)t' h \\
&\quad < \frac{(|\lambda_1| + |\lambda_2|)\pi}{2(|\lambda_{\alpha'} + |\lambda_{\beta'}|)}(1-h) + \frac{(|\lambda'_1| + |\lambda'_2|)\pi}{2(|\lambda'_{\gamma'} + |\lambda'_{\delta'}|)}h \\
&\quad \le \frac{\pi}{2},
\end{aligned}$$

so

$$s > A,$$

which means $\Gamma(h) \subset B_G(P_0)$, and we confirmed the first claim.

Now, we move the origin of the coordinates from P_0 to P'. The $n-$plane P' is spanned by n orthonormal vectors

$$f'_\alpha = \cos\frac{\lambda_\alpha t + \lambda'_\alpha t'}{2} e_\alpha + \sin\frac{\lambda_\alpha t + \lambda'_\alpha t'}{2} e_{n+\alpha}.$$

Define

$$f'_{n+\alpha} = -\sin\frac{\lambda_\alpha t + \lambda'_\alpha t'}{2} e_\alpha + \cos\frac{\lambda_\alpha t + \lambda'_\alpha t'}{2} e_{n+\alpha}$$

and the remaining $m-n$ vectors do not change. Thus, $\{f'_\alpha, f'_{n+i}\}$ forms an orthonormal base of $\mathbb{R}^{m+n}$. Each point $\Gamma(h)$ of the geodesic Γ is an $n-$plane in $\mathbb{R}^{m+n}$. It is spanned by

$$\cos(\lambda_\alpha t(1-h) + \lambda\alpha' t' h)e_\alpha + \sin(\lambda_\alpha t(1-h) + \lambda\alpha' t' h)e_{n+\alpha}.$$

By the above changed base, $\Gamma(h)$ is also spanned by

$$\cos(\lambda_\alpha t(\frac{1}{2}-h) - \lambda'_\alpha t'(\frac{1}{2}-h))f'_\alpha + \sin(\lambda_\alpha t(\frac{1}{2}-h) - \lambda'_\alpha t'(\frac{1}{2}-h))f'_{n+\alpha}$$

which means that the geodesic Γ in the coordinate neighborhood $\mathbb{U}'$ around P' can be described by

$$\left(\tan(\lambda_\alpha t(\frac{1}{2}-h) - \lambda'_\alpha t'(\frac{1}{2}-h))\delta_{\alpha i}\right).$$

Its tangent direction at P' is

$$((\lambda'_\alpha t' - \lambda_\alpha t)\delta_{\alpha i}).$$

By a similar argument we obtain the second property, that is $\Gamma \subset B_G(P')$.

These two properties of the geodesic Γ mean that any minimal geodesic γ from P to Q lies in $B_G(P_0)$ and has length less than $2t_X$, where X is the unit tangent vector of γ at P. On the other hand, we already computed all Jacobi fields along any geodesic in $G_{n,m}$, which means that any geodesic from P of length $< 2t_X$ has no conjugate points and that the squared distance function from P remains strictly convex along this geodesic for length $< t_X$. Thus, by a similar reasoning as Prop. 2.4.1 in [J] we conclude that $B_G(P_0)$ shares all the properties of the usual convex geodesic ball. In fact, the situation here is even simpler because, in a symmetric space, by a result of Crittenden [Cr], the first cut point along a geodesic always has to be a conjugate point. In summary, we have

Theorem 3.1. *In $B_G(P_0)$ the square of the distance function from its center P_0 is a smooth strictly convex function. Furthermore, $B_G(P_0)$ is a convex set, namely any two points in $B_G(P_0)$ can be joined in $B_G(P_0)$ by a unique geodesic arc. This arc is the shortest connection between its end points and thus in particular does not contain a pair of conjugate points.*

Remark. In the Grassmannian manifold there is the usual convex geodesic ball $B_R(P_0)$ of radius

$$R < \begin{cases} \frac{\pi}{2\sqrt{2}} & \text{when} \quad \min(m,n) > 1; \\ \frac{\pi}{2} & \text{when} \quad \min(m,n) = 1. \end{cases}$$

¿From (3.9) it is seen that $B_R(P_0) \subset B_G(P_0)$.

Let $P(t)$ be any $n-$plane in $\mathbb{U}$ of P_0 which is spanned by

$$f_\alpha = e_\alpha + z_{\alpha i} e_{n+i},$$

where $z_{\alpha i}$ is defined by (3.8). Let

$$\tilde{f}_1 = \cos(\lambda_1 t) f_1, \ \cdots, \ \tilde{f}_n = \cos(\lambda_n t) f_n.$$

Since $|f_\alpha| = \frac{1}{\cos(\lambda_\alpha t)}$, the vectors $\tilde{f}_1, \cdots, \tilde{f}_n$ are orthonormal.

Therefore, we can define the inner product $\langle P_0, P\rangle$ of $n-$planes $P_0 = e_1 \wedge \cdots \wedge e_n$ and $P = \tilde{f}_1 \wedge \cdots \wedge \tilde{f}_n$ by

$$\langle P_0, P\rangle = \det\left(\left\langle e_\alpha, \tilde{f}_\beta \right\rangle\right).$$

It follows that

$$\langle P_0, P(t)\rangle = \det\begin{pmatrix} \cos(\lambda_1 t) & & & 0 \\ & \cos(\lambda_2 t) & & \\ & & \ddots & \\ 0 & & & \cos(\lambda_n t) \end{pmatrix} = \prod_{\alpha=1}^{n} \cos(\lambda_\alpha t).$$

Theorem 3.2.

$$max\{\langle P_0, P\rangle\,; \quad P \in \partial B_G\} = \frac{1}{2}. \tag{3.13}$$

Proof. Suppose now for notational simplicity that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0, \quad \lambda_1^2 + \cdots + \lambda_n^2 = 1.$$

Let us maximize

$$f(\lambda_1, \cdots, \lambda_n) = \prod_{\alpha=1}^{n} \cos\left(\frac{\pi\lambda_\alpha}{2(\lambda_1+\lambda_2)}\right).$$

Let

$$F(\lambda_1, \cdots, \lambda_n, \mu) = \prod_{\alpha=1}^n \cos\left(\frac{\pi\lambda_\alpha}{2(\lambda_1+\lambda_2)}\right) + \mu(1 - \sum_{\alpha=1}^n \lambda_\alpha^2),$$

where μ is a Lagrange multiplier. Then

$$\begin{aligned}
F_{\lambda_1} =& -\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)^2}\sin\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)}\cos\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}\prod_{s=3}^n\cos\frac{\pi\lambda_s}{2(\lambda_1+\lambda_2)}\\
&+\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)^2}\cos\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)}\sin\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}\prod_{s=3}^n\cos\frac{\pi\lambda_s}{2(\lambda_1+\lambda_2)}\\
&+\sum_{t=1}^n\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)^2}\cos\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)}\cos\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}\sin\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)}\\
&\qquad\qquad\prod_{s\neq t, s=3}^n\cos\frac{\pi\lambda_s}{2(\lambda_1+\lambda_2)} - 2\mu\lambda_1\\
=& -\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)}f(\lambda) + \frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}f(\lambda)\\
&+\sum_{t=3}^n\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)}f(\lambda) - 2\mu\lambda_1.
\end{aligned}$$

At a critical point of f

$$\begin{aligned}
\frac{2\mu\lambda_1}{f(\lambda)} =& -\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)} + \frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}\\
&+\sum_{t=3}^n\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)}
\end{aligned}\tag{3.14}$$

and similarly,

$$\begin{aligned}
\frac{2\mu\lambda_2}{f(\lambda)} =& \frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)} - \frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}\\
&+\sum_{t=3}^n\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)}.
\end{aligned}\tag{3.15}$$

Adding both sides of (3.14) and (3.15) gives

$$\begin{aligned}
\frac{2\mu(\lambda_1+\lambda_2)}{f(\lambda)} =& \frac{\pi(\lambda_1-\lambda_2)}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)} - \frac{\pi(\lambda_1-\lambda_2)}{2(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}\\
&+\sum_{t=3}^n\frac{\pi\lambda_t}{(\lambda_1+\lambda_2)^2}\tan\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)}.
\end{aligned}\tag{3.16}$$

If $\lambda_2 < \lambda_1$, then (3.16) means

$$\mu > 0.$$

On the other hand, subtracting (3.15) from (3.14) gives

$$\frac{2\mu(\lambda_1-\lambda_2)}{f(\lambda)} = -\frac{\pi}{2(\lambda_1+\lambda_2)}\tan\frac{\pi\lambda_1}{2(\lambda_1+\lambda_2)} + \frac{\pi}{2(\lambda_1+\lambda_2)}\tan\frac{\pi\lambda_2}{2(\lambda_1+\lambda_2)}. \quad (3.17)$$

If $\lambda_2 < \lambda_1$, the left side of (3.17) is positive but its right side is negative which gives a contradiction. Consequently, the critical points of f occur only when $\lambda_1 = \lambda_2$. Now, from (3.16) we have

$$\frac{4\mu\lambda_1}{f(\lambda)} = \sum_{t=3}^{n}\frac{\pi\lambda_t}{4\lambda_1^2}\tan\frac{\pi\lambda_t}{4\lambda_1}. \quad (3.18)$$

We also have

$$F_{\lambda_t} = -\frac{\pi}{2(\lambda_1+\lambda_2)}\prod_{\alpha\neq t,\alpha=1}^{n}\sin\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)}\cos\frac{\pi\lambda_\alpha}{2(\lambda_1+\lambda_2)} - 2\mu\lambda_t,$$

where $t = 3,\cdots,n$. At a critical point $(\lambda_1,\cdots,\lambda_n)$ of f

$$-\frac{\pi}{2(\lambda_1+\lambda_2)}\prod_{\alpha\neq t,\alpha=1}^{n}\sin\frac{\pi\lambda_t}{2(\lambda_1+\lambda_2)}\cos\frac{\pi\lambda_\alpha}{2(\lambda_1+\lambda_2)} - 2\mu\lambda_t = 0.$$

It follows that

$$\frac{2\mu\lambda_t}{f(\lambda)} = -\frac{\pi}{4\lambda_1}\tan\frac{\pi\lambda_t}{4\lambda_1}. \quad (3.19)$$

If there exists t such that $\lambda_t > 0$, then from (3.19) we have $\mu < 0$ which contradicts with (3.18). Consequently, the only critical point of $f(\lambda)$ occurs when

$$\lambda_1 = \lambda_2 = \frac{\sqrt{2}}{2}, \qquad \lambda_3 = \cdots = \lambda_n = 0,$$

and

$$f(\lambda_1,\cdots,\lambda_n) = \prod_{\alpha=1}^{n}\cos\left(\frac{\pi\lambda_\alpha}{2(\lambda_1+\lambda_2)}\right) \leq \frac{1}{2}.$$

Q.E.D.

4. Proofs of the Main Theorems

Theorem 4.1. *Let M be an $n-$dimensional simple Riemannian manifold which is immersed in Euclidean space $\mathbb{R}^{m+n}$ with parallel mean curvature. Let $\gamma : M \to G_{n,m}$ be Gauss map. Suppose that there exists a fixed oriented $n-$plane P_0, and a number α_0,*

$$\alpha_0 > \begin{cases} \frac{1}{2} & \text{when} \quad min(m,n) \geq 2, \\ 0 & \text{when} \quad min(m,n) = 1; \end{cases}$$

such that

$$\langle P, P_0 \rangle \geq \alpha_0 \tag{4.1}$$

holds for all point $P \in \gamma(M) \subset G_{n,m}$. Then M has to be an $n-$dimensional affine linear subspace.

Proof. $\gamma : M \to G_{n,m}$ is harmonic by Thm 2.3. The point now is to show that such a harmonic map is constant if its range is contained in a region $B_G(P_0)$. In the case where $B_G(P_0)$ is replaced by a geodesiccally convex ball, this was shown by an iteration argument in [H-J-W]. The properties of $B_G(P_0)$ as established in §3 are strong enough to make that iteration technique still applicable (for example, a general version of that iteration technique that directly applies here has been given in [G-J]).

Alternatively, at least in the case of minimal graphs, one uses Allard's result [A] mentioned in the introduction that the tangent cone at infinity of M is the cone over a compact minimal submanifold M' of the sphere. The Gauss map of M' again is a harmonic map with values in a region $B_G(P_0)$. Since $B_G(P_0)$ supports a strictly convex function by Thm 3.1, the composition of that Gauss map with that function is a subharmonic function on M' which then has to be constant as M' is compact, as in [FC].

Possibly, there is even a third method to reach the conclusion. Namely, it is quite likely that Kendall's result [K] can be generalized to show that for any $Q \in B_G(P_0)$, there exists a strictly convex function on all of $B_G(P_0)$ with its minimum at Q. If that can be shown, the Hölder estimates and Liouville theorems for harmonic maps with values in $B_G(P_0)$ can be shown by a considerably simplified version of the method of [H-J-W], namely in a single step without the need to iterate. Although this would yield the simplest proof, we refrain here from studying the technical details as we have already described two other ways to reach the conclusion that the Gauss map is constant.

Obviously, if the Gauss map is constant, the submanifold has to be affine linear.
Q.E.D.

We are now in a position to prove Theorem 1 and Theorem 2 stated in the introduction.

Proof of Theorem 1

Since $M = (x, f(x))$ is a graph in $\mathbb{R}^{m+n}$ defined by m fuctions, the induced metric on M is

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta,$$

where

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \sum_i \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^i}{\partial x^\beta}.$$

It is obvious that the eigenvalues of the matrix $(g_{\alpha\beta})$ at each point are ≥ 1. The condition (1.2) implies that the eigenvalues of the matrix $(g_{\alpha\beta})$ are $\leq \beta_0^2$. The condition (2.6) is satisfied and M is a simple Riemannian manifold.

Let $\{e_\alpha, e_{n+i}\}$ be the standard orthonormal base of $\mathbb{R}^{m+n}$. Choose P_0 as an $n-$plane spanned by $e_1 \wedge \cdots \wedge e_n$. At each point in M its image $n-$plane P under the Gauss map is spanned by

$$f_\alpha = e_\alpha + \frac{\partial f^i}{\partial x^\alpha} e_{n+i}.$$

It follows that

$$|f_1 \wedge \cdots \wedge f_n|^2 = \det(\delta_{\alpha\beta} + \sum_i \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^i}{\partial x^\beta})$$

and

$$\Delta_f = |f_1 \wedge \cdots \wedge f_n|.$$

The $n-$ plane P is also spanned by

$$p_\alpha = \Delta_f^{-\frac{1}{n}} f_\alpha,$$

furthermore,

$$|p_1 \wedge \cdots \wedge p_n| = 1.$$

We then have

$$\begin{aligned} \langle P, P_0\rangle &= \det(\langle e_\alpha, p_\beta\rangle) \\ &= \begin{pmatrix} \Delta_f^{-\frac{1}{n}} & & 0 \\ & \ddots & \\ 0 & & \Delta_f^{-\frac{1}{n}} \end{pmatrix} \\ &= \Delta_f^{-1} \end{aligned}$$

which is $\geq \frac{1}{\beta_0} > \frac{1}{2}$ by (1.1) and (1.2). Thus Theorem 1 follows from Thorem 4.1.

Proof of Theorem 2

¿From Proposition 2.5 the normal gauss map $\gamma : M \to G_{n,m+1}$ is harmonic. The condition (1.5) means that the image $\gamma(M)$ under the normal Gauss map is contained inside of $B_G(P_0) \subset G_{n,m+1}$. Theorem 2.1 and Theorem 3.1 imply that γ is constant. We thus complete the proof of Theorem 2.

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