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Mapping problems, fundamental groups and defect measures

by

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Mapping Problems, Fundamental Groups and Defect Measures

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Abstract

We study all possible weak limits of a minimizing sequence, for p-energy functionals, consisting of continuous maps between Riemannian manifolds subject to a Dirichlet boundary condition or a homotopy condition. We show that if p is not an integer, then any such weak limit is a strong limit and, in particular, a stationary p-harmonic map which is $\mathbb{C}^{1,\alpha}$ continuous away from a closed subset of the Hausdorff dimension $\leq n - [p] - 1$. If p is an integer, then any such weak limit is a weakly p-harmonic map along with a $(n-p)$-rectifiable Radon measure μ. Moreover, the limiting map is $\mathbb{C}^{1,\alpha}$ continuous away from a closed subset $\Sigma =$ spt $\mu \cup S$ with $\mathcal{H}^{n-p}(S) = 0$. Finally, we discussed the possible varifolds type theory for Sobolev mappings.

1 Introduction

We start with the classical mapping problem. Let M be a smooth, compact Riemannian manifold with smooth boundary ∂M, and let N be a smooth, compact Riemannian manifold without boundary. Suppose g, $\partial M \to N$ is a Lipschitz continuous map, and $1 < p \leq n$, $n =$ dim M, we consider the variational problem:

$$\min E_p(u) \equiv \min \int_M |\nabla u|^p dx \tag{1.1}$$

among maps $u : M \to N$ such that $u\big|_{\partial M} = g$.

The first natural question one has to address is whether or not the following set,

$$W_g^{1,p}(M,N) \equiv \left\{u : M \to N, u\big|_{\partial M} = g \text{ and } \nabla u \in L^p(M)\right\}, \tag{1.2}$$

is not empty. For this we have:

Theorem [HL]. *If N is $[p-1]$-connected, then $W_g^{1,p}(M,N) \neq \varnothing$. In fact,*

$$\begin{aligned} &\inf\left\{\int_M |\nabla u|^p dx : u \in W_g^{1,p}(M,N)\right\} \\ &\leq c(p,N) \inf\left\{\int_M |\nabla u|^p dx : v \in W_g^{1,p}(M,\mathbb{R}^K)\right\}. \end{aligned} \tag{1.3}$$

Here we assume that N is isometrically embedded in $\mathbb{R}^K$, and $c(p,N)$ is a constant depending only on N and p. In particular, if g is a trace of a $W^{1,p}(M,\mathbb{R}^K)$ map on ∂M, with $g(\partial M) \subset N$, then $W_g^{1,p}(M,N) \neq \varnothing$.

In general, B. White [W] showed that $W_g^{1,p}(M,N) \neq \varnothing$ if and only if g has continuous extension on $\partial M \cup M^{[p]}$, where $M^{[p]}$ is a $[p]$-dimensional skeleton of M. However, the estimate (1.3) may not be valid in this general situation.

If under some topological conditions that the space $C_g^o(M,N)$, continuous maps from M into N with trace g on ∂M, is not empty, then one is interested in the following mapping spaces:

$$\begin{aligned} H_s^{1,p}(M,N) &\equiv \text{ the strong closure of } C_g^o(M,N) \text{ in } W_g^{1,p}(M,N), \\ H_s^{1,p}(M,N) &\equiv \text{ the weak closure of } C_g^o(M,N) \text{ in } W_g^{1,p}(M,N). \end{aligned}$$

Obviously one has

$$H_s^{1,p}(M,N) \subseteq H_w^{1,p}(M,N) \subseteq W_g^{1,p}(M,N). \tag{1.4}$$

We also define $R_g^{1,p}(M,N)$ to be the subset of $W_g^{1,p}(M,N)$ consisting of all such maps u that are smooth away from a $(n-[p]-1)$-dimensional skeleton of M. It follows from the proof of [HL, §6] that $R_g^{1,p}(M,N)$ is dense in $W_g^{1,p}(M,N)$ in the strong topology. Later Bethuel [B] showed that $R_g^{1,p}(M,N)$ is always dense in $W_g^{1,p}(M,N)$ without the assumption of the $[p-1]$ simply connectedness of N. On the other hand, we have the following so called "Gap-phenomena".

Theorem [HL2]. *There are smooth maps g from $\mathbb{S}^2 = \partial \mathbb{B}^3$ into $\mathbb{S}^2$ of degree zero such that*

$$\min \left\{ \int_{\mathbb{B}^3} |\nabla u|^2(x)\, dx : u \in W_g^{1,2}(\mathbb{B}^3, \mathbb{S}^2) \right\}$$
$$< \inf \left\{ \int_{BB^3} |\nabla u|^2(x)\, dx : v \in H_s^{1,2}\left(\mathbb{B}^3, \mathbb{S}^2\right), v\big|_{\partial \mathbb{B}^3} = g \right\}.$$

This gap-phenomena implies, in particular, that $H_s^{1,2}(\mathbb{B}^3, \mathbb{S}^2) \subsetneq W_g^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$. On the other hand, it is relatively easy to show $H_w^{1,2}(\mathbb{B}^3, \mathbb{S}^2) = W_g^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ (cf. [BBC]).

Some more general gap-phenomena were established in [GMS2]. The most remarkable result in this direction is probably the following theorem of Bethuel:

Theorem [B]. *$C_g^o(M, N)$ is dense in $W_g^{1,p}(M, N)$ with respect to the strong topology on $W_g^{1,p}(M, N)$ if and only if $\pi_{[p]}(N) = 0$.*

As a consequence, one can deduce that from p-energy functional that no gap-phenomena for an arbitrary boundary map $g \in C^o(\partial M, N)$ if and only if $\pi_{[p]}(N) = 0$.

This leads to the following long standing problem (cf. [HL2]):

Open Problem I.

$$\text{Is } \inf \left\{ \int_M |\nabla u|^p dx : u \in H_s^{1,p}(M, N), u\big|_{\partial M} = g \right\}$$

achieved?

Similar questions were also posed by Schoen-Uhlenbeck, Schoen-Yau before the work [HL2] for maps in Homotopy classes. Indeed, in the case $\partial M = \varnothing$, it was shown in [SU2] that any map $f \in W^{1,p}(M, N)$ induces $f^{\#} : H^k(N, R) \to H^k(M, R)$, for any $0 \le k \le [p-1]$, a homomorphism between cohomology classes. We also note that Burstall [Bu] proved $W^{1,2}(M, N)$ maps induce conjugate classes of homomorphisms of $\pi_1(M) \to \pi_1(N)$, and Schoen-Yau [SY] showed, one can define conjugate classes of homomorphisms of $\pi_{n-1}(M) \to \pi_{n-1}(N)$ for $W^{1,n}(M, N)$ maps. In [W], White established the following results.

Theorem [W].

(i) *Let* $d = \begin{cases} [p] & \text{if } p \text{ is not an integer} \\ (p-1) & \text{if } p \text{ is an integer} \end{cases}$ *Then each* $f \in H_W^{1,p}(M,N)$ *has well-defined* d*-homotopy type which is preserved under weak convergence;*

(ii) *if* $d = [p-1]$*, then each* $f \in W^{1,p}(M,N)$ *has a well-defined* d*-homotopy type and it is preserved under weak convergence of maps;*

(iii) *Each map* $f \in H_s^{1,p}(M,N)$ *has a well-defined* $[p]$*-homotopy type that is preserved under the strong convergence of maps.*

Very recently, Duzaar-Kuwert [DK] improved the statement (i) of the above theorem by introducing the suitable notion of p-homotopy classes of maps and the weak limit sets for maps in $H_w^{1,p}(M,N)$ when p is an integer.

These results also indicate the delicate difficulties of the open problem aforementioned. Indeed there is very little progress being made toward the solution of this open problem in general.

For the special case of maps from $\mathbb{B}^3$ into $\mathbb{S}^2$, Bethuel-Brezis-Coron [BBC] introduced first the so-called relaxed energy:

$$F(u) = E(u) + 8\pi\, L(u), \tag{1.5}$$

where $E(u) = \int_{B^3} |\nabla u|^2(x)\, dx$,

$$L(u) = \frac{1}{4k} \sup_{\varrho:\mathbb{R}^3\to\mathbb{R}, |\nabla\varrho|\le 1} \left\{ \int_{\mathbb{B}^3} D(u)\cdot\nabla\varrho\, dx - \int_{\partial\mathbb{B}^3} D(u)\cdot n\varrho d\sigma \right\},$$

$D(u)$ is the dual vector of the 2-form $u^*\omega$, ω is the area-form(?) on $\mathbb{S}^2$ (cf. [BBC] for a geometric interpretation of $L(u)$). They proved the following

Theorem [BBC].

(i) $|L(u) - L(u)| \le C_0 \|\nabla u - \nabla v\|_{L^2(B^3)} (\|\nabla u\|_{L^2(B^3)} + \|\nabla v\|_{L^2(B^3)})$,
for all u, $v \in W_g^{1,2}(B^3, \mathbb{S}^2)$*;*

(ii) $\inf\{E(u) : u \in H_s^{1,2}(\mathbb{B}^3, \mathbb{S}^2), u\big|_{\partial\mathbb{B}^3} = g\} = \inf\{F(u) : u \in W_g^{1,2}(\mathbb{B}^3, \mathbb{S}^2)\}$.
Moreover, $L(u) = 0 \Longleftrightarrow u \in H_s^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$*;*

(iii) *$F(\cdot)$ is sequentially lower-semicontinuous with respect to the weak convergence of maps in $W^{1,2}_g(\mathbb{B}^3, \mathbb{S}^2)$.*

In particular the infimum of F is achieved.

It is also noted in [BBC] that $F(\cdot)$ minimizers are in general different from absolute energy minimizers. The result of [BBC] leads to the following:

Open Problem II. Are F-minimizers continuous?

In [HLP] it is showed that the answer to the above problem II is "No" for general maps in the restricted axially symmetric class. Indeed, it was proven that when $F(\cdot)$ is restricted to the axially symmetric maps from $\mathbb{B}^3$ with $\mathbb{S}^2$, the minimizers of $F(\cdot)$ may have isolated degree zero singularities. One does not know if this result remains true for $F(\cdot)$ minimizers among all maps in $W^{1,2}_g(\mathbb{B}^3, \mathbb{S}^2)$.

In a series of very general works, Giaquinta-Modica-Souček studied the so-called Cartesian currents. As an application, they deduced the above result of [BBC]. Moreover, they showed F-minimizers are smooth away from a closed rectifiable set of finite $\mathcal{H}^1$ measure (see [GMS] for the details). It seems that these arguments in [GMS] works only when the target or the domain manifold is 2-dimensional. We should also remark that to characterize whose maps in $H^{1,p}_s(M, N)$ is also an interesting and difficult question, see [B2], [BCDH].

Now we can describe the main results of the present paper.

Theorem 1. *Suppose the space of continuous maps from M into N with trace g, $C^o_g(M, N)$, is not empty. Then any energy minimizing sequence $\{v_i\}$, $v_i \in C^o_g(M, N)$ of $E(\cdot)$ contains a subsequence converging weakly to a harmonic map u, $M \to N$, which is smooth away a closed subset Σ of M of finite $\mathcal{H}^{n-2}$-measure. Moreover Σ is rectifiable, and the pair (u, ν) is stationary for the energy. Where ν is the corresponding defect measure with spt $\nu \subset \Sigma$. If, in addition, $\pi_2(N) = 0$, then $\nu \equiv 0$, and u is absolutely energy minimizing.*

Remark. The above theorem can be viewed as a generalization of the aforementioned results of [BBC] and [GMS]. The defect measure is defined in the next section, and so is the stationarity of the pair (u, ν). In the case $\nu \equiv 0$, one has the strong convergence of the minimizing sequences to the limiting minimizers.

With the same proof as that for Theorem 1, one can deduce the following results for p equals an integer.

Theorem 1′. *Suppose $C^o_g(M,N) \neq \varnothing$, then any minimizing sequence for $E_p(\cdot)$ over the space $C^o_g(M,N)$ contains a subsequence converging weakly to a p-harmonic map u (here p is an integer) which is $\mathbb{C}^{1,\alpha}$ smooth away from a closed subset Σ of M of finite $(n-p)$-dimensional Hausdorff measure. Moreover, Σ is $\mathcal{H}^{n-p}$-rectifiable and the pair (ν, u) is stationary for $E_p(\cdot)$. If, in addition, $\pi_p(N) = 0$, then $\nu \equiv 0$, and u is an absolute $E_p(\cdot)$ minimizing map.*

We also have a homotopy version with the identical proof as previous two theorems.

Theorem 1″. *Suppose $\partial M = \varnothing$ and $g : M \to N$ is a map in $C^o_g(M,N)$. Then any minimizing sequence of $E_p(\cdot)$ in the space $C^o(M,N)$ with the same homotopy class as g contains a weakly converging subsequence such that the weak limit u along with the defect measure ν have the following properties:*

- *(a) u is $\mathbb{C}^{1,\alpha}$, p harmonic map away from a closed $\mathcal{H}^{n-p}$-rectifiable set $\Sigma \subseteq M$ with $\mathcal{H}^{n-p}(\Sigma) < \infty$;*
- *(b) spt $\nu \subseteq \Sigma$ and u has the same $(p-1)$-homotopy type as g;*
- *(c) if, in addition, $\pi_p(N) = 0$, then $\nu \equiv 0$ and u is an absolute $E_p(\cdot)$ minimizer which has the same p-homotopy type as g. Moreover u is $\mathbb{C}^{1,\alpha}$ away from a closed subset of M of Hausdorff dimension $\leq n-p-1$.*

Next we consider the case p is not an integer. We have somewhat stronger statements.

Theorem 2. *Under the same assumption as in Theorem 1. Let U_p be a weak limit of a minimizing sequence for $E_p(\cdot)$ over $C^o_g(M,N)$ with $p \neq$ integer . Then $U_p \in H^{1,p}_s(M,N)$ and hence U_p achieves the value* $\inf\left\{E_p(v) : v \in H^{1,p}_s(M,N), v\big|_{\partial M} = g\right\}$. *In particular, U_p is stationary for $E_p(\cdot)$. Moreover, U_p is $\mathbb{C}^{1,\alpha}$ away from a closed subset of Hausdorff dimension $\leq n-[p]-1$.*

Remark. (a) If for some $k \in \{2, 3, \ldots, n-1\}$ that $p < k$, and $k - p$ sufficiently small, then the singular set has Hausdorff dimension $\leq n - [p] - 2$. This follows from the global energy bound and analysis in [HLW].

(b) Note that there is no defect measure in the case p is not an integer. Indeed such minimizing maps U_p obtained in Theorem 2 above form a compact family in $W^{1,p}(M, N)$ whenever their energies remain bounded.

(c) The regularity of limiting maps obtained in both Theorem 1 and Theorem 2 was an open issue in previous works [B] and [W].

When $\partial M = \varnothing$, we have also the following.

Theorem 2′. *Let $g : M \to N$ be a continuous map between compact Riemannian manifolds without boundary. Let $[g] \subset C^o(M, N)$ be the set of all maps homotopy to g. Then for any minimizing sequence of maps in $[g]$ for the p-energy $E_p(\cdot)$, for some noninteger $p \in (1, n)$, there is a strong converging subsequence such that the limiting map U_p is a stationary p-harmonic map. The map U_p is $C^{1,\alpha}$ away from a closed subset Σ with Hausdorff dimension $\leq n - [p] - 1$. Moreover, U_p has the same $[p]$-homotopy type as g.*

We have also made preliminary analysis on the defect measure ν arises in various situations stated above. They are sum of integral multiplicity rectifiable Radon measures.

For the behavior of minimizing sequence near the smooth boundary ∂M on which they take the smooth boundary value $g : \partial M \to N$, we have the following statements.

Theorem 3. *For any $p \in (1, n)$, and $p \neq$ integer, ($p > n$ is trivial by the Sobolev embedding theorem), let $\{U_i\} \subset C^o_g(M, N)$ be a $E_p(\cdot)$ minimizing sequence such that U_i weakly converge to U_p along with a defect measure ν. Then U_p is regular near ∂M and $\nu \equiv 0$.*

Here we say U_p is regular near ∂M if there is a neighborhood O of ∂M in M such that $U_p \in C^{1,\alpha}(O \setminus M) \cap C^\alpha(\overline{O})$.

Finally we like to state two consequences of our results.

Corollary 1. *Let $g : \partial \mathbb{B}^4 \to \mathbb{S}^2$ be a smooth map and let Hopf-invariant of g, $H(g)$, be zero. Then the* $\inf\{E_p(u) : u \in C^o_g(\mathbb{B}^4, \mathbb{S}^2)\}$ *is achieved by a map $U_p \in H^{1,p}_s(\mathbb{B}^4, \mathbb{S}^2)$ for $3 < p < 4$. Moreover U_p is a stationary p-harmonic map which is $\mathbb{C}^{1,\alpha}$ inside $\mathbb{B}^4$, $\mathbb{C}^\alpha$–up to $\partial \mathbb{B}^4$, away* *from a finitely many points inside* $\mathbb{B}^4$, *say* $\{x_j, j = 1, \ldots, N\}$. *Moreover the Hopf invariants* $H(U_p \mid \partial B_r(x_j)) = 0$, *for* $j = 1, \ldots, N$ *and for all sufficiently small* $r > 0$.

Corollary 2. *Let* $g : \partial B^3 \to \mathbb{S}^2$ *be a smooth degree zero map. Then the* $\inf\{E_p(u) : u \in C^o_g(\mathbb{B}^3, \mathbb{S}^2)\}$ *is achieved by a map* $U_p \in H^{1,p}_s(\mathbb{B}^3, \mathbb{S}^2)$, *for* $2 < p < 3$. *Moreover,* U_p *is stationary p-harmonic map which is* $\mathbb{C}^{1,\alpha}$ *in* $\mathbb{B}^3$, $\mathbb{C}^\alpha$*–up to boundary* ∂B^3, *away from a finite many points* x_j, $j = 1, \ldots, k$, *inside* $\mathbb{B}^3$. *Moreover, deg* $(U_p, \partial B_r(x_j)) = 0$ *for all* j *and all sufficiently small* $r > 0$.

Most of results presented here were announced in [L] with sketched proofs.

The present paper is written as follows, In Section 2, we establish the partial regularity of the weak limiting maps. To do so, we have to generalize the Schoen-Uhlenbeck construction for absolute energy minimizing maps to our situation of minimizing energy among continuous maps (see Lemma 2.4 below). This was done by special methods for the case that the domain manifold is two or three dimensional, and then by an inductive argument for the case of dimensions ≥ 4. This key Lemma 2.4 leads to the so-called small energy regularity Theorem 2.3.

Section 3 of the paper is devoted to the study of defect measures. The first important step is Theorem 3.1 which leads to the fact that defect measures are supported in a closed subset of suitable Hausdorff dimension, and that the corresponding Hausdorff measures are also finite. Then we show that these defect measures are rectifiable (cf. Theorem 3.5). To do so we will need the key lemmas, Lemma 3.6 and Lemma 3.8. These two lemmas will be proved in Section 4 of the paper. Then we show in various cases that such defect measures may be vanish. If that is the case, various compactness results follow.

In Section 4, we introduced the so-called generalized varifolds. The general varifold type theory for mappings will be the subject of a forthcoming work. Here we show an energy minimizing sequence leads to a stationary generalized varifold. Then we use results in Section 3 to show such varifold can be nicely decomposed into two parts. One part is given by the weak limiting map, the other part is given by the defect measure. We show the latter is a classical integral rectifiable varifold. In some case there are integral multiplicity currents. Several remarks concerning the boundary regularity are discussed in the final Section 5. We should present complete proofs of Theorem 1, Theorem 2 and Theorem 3. The proofs of Theorem $1'$, Theorem $1''$ as well as Theorem $2'$ are very similar and we shall thus omit them beside a few remarks.

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2 Partial Regularity of the Limiting Maps

Let $\{u_i\}$ be a minimizing sequence for $E(\cdot)$ over $C_g^o(M,N)$. Consider a sequence of Radon measure on M:

$$\mu_i = |\nabla u_i|^2 \, dx, \quad i = 1, 2 \dots \ .$$

We may assume, by taking a subsequence if necessary, that $u_i \rightharpoonup u$ in $W_g^{1,2}(M,N)$ weakly, and $\mu_i \rightharpoonup \mu$ as Radon measures. By Fatou's theorem, we may write $\mu = |\nabla u|^2 \, dx + \nu$, here $\nu \geq 0$ is also a Radon measure. We will call it the defect measure associated with the weakly converging subsequence.

Lemma 2.1. *The weak limit u is a weakly harmonic map in $W_g^{1,2}(M,N)$.*

Proof. For the simplicity, we assume that N Is isometrically embedded in $\mathbb{R}^K$. Let $\psi \in C_0^1(M, \mathbb{R}^K)$, and let $t \in \mathbb{R}$ be sufficiently small. Consider $u_i(t) = \pi_N(u_i + t\psi)$, here π_N is the nearest point projection of a neighborhood of N in $\mathbb{R}^K$ onto N, then $u_i(t) \in C_g^o(M,N)$. Thus

$$\lim_{i\to\infty} E(u_i(t)) \geq \lim_{i\to\infty} E(u_i) = \int_M |\nabla u|^2 dx + \nu(M).$$

On the other hand, a direct computation shows

$$E(u_i(t)) = E(u_i) + t \int_M \nabla u_i \cdot D\pi_n(u_i) \nabla\psi \, dx + O_i(|t|^2|.$$

where $O_i(|t|^2)$ is a quantity bounded by $C|t|^2$, for a constant C depending only on N, ψ and the uniform energy bound on u_i's. Therefore, $t \int_M \nabla u \cdot (D$
$pi_N(u)) \, \nabla\psi \, dx + O(t^2) \geq 0$, for all sufficiently small $t \in \mathbb{R}$. In other words, $u \in W_g^{1,2}(M,N)$ is a weakly harmonic map. □

Next is the usual energy monotonicity property.

Lemma 2.2. *For $a \in M$, $0 < r < d_a \equiv dist_M(a, \partial M)$, the function $\frac{\mu(B_r(a))}{r^{n-2}}$ is monotone nondecreasing in r, here $n = dim\ M$.*

For simplicity we shall assume M is an n-dimension domain in $\mathbb{R}^n$. The modifications for general M are standard.

Proof. For almost all ρ, $\rho + \delta \in (0, d_a)$, one has

$$\frac{\mu(B_{\rho+\delta}(a)) - \mu(B_\rho(a))}{\delta} = \frac{1}{\delta} \lim_i \int_{B_{\delta+\rho}(a) \setminus B_\rho(a)} |\nabla u_i|^2 \, dx$$
$$= \frac{1}{\delta} \lim_i \int_\rho^{\rho+\delta} \int_{\partial B_r(a)} |\nabla u_i|^2 \, d\sigma \, dr$$
$$\geq \frac{1}{\delta} \underline{\lim}_i \int_\rho^{\rho+\delta} \left(\frac{n-2}{r} \int_{B_r(a)} |\nabla u_i^r|^2 \; dy \right) dr.$$

Here u_i^r is the homogeneous degree zero extension of u_i on $\partial B_r(a)$.

On the other hand, we may replace u_i^r, for each $r \in [\rho, \rho + \delta]$, by a map $\overline{u}_i$ which is equal to u_i^r on $B_r(a) \setminus B_{1/i}(a)$ and, which is also continuous on $B_r(a)$. Indeed, we simply let

$$\overline{u}_i(x) = u_i^r(x), \text{ for } \frac{1}{i} < |x| \leq r, \text{ and}$$
$$\overline{u}_i(x) = u_i(irx), \text{ for } |x| \leq \frac{1}{i}.$$

Since $\{u_i\}$ is an energy minimizing sequence, we have, for almost all $r \in (\rho, \rho+\delta)$, $\mu(B_r(a)) \leq \underline{\lim}_i E(\overline{u}_i) \leq \underline{\lim}_i E(u_i^r)$.

Therefore we obtain

$$\frac{1}{\delta} \left[\mu\left(B_{\rho+\delta}(a)\right) - \mu\left(B_\rho(a)\right)\right] \geq \frac{1}{\delta} \int_\rho^{\rho+\delta} \frac{n-2}{r} \mu\left(B_r(a)\right) dr.$$

Let $\delta \to o^+$, we conclude that $\frac{\mu(B_r(a))}{r^{n-2}}$ is a monotone nondecreasing function of r. ☐

The main result of this section is the following:

Theorem 2.3. *There is an $\epsilon_0 = \epsilon_0(n, M, N) > 0$ such that if $\frac{\mu(B_r(a))}{r^{n-2}} \leq \epsilon_0$, $B_r(a) \subseteq M$, then u is smooth inside $B_{r/2}(a)$.*

The key point of the proof of the above theorem is the following Schoen-Uhlenbeck type lemma (cf. [SU, §4]).

Lemma 2.4. *For any $\theta \in (0, \frac{1}{16})$, one has that*

$$\frac{\mu\left(B_r(a)\right)}{r^{n-2}} \le \theta\, \frac{\mu\left(B_{2r}(a)\right)}{(2r)^{n-2}} + C(\theta)\, \frac{\int_{B_{2r}(a)} |u-\overline{u}|^2\, dx}{(2r)^n}$$

whenever $\frac{\mu(B_{2r}(a))}{(2r)^{n-2}} \le \epsilon_1$, *for a sufficiently small* $\epsilon_1 > 0$.

Here and later on, we will use $\overline{u}$ to denote the average of u over the ball of integration. Thus, $\overline{u} = \fint_{B_{2r}(a)} u$ in the Lemma 2.4.

Let us assume, for a moment, that Lemma 2.4 is true, and proceed our proof of Theorem 2.3. By a scaling and a translation, we shall assume $a = 0$, $r = 1$ in the statement of Theorem 2.3. Thus $\mu(B_1(0)) \le \epsilon_0$. As in [HL] we show first that

$$\frac{\mu(B_r(0))}{r^{n-2}} \le C_0\, \epsilon_0\, r^{\alpha}, \text{ for some constant } C_0,\ \alpha > 0, \tag{2.1}$$

and for all $r \in (0, 1/2)$.

As usual, it reduces to show the following discrete version of the decay estimate.

$$\text{if } \mu(B_1(0)) \le \epsilon_0, \text{ then } \frac{\mu(B_{\theta_0}(0))}{\theta_0^{n-2}} \le \frac{1}{2}\, \mu(B_1), \tag{2.2}$$

for some $\theta_0 \in (0, 1/4)$.

Suppose, to the contrary, that the conclusion (2.2) is not valid. That is, such θ_0 does not exist no matter how small ϵ_0 is. Then there would be a sequence of Radon measures of the form $\mu_i = |\nabla v_i|^2\, dx + \nu_i$, such that each μ_i is a weak limit of $|\nabla u_i^j|^2\, dx$, $j = 1, 2, \dots$, for some continuous energy minimizing sequence $\{u_i^j\}$, $j = 1, 2, \dots$, and such that $\mu_i(B_1(0)) = \epsilon_i^2 \to 0^+$, $\theta_0^{2-n}\, \mu_i(B_{\theta_0}(0)) \ge \frac{1}{2}\, \mu_i(B_1(0))$.

We consider normalized sequence of Radon measures $\overline{\mu}_i$, $i = 1, 2, \dots$, such that $\overline{\mu}_i(A) = \mu_i(A) \setminus \mu_i(B_1(0))$, for Borel sets $A \subseteq B_1$. We also consider the blow-up sequence $v_i^* = \frac{v_i - \overline{v}_i}{\epsilon_i}$, $\overline{v}_i = \fint_{B_1(0)} v_i$. Then $v_i^* \rightharpoonup v$ in $W^{1,2}(B_1)$ weakly, $\fint_{B_1(0)} v = 0$, $\int_{B_1(0)} |\nabla v|^2\, dx \le 1$.

Since each v_i is a weakly harmonic map, as in [HL], it is easy to deduce $\Delta v = 0$ in $\mathcal{D}'(B_1(0)$.

Let ℓ_0, k_0 be two positive integers to be chosen later. We apply Lemma 2.4 repeatedly, k_0 times to obtain

$$\begin{aligned} \theta_0^{2-n} \overline{\mu}_i\left(B_{\theta_0}\right) &\le \theta^{k_0}\, \overline{\mu}_i\left(B_{2^{-\ell_0}}\right) 2^{\ell_0(n-2)} \\ &+ \quad C(\theta) \sum_{k=0}^{k_0-1} \theta^k \fint_{B_{r_k}} |v_i^* - \overline{v}_i^*|^2\, dy. \end{aligned} \tag{2.3}$$

Here $\theta_0 = 2^{-(k_0+\ell_0)}$, $r_k = 2^{-k}\, 2^{-\ell_0}$.

We assume k_0 is larger than ℓ_0 so that

$$\theta^{k_0}\, 2^{\ell_0(n-2)} < \frac{1}{4}. \tag{2.4}$$

Hence the first term on the right-hand side is less than 1/4.

Next, since $v_i^* \rightharpoonup v$ in $W^{1,2}(B_1)$ weakly, one may assume $v_i^* \to v$ in $L^2(B_1)$ strongly. Thus

$$\fint_{B_{r_k}} |v_i^* - \overline{v}_i^*|^2\, dy \to \fint_{B_{r_k}} |v - \overline{v}|^2\, dy, \text{ as } \imath \to \infty, \tag{2.5}$$

for each $k = 0, 1, \dots, k_0$. Since v is harmonic and $\int_{B_1} |\nabla v|^2\, dx \le 1$, we have

$$\fint_{B_{r_k}} |v - \overline{v}|^2\, dy \le C_0\, r_k^2 \le \frac{1}{2}\, r_k, \quad \text{for all } k = 0, 1, \dots, k_0 \tag{2.6}$$

whenever $\ell_0 = \ell_0(n)$ is chosen suitably large.

Therefore, for all i large, one has, from (2.3)–(2.6) that

$$\theta_0^{2-n}\, \overline{\mu}_i(B_{\theta_0}) \le \frac{1}{4} + C(\theta_0)\, 2^{-\ell_0}. \tag{2.7}$$

Now we assume ℓ_0 suitably large that the right-hand side of (2.7) is less that 1/3 to obtain a contradiction. This proves (2.2) and hence also (2.1).

Next, by the monotonicity Lemma 2.2, we see that if $\mu(B_1(0))$ is sufficiently small, then $4^{n-2}\, \mu(B_{1/4}(x)) \le 4^{n-2}\, \mu(B_1(0))$ is also small, for all $x \in B_{2/3}(0)$. We thus can apply above arguments to the ball $B_{1/4}(x)$ with center at x to obtain (as (2.1)) that

$$\frac{\mu(B_r(x))}{r^{n-2}} \le C_1(n)\, \epsilon_0\, r^\alpha \tag{2.8}$$

for all $0 < r < \frac{1}{8}$, $x \in B_{2/3}(0)$.

Since $\mu = |\nabla u|^2\, dx + \nu$, we first conclude, from Morrey's Lemma, that u is uniformly Hölder continuous in $B_{2/3}(0)$. Since u is also weakly harmonic, then the usual arguments imply that u is also smooth in $B_{2/3}(0)$. This proves Theorem 2.3.

Remark 2.5. After proving the smoothness of u, we can go back to Lemma 2.4 to obtain, after dividing by r^2, that

$$\frac{\mu(B_r(a))}{r^n} \le 4\theta\, \frac{\mu(B_{2r}(a)}{(2r)^n} + C(\theta) \fint_{B_{2r}(a)} \frac{1}{r^2} |u - \overline{u}|^2\, dx,$$

for all $r \in (0, 1/4)$, $a \in B_{2/3}(0)$.

Using iterations similar to (2.3), we obtain

$$\frac{\mu(B_{2^{-k}}(a))}{2^{-kn}} \leq C(\theta) \sum_{j=1}^{k-1} (4\theta)^{j-1} \fint_{B_{2^{-j}}(a)} 2^{-2j} |u - \overline{u}|^2 \, dy$$
$$(4\theta)^{k-1} \frac{\mu(B_{1/2}(a))}{\left(\frac{1}{2}\right)^n}, \quad \text{for all} \quad k = 1, 2, \dots .$$

Since $|u - \overline{u}| \leq C_0 \, 2^{-j}$ on $B_{2^{-j}}(a)$, note here $\overline{u} = \fint_{B_{2^{-j}}(a)} u$, we thus have

$$\frac{\mu(B_{2^{-k}}(a)}{2^{-kn}} \leq C(\theta, n), \text{ for all } a \in B_{2/3}(0) \text{ and } k = 2, 3, \dots .$$

In other words μ is absolutely continuous with respect to the Lebesgue measure on $B_{2/3}(0)$. In other words $\nu = f(x) \, dx$ for some $f \in L^\infty(B_{2/3}(0))$ on $B_{2/3}(0)$. Later in the paper, we shall show $\nu \equiv 0$ inside $B_{2/3}(0)$.

Here we have shown that

$$\text{If } \circledH^{n-2}(\mu, x) = \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{r^{n-2}} \leq \epsilon_0, \tag{2.9}$$

then $\circledH^{n-2}(\mu, x) = 0$.

Now we modify arguments in [SU, §4] to prove Lemma 2.4. We shall use the exact same notations as in [SU, §4] for convenience. Let u^* be a point in $\mathbb{R}^k$,

$$W_\sigma(u) = \int_{B_\sigma} |u - u^*|^2 \, dx, \; E_\sigma(u) = \int_{B_\sigma} |\nabla u|^2 \, dx,$$
$$C_\sigma^n = B_\sigma^{n-1} \times [-\sigma, \sigma], \; \sigma > 0.$$

We first have a version of Lemma 4.1 of [SU] for $n \geq 4$.

Lemma 2.6. *Let $u \in C^0(\partial C_\sigma^n, N) \cap W^{1,2}(\partial C_\sigma^n, N)$ be given such that $u(x, -\sigma) = u_1(x)$, $u(x, \sigma) = u_2(x)$ for $x \in B_\sigma^{n-1}$ with $u^1(x) = u^2(x) = u(x, t) = u^0(x)$ for $(x, t) \in S_\sigma^{n-2} \times [-\sigma, \sigma]$, $S_\sigma^{n-2} = \partial B_\sigma^{n-1}$. Suppose $n \geq 4$, then there is an extension $\overline{u}_\delta \in C^0(C_\sigma^n, N)$ with $\overline{u}_\delta = u$ on $(S_\sigma^{n-2} \times [-\sigma, \sigma]) \cup (B_\sigma^{n-1} \times [\sigma])$ and $\overline{u}_\delta = u_1$ on $(B_\sigma^{n-1} \setminus B_\delta^{n-1}) \times [\sigma]$ such that*

$$E(\overline{u}_\delta) \leq C \, \sigma(E_\sigma(u_1) + E_\sigma(u_2) + \sigma \, E(u^0)) + \delta,$$
$$W(\overline{u}_\delta) \leq C \, \sigma(W_\sigma(u_1) + W_\sigma(u_2) + \sigma \, W(u^0)) + \delta$$

$0 < \delta \ll \sigma$ (δ can be arbitrary small) and that

$$E_\sigma(u_\delta(\cdot, -\sigma)) \leq E_\sigma(u_1) + \delta.$$

Proof. Suppose u has a continuous extension $\tilde{u}$ from C^n_σ into N. Then we may simply take $\overline{u}_\delta = u_1$ on B^{n-1}_σ in the above lemma, and follow the proof of [SU, Lemma 4.1]. The homogeneous degree zero extension is essentially allowed in our construction as in the proof of the monotonicity Lemma 2.2. The conclusion of Lemma 2.6 follows. Suppose u has no continuous extension from C^n_σ into N, then u must represent a nontrivial homotopy class $\alpha \in \pi_{n-1}(N)$. Since $n \geq 4$, $n-1 \geq 3$, the homotopy class $\alpha^{-1} \in \pi_{n-1}(N)$ can be represented by maps from $\mathbb{S}^{n-1}$ into N of arbitrary small energy, (cf. [W]), and has their support contained in an arbitrary small ball, $B_{\delta/2} \subseteq \mathbb{S}^{n-1}$. Indeed, we may first modify u_1 on the ball B^{n-1}_δ to obtain a new map $\tilde{u}_1$ such that $\tilde{u}_1 \equiv$ const on the ball B^{n-1}_δ, and $\tilde{u}_1 = u_1$ on ∂B^{n-1}_δ. Moreover $E_\sigma(\tilde{u}_1) \approx E_\sigma(u_1)$. Next, we view ∂C^n_σ as $\mathbb{S}^{n-1}$ (after a suitable bi-Lipschitz map $f : \partial C^n_\sigma \to \mathbb{S}^{n-1}$ and so that $f(B^{n-1})_\delta)$ become a ball B_δ in $\mathbb{S}^{n-1}$. Then α^{-1} is simply represented by the inversion of the map $\tilde{u}_1 \circ f^{-1}$ along ∂B_δ in $\mathbb{S}^{n-1}$. Denote the resulting map by $\tilde{u}_2$. We then let $u_\delta : B^{n-1}_\delta \to N$ be such that $u_\delta = \tilde{u}_1$ on $B^{n-1}_\sigma \setminus B^{n-1}_\delta$, $u_\delta = \tilde{u}_2 \circ f$. Then u_δ represents a trivial homotopy class of continuous map from ∂C^n_σ into N. We thus can apply again the arguments in [SU, Lemma 4.1] and our proof of Lemma 2.2 to obtain a continuous extension with desired property. We note that one may assume $E_\sigma(\tilde{u}_2 \circ f) < \delta$, otherwise we simply do an additional rescaling of $\tilde{u}_2 \circ f$ on B^{n-1}_δ. $\square$

Lemma 2.7. *If $u \in C^0(S^n_\sigma, N) \cap W^{1,2}(S^n_\sigma, N)$ for $n = 1, 2$, and if $E(u)\, W(u) \leq \delta_1^2$, $E(u) \leq \delta_1$, for a number $\delta_1 = \delta_1(N)$, then there exists*

$$\overline{u} \in C^0(B^{n+1}_\sigma, N) \text{ with } \overline{u}\big|_{\partial B^{n+1}_\sigma} = u \text{ and}$$
$$E_0(\overline{u}) \leq C_1(E(u)\, W(u))^{1/2},\ W_\sigma(\overline{u}) \leq C_1\, \sigma\, W(u).$$

Proof. For $n = 1$, this lemma is exactly Lemma 4.2 of [SU]. For $n = 2$, as usual, we take $\sigma = 1$. Let v be the harmonic (vector-valued) function from B^3_1 into $\mathbb{R}^K$ (here N is isometrically embedded in $\mathbb{R}^K$) such that $v\big|_{S^2} = u$. It is obvious that $H^{1/2}$-estimate implies that $E_1(v) \leq (E(u)\, W(u))^{1/2} \leq \delta_1$. Since δ_1 is small, we shall conclude that

$$|v(x) - v(0)| \leq C(n, \delta)\, \sqrt{\delta_1} \text{ whenever } |x| \leq 1 - \delta,$$

for a $\delta \in (0, 1/2)$. Note that $v(0) = ⨍_{\mathbb{S}^2} u$. Since $E(u) \leq \delta_1$, we obtain that $\operatorname{dist}^2(v(0), N) \leq ⨍_{\mathbb{S}^2} |u - v(0)|^2 \leq C_0\, E(u) \leq C_0\, \delta_1$. Next, for $r < 1$ and close to 1, $\theta \in \mathbb{S}^2$, we let $a_{m,r}(\theta) =$

$\fint_{B_{m(1-r)}(\theta)} u$, here $B_{m(1-r)}(\theta)$ is the ball in $\mathbb{S}^2$ centered at θ with radius $m(1-r)$, m is a large number to be chosen below.

$$
\begin{aligned}
\text{(2.10)} \qquad \text{Then} \quad & |v(r,\theta) - a_{m,r}(\theta)| \\
\leq & \int_{\mathbb{S}^2} \mathbf{P}(r, \theta - \phi)\, |u(\phi) - a_{m,r}(\theta)|\, d\phi \\
\leq & \, C_N \int_{\mathbb{S}^2 \setminus B_{m(1-r)}(\theta)} \mathbf{P}(r, \theta - \phi)\, d\phi \\
& + C(m) \fint_{B_{m(1-r)}(\theta)} |u - a_{m,r}(\theta)|\, d\phi.
\end{aligned}
$$

Here $\mathbf{P}$ is the Poisson kernel, C_N is a constant depending only on N.

Now we choose m suitably large so that the first term on the right hand side of (2.10) is small. The second term on the right hand side of (2.10) is bounded by $c(m) \fint_{B_{m(1-r)}(\theta)} |u - a_{m,r}(\theta)|^2\, d\phi \leq C(m)\, C_0\, \delta_1$ which for any fixed m, can be made small if δ_1 is sufficiently small.

Suppose the nearest point projection from $N_{2\delta_0} = 2\delta_0$-neighborhood of N in $\mathbb{R}^K$ onto N is smooth, for some $\delta_0(N) > 0$. For this given δ_0, we may find $\delta_1 > 0$ and $r_1 < 1$ such that dist $(v(x), N) \leq \delta_0$ whenever $|x| \geq r_1$. Note dist $(a_{m,r}(\theta), N)$ is also small.

Now by taking δ_1 further small if needed, we have $C(n, 1 - r_1)\, \delta_1 \leq \delta_0$. In other words, dist $(v(x), N) \leq \delta_0$ whenever $E(u) \leq \delta_1$.

Let$\overline{u}(x) = \pi_N\, v(x)$, here π_N is the nearest point projection from $N_{2\delta_0}$ onto N. Then all conclusions of Lemma 2.7 follows exactly as in Lemma 4.2 of [SU].

The proofs of Lemma 4.3 and Lemma 4.4 of [SU] can be easily carried over here also. Except in the statements of these lemmas, all map involved are also continuous. To do so in the Lemma 4.4, the maps $\overline{v}$ and v' have to be slightly modified (for $n \geq 4$ case) as in the proof of Lemma 2.6 in order to guarantee that such continuous maps $\overline{v}$ exist. All the estimates remain valid when arbitrary small given error δ as in Lemma 2.6. Finally, to make all statements consistent, we add one additional assumption in the statement of Lemma 4.3 of [SU] that $E(u) \leq \delta^2\, \epsilon^q$ (see the proof of Lemma 2.7 for the case $n = 3$). This assumption can be easily verified in the *inductive* proof of Lemma 4.3 and Lemma 4.4 of [SU]. We leave these details for the cautious readers. Later in the study of defect measures, a more effective construction which is relatively simpler than constructions in [SU, §4] will also be introduced. Thus we completed the proof of Lemma 2.4. $\square$

3 Rectifiability of Defect Measures

As in the previous section, we let $u_i : B_1^n \to N$ be an energy minimizing sequence among maps in $W^{1,2}(B_1^n, N) \cap C^0(B_1^n, N)$ such that $\mu_i = |\nabla u_i|^2\,dx \rightharpoonup \mu = |\nabla u|^2\,dx + \nu$ as Radon measures. Here $u_i \rightharpoonup u$ in $W^{1,2}(B_1^n, N)$ and $\nu \geq 0$ is also a Radon measure. The main result of the previous section is that

$$\text{if } \mu(B_1^n) \leq \epsilon_0, \text{then } u \text{ is smooth inside } B_{2/3}^n. \tag{3.1}$$

The first important result of this section is the following:

Theorem 3.1. *If $\mu(B_1^n) \leq \epsilon_0$, then $\nu \equiv 0$ in $B_{2/3}^n$.*

Though we should give a proof of the above theorem for all $n \geq 2$ cases, we would like to present a direct proof for the case $n = 2$.

Proof of Theorem 3.1, $n = 2$ case. First we observe the following facts for $n = 2$ case. Suppose $u : \mathbb{S}^1 \to N$ be map with $E(u) \leq \epsilon_0$, then Morrey's Theorem says that $\overline{u}$ is in $C^\infty(B_1^2, N) \cap C^\alpha(\overline{B_1^2}, N)$ for some $\alpha > 0$. Here $\overline{u}$ is an energy minimizer on B_1^2 with $\overline{u} = u$ on $\mathbb{S}^1$. Suppose, instead of a single map u, we have a sequence $\{u_i\}$ with the same property as u i.e. $\int_{\partial B_1} |\nabla u_i|^2 \leq \epsilon_0$. Let $\{\overline{u}_i\}$ be a corresponding sequence of minimizers with $\overline{u}_i = u_i$ on $\mathbb{S}^1$. Suppose $\overline{u}_i \rightharpoonup \overline{u}$ weakly in $W^{1,2}(B_1^2, N)$. Then $\overline{u}_i \to \overline{u}$ strongly in $W^{1,2}(B_1^2, N)$. Indeed, it is well-known that $\overline{u}_i \to \overline{u}$ strongly in $W^{1,2}_{\text{loc}}$ (cf. [SU, §4]). Let $\sigma \in (0, 1/2)$, and let v_i be the linear harmonic extension of $\overline{u}_i$ from $\partial(B_1^2 \setminus B_{1-\sigma}^2)$into $B_1^2 \setminus B_{1-\sigma}$. Then it is easy to see $E(v_i, B_1^2 \setminus B_{1-\sigma}^2) \to E(\overline{v}, B_1^2 \setminus B_{1-\sigma}^2)$. Here $\overline{v}_i \to \overline{v}$ in $W^{1,2}(B_1^2 \setminus B_{1-\sigma}^2)$. Moreover, $\overline{v}$ is the harmonic extension of $\overline{u}$ from $\partial(B_1^2 \setminus B_{1-\sigma}^2)$ into $B_1^2 \setminus B_{1-\sigma}^2$. Therefore, $E(\overline{v}, B_1^2 \setminus B_{1-\sigma}^2) \to 0$ as $\sigma \to 0$. On the other hand, the images of v_i are in a small neighborhood of N, so after nearest point projection π_N onto N, we see

$$\begin{aligned} &E(u_i, B_1^2 \setminus B_{1-\sigma}^2) \leq E(\pi_N\, v_i, B_1^2 \setminus B_{1-\sigma}^2) \\ &\leq C\,E(v_i, B_1^2 \setminus B_{1-\sigma}^2) \to C\,E(\overline{v}, B_1^2 \setminus B_{1-\sigma}^2) \to 0 \text{ as } \sigma \to 0. \end{aligned}$$

Since $\mu(B_1) \leq \epsilon_0$, we may find a $\rho \in (\frac{1}{2}, 1)$ and infinitely many i's such that $\int_{\partial B_\rho^2} |\nabla u_i|^2 \leq 3\epsilon_0$, for these i's. This is a consequence of the Fabini's Theorem (see [SL]). We let u_i^* be such that $u_i^* = u_i$ on $B_1^2 \setminus B_\rho^2$, $u_i^* = \overline{u}_i$ on B_ρ^2, here $\overline{u}_i$ is an energy minimizing map with $\overline{u}_i = u_i$ on ∂B_ρ^2.

It is obvious that $\{u_i^*\}$ form a new minimizing sequence in $C^0(\overline{B_1^2}, N)$ with $u_i^* = u_i$ on ∂B_1^2. Moreover, $|\nabla u_i^*|^2\,dx \rightharpoonup |\nabla u^*|^2\,dx + \nu^*$ with $\nu^* = \nu$ on $B_1 \setminus B_\rho$ and $\nu^* = 0$ on B_ρ by above arguments. Note also that $u^* = u$ on $B_1 \setminus B_\rho$ and $E(u^*) \le E(u)$ as $u^* = u$ on ∂B_ρ and u^* is also energy minimizing (an easy consequence of the strong convergence) statement above), we obtain $\nu \equiv 0$ on B_ρ. This proved Theorem 3.1 for the case $n = 2$. $\square$

Proof of Theorem 3.1, for $n \ge 3$. We first prove the $n = 3$ case. It is the important first inductive step.

First we pick up $\rho \in (1/2, 2/3)$ such that $\int_{\partial B_\rho^3} |\nabla u_i|^2 \le 8\epsilon_0$ for infinitely many i's.

Since $|\nabla u_i|^2\,dx = \mu_i \rightharpoonup \mu = |\nabla u|^2\,dx + \nu$ in B_1 as Radon measures, to show $\nu \equiv 0$ in $B_{1/2}$, it suffices to show $E(u_i, B_\rho^3) \le E(u, B_\rho^3) + \delta$ for any $\delta > 0$ and all sufficiently large i's.

To do so, we use the fact that u_i is a minimizing sequence in $C^0(B_1^3, N)$. We shall construct a new sequence $\{\tilde{u}_i\}$ in $C^0(B_1, N)$ such that $\tilde{u}_i = u_i$ on $B_1^3 \setminus B_\rho^3$ and such that $\tilde{u}_i = u$ on $B_{\rho(1-\epsilon)}^3$, for a very small positive ϵ and $E(\tilde{u}_i, B_\rho^3 \setminus B_{\rho(1-\epsilon)}^3) \le C(\epsilon) < \delta$.

Since $\underline{\lim}_{i\to\infty} E(\tilde{u}_i, B_\rho^3) \ge \lim_{i\to\infty} E(u_i, B_\rho^3)$ we obtain $E(u_i, B_\rho) \le E(u, B_\rho^3) + C(\epsilon)$ for all sufficiently large i, and that will be what we wanted.

We already proved in the previous section that u is smooth inside $B_{2/3}$, say $\|u\|_{C^{1,\alpha}} \le C\sqrt{\epsilon_0}$. We first consider linear harmonic extension v_i on $B_\rho^3 \setminus B_{\rho(1-\epsilon)}^3$ such that $v_i = u$ on $B_{\rho(1-\epsilon)}^3$ and $v = u_i$ on ∂B_ρ^3, $i = 1, 2, \ldots$. We observe that $v_i$is uniformly smooth away from outside boundary and $v_i \to v$ strongly in $W^{1,2}(B_\rho^3 \setminus B_{\rho(1-\epsilon)}^3)$, here v is the linear harmonic extension of u on $\partial(B_\rho^3 \setminus B_{(1-\epsilon)\,\rho}^3)$. In particular, the image of v stays inside a very small neighborhood of N in $\mathbb{R}^K$. Thus, for all sufficiently large i, the images of $v_i(x)$ stay in a very small neighborhood of N in $\mathbb{R}^K$ so long as $|x| \le (1 - \epsilon_i)\,\rho$, here $0 < \epsilon_i \ll \epsilon$, $\epsilon_i \to 0^+$ as $i \to \infty$.

On the other hand, v_i can be represented as a Poisson integral of boundary values u on $\partial B_{\rho(1-\epsilon)}$ (here $\epsilon < 0$ small but fixed) and u_i on ∂B_ρ. Since $\int_{\partial B_\rho} |\nabla u_i|^2 \le 8\epsilon_0$ is very small, by then proof of Lemma 2.7 in the previous section, we get the images of $v_i(x)$, for $\rho(1-\lambda) \le |x| \le \rho$, for some $0 < \lambda < \epsilon$, stay inside a very small neighborhood of N. In other words, $v_i(B_\rho^3 \backslash B_{\rho(1-\epsilon)}^3) \subseteq N_{\delta_0}$ – δ_0 neighborhood of N in $\mathbb{R}^K$ on which π_N (the nearest point projection into N is smooth). We let $\tilde{u}_i = \pi_N \cdot v_i$ on $B^3_\rho \setminus B^3_{\rho(1-\epsilon)}$. Then $\tilde{u}_i$ is in $C^\circ\overline{(B^3_\rho \setminus B^3_{\rho(1-\epsilon)})}$. Moreover, $E(\tilde{u}_i, B^3_\rho \setminus B^3_{\rho(1-\epsilon)}) \leq C_N\, E(v_i, B^3_\rho \setminus B^3_{\rho(1-\epsilon)}) \to C_N\, E(v, B^3_\rho \setminus B^3_{\rho(1-\epsilon)})$ as $i \to \infty$. Finally, $E(v, B^3_\rho \setminus B^3_{\rho(1-\epsilon)}) \leq E(u, B^3_\rho \setminus B^3_{\rho(1-\epsilon)}) \leq C\, \epsilon_0\, \epsilon = C(\epsilon)$. This completes the proof of the $n = 3$ case. $\square$

To prove the theorem for the case $n \geq 4$, it suffices to construct $\tilde{u}_i$ as in the proof for the case $n = 3$. That is the following:

Lemma 3.2. *Let $u \in C^1(B^n_1, N)$ with $|\nabla u|_{C^1(B_1)} \leq \epsilon_0$. Let $\{u_i\} \in C^0(B^n_1, N) \cap W^{1,2}(B^n_1, N)$ such that $u_i \rightharpoonup u$ in $W^{1,2}(B_1, N)$ weakly as $i \to \infty$, and that $\int_{\partial B^n_1} |\nabla u_i|^2 \leq \epsilon_0$, for $i = 1, 2, \ldots$. Then, for any $\epsilon \in (0, \frac{1}{2})$, there is a sequence $\tilde{u}_i \in C^0(B^n_1 \setminus B^n_{1-\epsilon}, N)$ defined on $B^n_1 \setminus B^n_{1-\epsilon}$ such that $\tilde{u}_i = u$ on $\partial B^n_{1-\epsilon)}$, $\tilde{u}_i = u_i$ on ∂B^n_1. Moreover, $E(\tilde{u}_i, B^n_1 \setminus B^n_{1-\epsilon}) \leq C(n, N, \epsilon) \to 0$ as $\epsilon \to 0$ uniformly in i whenever $0 < \epsilon_0 \leq \epsilon_0(n, N)$.*

Proof. By induction, we have already shown the conclusion of Lemma 3.2 when $n = 2$ and $n = 3$. Suppose Lemma 3.2 is true for $n \leq k$, for some $k \geq 3$. Then we consider the case $n = k + 1 \geq 4$. First, we can divide $\mathbb{S}^k$ into 2 disjoint subdomain Ω_j $j = 1, 2$ such that each spherical shell, $j = 1, 2$, $S_j = \{(r, \theta) \in \mathbb{R}^{k+1} : 1 - \epsilon \leq r \leq 1,\ \theta \in \Omega_j \leq \mathbb{S}^k\}$ is bi-Lipschitz to equivalent to $C = [1 - \epsilon, 1] \times B^k_1$. Moreover, by the Fubini's Theorem, we can arrange such Ω_j that $\int_{\partial \Omega_j} |\nabla u_i|^2 \leq C(n)\, \epsilon_0$, $j = 1, 2$.

We should simply construct a suitable extension on C (after composition with suitable bi-Lipschitz maps from Ω_j into C).

By induction, we can find an extension $\overline{u}_i$ on $[1 - \epsilon, 1] \times \partial B^k_1$ such that $\overline{u}_i \in C^0([1 - \epsilon, 1] \times \partial B^k_1, N)$ with $\overline{u}_i = u$ on $[1 - \epsilon] \times \partial B^k_1$ and $\tilde{u}_i = u_i$ on $[1] \times \partial B^k_1$ such that $E(\tilde{u}_i) \leq C(k, N, \epsilon) \to 0$ as $\epsilon \to 0^+$ uniformly in ϵ whenever ϵ_0 is suitably small.

Next, we simply let $\tilde{u}_i$ on C to be the homogeneous degree zero extension of $\tilde{u}_i$ on $[1 - \epsilon, 1] \times \partial B^k_1$, u on $[1 - \epsilon] \times B^k_1$ and u_i on $[1] \times B^k_1$. Such homogeneous degree zero extension can be approximate by maps in $C^0(C, N)$ in the strong $W^{1,2}(C, N)$ topology (with the same Dirichlet boundary conditions) whenever the boundary value of $\tilde{u}_i$ represents a trivial homotopy class of $\pi_k(N)$. If such boundary value of $\tilde{u}_i$ represents a nontrivial homotopy class of $\pi_k(N)$, then we modify $\overline{u}_i$ in a neighborhood of $\frac{1}{2}(2 - \epsilon) \times \{p_0\}$ for some $p_0 \in \partial B^k_1$ as in the proof of Lemma 2.6 such that the modified maps has the same stated properties as

$\overline{u}_i$. Moreover the resulting maps will give a topological trivial class and so the homogeneous degree zero extension will be allowed. In order to make this process as whole to be possible, one should do the following. We view the whole spherical shell as one piece with topological trivial boundary value. Then subdivide into 2 pieces after inductive step of extensions then modify the extension maps near one point will make both 2 pieces have boundary values in trivial topological class.

In any event, we need to estimate the energy of resulting map $\tilde{u}_i$. The part from the homogeneous extension of $\overline{u}_i$ is by induction has energy smaller than $C(k, N, \epsilon)$. The part from the homogeneous extension of u is obviously small as $|\nabla u| \leq \epsilon_0$. The part from the homogeneous extension of u_i can be calculated directly and can be estimated by $C(n)\,\epsilon^{k-2}\int_{B_1^k}|\nabla u_i|^2$. Here ϵ^{k-2} is a natural scaling factor. Since $k \geq 3$, this last quantity is again bounded by $C(n)\,\epsilon\,\epsilon_0$, This completes the proof of Lemma 3.2, and hence also Theorem 3.1. $\square$

Remark 3.3. Similar statement as Lemma 3.2 was established by Luckhaus [Lu] under weak assumptions on u. It is a key ingredient in his proof of weak limits of energy-minimizing maps are also energy minimizing. However, his construction yields a comparison map which may not be continuous in general.

Let us describe a few consequence of Theorem 2.3 and Theorem 3.1. Define

$$\Sigma = \{x \in M : Ⓗ^{n-2}(\mu, x) > 0\}.$$

Corollary 3.4. *For all $x \in \Sigma$, one has $Ⓗ^{n-2}(\nu, x) \geq \epsilon_0$, for some $\epsilon_0 > 0$. Hence Σ is closed in M, and $\mathcal{H}^{n-2}(\Sigma) \leq \mu(M)/\epsilon_0$. Moreover, $\nu = Ⓗ^{n-2}(\mu, \cdot)\,\mathcal{H}^{n-2}\lfloor\epsilon$.*

Proof. We have already shown that if $Ⓗ^{n-2}(\nu, x) < \epsilon_0$, then $Ⓗ^{n-2}(\mu, x) = 0$. Thus the first statement follows. By a usual covering argument, one then deduces that $\mathcal{H}^{n-2}(\Sigma) \leq \mu(M) \setminus \epsilon_0 < \infty$. Since $Ⓗ^{n-2}(\mu, x)$ is upper semi-continuous in x (which is an easy consequence of the monotonicity Lemma 2.2), Σ is also closed. Finally, for $\mathcal{H}^{n-2}$-a.e. $x \in M$, $\lim_{r\to\infty}\frac{1}{r^{n-2}}\int_{B_r(x)}|\nabla u|^2\,dy = 0$, by a theorem of Federer and Ziemer, we obtain $Ⓗ^{n-2}(\mu, x) = Ⓗ^{n-2}(\nu, x)$ for $\mathcal{H}^{n-2}$-a.e. $x \in M$. Since $\nu \equiv 0$ away from Σ, and since Lemma 2.2 implies ν is absolutely continuous with respect to $\mathcal{H}^{n-2}$-measure, we conclude that $\nu = Ⓗ^{n-2}(\mu, x)\;\mathcal{H}^{n-2}\lfloor\Sigma$. $\square$

Similar to [L2], we have the following:

Theorem 3.5. *Σ is a $\mathcal{H}^{n-2}$-rectifiable set. That is ν is a $\mathcal{H}^{n-2}$-rectifiable measure.*

The proof of this theorem is very similar to the proof of the rectifiability of defect measures arise in the weak limits of stationary harmonic maps (cf. [L2]). It follows from the following three lemmas.

Lemma 3.6. *For ν-a.e. $x \in \Sigma$, and any given $\delta > 0$, there is a $\delta_x > 0$ such that, if $0 < r < \delta_x$, then there is a $(n-2)$-dimension plane V_r passes through x with property that*

$$\frac{\nu(B_r(x) - V_r^\delta)}{r^{n-2}} \le \epsilon(r) \to 0, \quad as \ r \to 0^+.$$

Here V_r^δ is the δ-neighborhood of V_r in $\mathbb{R}^n$.

Lemma 3.7. *Suppose $E \subset \Sigma$ is purely unrectifiable, then $H^{n-2}(\mathbf{P}_V(E)) = 0$ for any $(n-2)$-dimensional plane V in $\mathbb{R}^{n-2}$. Here $\mathbf{P}_V$ denotes the orthonormal projection onto V in $\mathbb{R}^n$.*

Lemma 3.8. *For $\mathcal{H}^{n-2}$-a.e. $x \in E$ (E any $\mathcal{H}^{n-2}$-measurable subset of Σ), one has*

$$\limsup_{r\to 0} \sup_{V \in G\lfloor(n,n-2)} \mathcal{H}^{n-2}(\mathbf{P}_V(E \cap B_r(x)) \setminus r^{n-2} = |B_1^{n-2}| > 0.$$

The property stated in the Lemma 3.6 is called the weak-tangent plane property. Whenever $\mathcal{H}^{n-2}$-measurable set E satisfies the weak-tangent property, then the conclusion of Lemma 3.7 is valid as shown in [L2].

The proofs of Lemma 3.6 and Lemma 3.7 are similar to that in [L2] with some necessary modifications given in section 4 below. These give us motivations to develop a general varifold type theory for Sobolev mappings that will be discussed also in section 4. Therefore, we shall postpone the proofs of Lemma 3.6 and Lemma 3.8 in section 4.

To end this section, we discuss cases $p \neq 2$.

Suppose p is an integer, $p \ge 2$. Then all the results as well as their proofs can be easily carried over with some obvious modifications. The modifications for the proof of Lemma 2.4 are somewhat not obvious. The case when $\dim M \le p+1$ can be done exactly as that for Lemma 2.7. The inductive proof of Lemma 2.4 where $\dim M \ge p+2$ uses the same observation as in Lemma 2.6 and arguments in [HL]. The derivation of Theorem 2.3 from Lemma 2.4 can be found also in [HL] for any $p > 1$.

For the proof of Theorem 3.1, the case $\dim M \leq p$ is again trivially carried over (as for $n = 2$ case). The case $\dim M = p + 1$ should be handled exactly as our proof of the $n = 3$ case. Note that linear harmonic extensions satisfies all desired inequalities for p-energy up to a constant $c(n, p)$. Finally the inductive proof of Lemma 3.2 works when $\dim M \geq p + 2$. When p is an integer, $p \geq 2$, and if $\pi_p(N) = 0$ then, by a theorem of Bethuel [B], we know smooth maps are dense in $W^{1,p}(M, N)$ with the strong topology. Thus minimizing p-energy among $C^0(M, N) \cap W^{1,p}(M, N)$ maps is the same as minimizing p-energy over all space $W^{1,p}(M, N)$. In particular, the defect measures, $\nu \equiv 0$ in this situation.

Finally, let us discuss the case p is not an integer and $1 < p < n = \dim M$. Again, the proof of Theorem 3.1 and Theorem 2.3 are very similar and we should simply omit here. However we have the following additional conclusion concerning the defect measure ν.

Theorem 3.9. *Suppose $1 < p < n = \dim M$, and p is not an integer. Suppose $\{u_i\}$ is a p-energy minimizing sequence in $C^0(B_1^n, N)$ such that $u_i \rightharpoonup u$ weakly in $W^{1,p}(B_1^n, N)$ and that $\mu_i \equiv |\nabla u_i|^p \, ex \to u \equiv |\nabla u|^p \, dx + \nu$ as Radon measures in B_1^n. Then $\nu \equiv 0$ in B_1^n. In particular, $u_i \to u$ strongly in $W^{1,p}_{\mathrm{loc}}(B_1^n, N)$.*

Proof. Via the monotonicity Lemma 3.2, $\frac{\mu(B_r(x))}{r^{n-p}}$ is a monotone nondecreasing function of r, whenever $B_r(x) \subseteq B_1$. In particular, $\textcircled{H}^{n-p}(\mu, x) = \lim_{r \to 0^+} \frac{\mu(B_r(x))}{r^{n-p}}$ exists and, as a function of x it is upper semi-continuous. Theorem 2.3 and Theorem 3.1 imply that if $\frac{\mu(B_r(x))}{r^{n-p}} \leq \epsilon_0$, then u is $C^{1,\alpha}$ (for some $\alpha > 0$, cf. [HL]) in $B_{r/2}$ and $\nu \equiv 0$ on $B_{r/2}$. Moreover monotonicity lemma implies ν is absolutely continuous with respect to the $(n-p)$-dimensional Hausdorff measure, $\mathcal{H}^{n-p}$. As before, we let $\Sigma = \{x \in B_1 : \textcircled{H}^{n-p}(\mu, x) > 0\}$. Then $\textcircled{H}^{n-p}(\mu, x) \geq \epsilon_0$ for every $x \in \Sigma$. Σ is closed in B_1 with $\mathcal{H}^{n-p}(\Sigma) \leq \mu(B_1)/\epsilon_0$. Moreover $\nu = \textcircled{H}^{n-p}(\mu, x) \, \mathcal{H}^{n-p} \lfloor \Sigma$.

To prove such $\nu \equiv 0$, we follow the idea of J. Marstrand (see [P]). In fact, it follows from the Marstrand's Theorem. But for the sake of completeness, we sketch a relatively simpler proof here.

Since $\textcircled{H}^{n-p}(\mu, x)$, $x \in \Sigma$ is a Borel-measurable (as easy fact to check), it is approximate continuous $\mathcal{H}^{n-p}$-a.e. on Σ. We let $x_0 \in \Sigma$ be such that $\textcircled{H}^{n-p}(\mu, x)$ is approximate continuous at x_0 as a function on Σ. Here we note that $\textcircled{H}^{n-p}(\mu, x) = \textcircled{H}^{n-p}(\nu, x)$, for $\mathcal{H}^{n-p}$-a.e. $x \in \Sigma$, by a theorem of Federer and Ziemer. We thus assume also that at x_0, $\textcircled{H}^{n-p}(\mu, x_0) = \textcircled{H}^{n-p}(\nu, x_0)$. Let $\{\lambda_i\} \downarrow 0$ be a sequence of numbers, we consider a sequence of Radon measures ν_i, $\nu_i(A) = \frac{\nu(x_0+\lambda_i A)}{\lambda_i^{n-p}}$, for all Borel subset $A \subset \mathbb{R}^n$, such that $x_0 + \lambda_i A \subseteq B_1$. Since $\nu_i(B_\rho(0)) \to \rho^{n-p} \textcircled{H}^{n-p}(\nu, x_0)$, we may assume $\nu_i \to \eta$ as Radon measure on $\mathbb{R}^n$ when $i \to \infty$ (by taking subsequences if needed). η is called a tangent measure of ν at x_0. It is obvious $\eta(B_\rho(0)) = \rho^{n-p} \textcircled{H}^{n-p}(\nu, x_o)$, we want to show for any $x \leftarrow$ spt η, $\eta(B_\rho(x)) = \textcircled{H}^{n-p}(\nu, x_0)\, \rho^{n-p}$. Indeed, for $x \in$ spt η, $\rho > 0$ is such that

$$\eta(B_\rho(x)) = \lim_{i\to\infty} \nu_i(B_\rho(x)) = \lim_{i\to\infty} \frac{\nu(B_{\lambda_i\rho}(x_0 + \lambda_i x))}{\lambda_i^{n-p}}.$$

We first estimate $\eta(B_\rho(x))$ from above.

$$\frac{\nu(B_{\lambda_i\rho}(x_0 + \lambda_i x))}{\lambda_i^{n-p}} \leq \rho^{n-p} \frac{\nu(B_\delta(x_0 + \lambda_i x))}{\delta^{n-p}}$$

for any $\delta > 0$ fixed (since $\lambda_i \to 0$ and since Lemma 2.2)

$$\leq \rho^{n-p}\, \nu(B_{\delta+\lambda_i|x|}(x_0)) \setminus \delta^{n-p}.$$

As $i \to \infty$ the right hand side of the last inequality tends to $\rho^{n-p}\, \nu(B_\delta(x_0)) \setminus \delta^{n-p}$. Since $\delta > 0$ is arbitrary and $\textcircled{H}^{n-p}(\nu, x_0) = \lim_{\delta\to 0^+} \frac{\nu(B_\delta\,(x_0))}{\delta^{n-p}}$, we obtain

$$\eta(B_\rho(x)) \leq \textcircled{H}^{n-p}(\nu, x_0)\, \rho^{n-p}.$$

To estimate $\eta(B_\rho(x))$ from below we use the fact that $\textcircled{H}^{n-p}(\nu, x)$ is $\mathcal{H}^{n-p}$-approximate continuous and has definite positive lower bound. Note also $\textcircled{H}^{n-p}(\nu, x)$ has an upper bound for all x near x_0. Therefore, inside the ball $B_{\lambda_i(\rho+|x|)}(x_0)$, the set, for any small positive ϵ,

$$B_\epsilon = \{y \in \Sigma \cap B_{\lambda_i(\rho+|x|)}(x_0) : \textcircled{H}^{n-p}(\nu, y) \leq \textcircled{H}^{n-p}(\nu, x_0) - \epsilon\},$$

$$\text{satisfies } \nu(B_\epsilon) \leq \delta_i\, \epsilon_0(\lambda_i(\rho + |x|))^{n-p}.$$

Here $\delta_i \to 0^+$.

Since $x \in$ spt η, there is a sequence $x_i \in$ spt ν_i, $x_i \to x$. That is $y_i = x_0 + \lambda_i x_i \in$ spt ν. Then $\eta(B_\rho(x)) \geq \lim_{i\to\infty} \nu_i(B_{\rho-\epsilon}(x_i)) = \lim_i \frac{\nu(B_{\lambda_i(\rho-\epsilon)}(y_i))}{\lambda_i^{n-p}}$.

Since $\nu(B_\epsilon) = o_i(1)\,\lambda_i^{n-p}$, $o_i(1) \to 0$ as $i \to \infty$, we must be able to find a point $\overline{y}_i \in$ spt ν $\overline{y}_i \notin B_\epsilon$, $|\overline{y}_i - y_i| \le \lambda_i\,\epsilon$, $\text{Ⓗ}^{n-p}(\nu, \overline{y}_i) \ge \text{Ⓗ}^{n-p}(\nu, x_0) - \epsilon$. Hence

$$\begin{aligned}\frac{\nu(B_{\lambda_i(\rho-\epsilon)}(y_i))}{\lambda_i^{n-p}} &\ge \frac{\nu(B_{\lambda_i(\rho-2\epsilon)}(\overline{y}_i))}{\lambda_i^{n-p}} \\ &\ge (\rho - 2\epsilon)^{n-p}\,\text{Ⓗ}^{n-p}(\nu, \overline{y}_i) \ge (\rho - 2\epsilon)^{n-p}[\text{Ⓗ}^{n-p}(\nu, x_0) - \epsilon]\end{aligned}$$

since $\epsilon < 0$ is arbitrary, we obtain for any $\rho > 0$ that $\eta(B_\rho(x)) \ge \rho^{n-p}\,\text{Ⓗ}^{n-p}(\nu, x_0)$, whenever $x \in$ spt η. ☐

To complete the proof of Theorem 3.9 we just need the following:

Lemma 3.10. *Let $\eta \ge 0$ be a Radon measure on $\mathbb{R}^n$ such that $\eta(B_\rho(x)) = \rho^\alpha$, for all $x \in$ spt η and $\rho > 0$. Here α is a nonnegative real number. Then α is an integer.*

Proof. Given an $\alpha \ge 0$, let n be the smallest positive integer such that $\mathbb{R}^n$ supports a Radon measure η with the property that $\eta(B_\rho(x)) = \rho^\alpha$ for all $x \in$ spt η and $\rho > 0$.

If support $(\eta) = \mathbb{R}^n$, then η would be a rotation and translation in variant measure on $\mathbb{R}^n$, hence it is a constant multiple of the Lebesgue measure. In particular $\alpha = n$ and we are done.

If spt $(\eta) \ne \mathbb{R}^n$. Let $x_0 \notin$ spt (η) and let $B_R(x_0)$ be the largest open ball such that spt $(\eta) \cap B_R(x_0) = \varnothing$. Then there must be a point, say $\underline{0} \in$ spt $(\eta) \cap \overline{B_R}(x_0)$. Let μ be a tangent measure of η at $\underline{0}$. Then we obtain a Radon measure μ has the same property as η (by argument earlier). Moreover we may assume spt (μ) is contained in a closed half space, say $\overline{R^n_+}$. Note $\mathbb{R}^n$ is the blow-up of $B_R(x_0)$ at $\underline{0}$.

Now we define $\vec{b} = \int_{B_1^n(\underline{0})} x\,d\,\mu(x)$. Suppose $b_n = 0$, then spt $(\mu) \subseteq \{x \in \mathbb{R}^n : x_n = 0\}$, that will contradicts to the fact that $\mathbb{R}^n$ is the smallest (n is the smallest) Euclidean space which can support such Radon measures. Thus $b_n \ne 0$.

Let $y \in$ spt μ and y near $\underline{0}$, then we have $\int_{B_1^n(y)}(1^2 - |x-y|^2)\,d\mu(x) = \int_{B_1^n(0)}(1-|x|^2)\,d\mu(x)$ by the property of $\mu(B_r(y)) = \mu(B_r(0)) = r^\alpha$, for all $r > 0$. This identity leads to

$$\begin{aligned}&\int_{B_1^n(y)\cap B_1^n(0)} 2x \cdot\, y\,d\mu(x) = \int_{B_1^n(y)\cap B^{n-1}(0)} |y|^2\,d\mu(x) + \\ &+ \int_{B_1^n(0)\setminus B_1^n(y)} (1 - |x|^2)\,d\mu(x) - \int_{B_1^n(y)\setminus B_1^n(0)} (1 - |x-y|^2)\,d\mu(x) \\ &= O(|y|^2) \quad \text{as} \quad y \to 0^+.\end{aligned}$$

In other word, $\vec{b} \cdot y = O(|y|^2)$ for all $y \in$ spt μ and y near $\underline{0}$. Let λ be a tangent measure of μ at $\underline{0}$, then spt (λ) will be contained in a a hyperplane orthogonal to $\vec{b} \cdot \lambda$ has the same stated property as μ, i.e., as η. Thus we again obtain a contradict ion as $\mathbb{R}^n$ is the smallest Euclidean space which can support such Radon measures. The only possibility is therefore $\alpha = n$. $\square$

Theorem 3.9 has several interesting consequences. First the map u given in the statement of Theorem 3.9 is a p-energy stationary map, i.e., for any smooth diffeomorphisms $\phi_t : B_1 \to B_1$ with $\phi_t = id$ near ∂B_1, $-\delta \leq t \leq \delta$, $\delta > 0$, and $\phi_0 = id$ on B_1, one has $\frac{d}{dt} E_p(u(t))\big|_{t=0} = 0$. Here$u(t)(x) = u \circ \phi_t(x)$. Indeed, $u(t)$ is the strong limit of $u_i(t) = u_i \circ \phi_t$ in $W^{1,p}(B_1, N)$ with $u_i(t) \equiv u_i$ on ∂B_1. Since $\{u_i\}$ is a minimizing sequence in $W^{1,p}(B_1, N) \cap C^0(B_1, N)$, one has $\lim_{i\to\infty} E_p(u_i(t)) \geq \lim_{i\to\infty} E_p(u_i) = E_p(u)$. By strong convergence of u_i to u, one has

$$E_p(u(t)) \geq E_p(u), \quad \text{for} \quad t \in (-\delta, \delta).$$

The stationary follows.

The second consequence is that such maps u as in the statement of Theorem 3.9 with a given $E_p(\cdot)$ energy bound automatically compact in $W^{1,p}_{\text{loc}}(B_1, N)$. Indeed, suppose $\{v_i\}$ be a sequence of such map, with $E_p(v_i, B_i) \leq C$, $i = 1, 2, \ldots$, proof of Theorem 3.9 given (up to a subsequence) $v_i \to v$ strongly in $W^{1,p}(B_1, N)$. This compactness combine with usual dimension reduction argument (cf. [HL]) gives the conclusions of Theorem 2 and Theorem 2′ described in the introduction. In particular, the Hausdorff dimension of the sing (u), for u given in Theorem 3.9, is not larger than $n - [p] - 1$.

4 Generalized Varifolds and Defect Measures

The theory of varifolds were motivated by the study of certain "weakly converging" surfaces. e.g., the study of a weakly converging sequence of minimal surfaces. It is a well-known fact a weakly converging sequence of stationary integral currents may have a limit which is not a stationary current. However, a weakly limit of stationary varifolds is [A] and also a stationary varifold. For the general theory of varifolds and related subjects we refer to [SL].

Here we introduce the following definitions.

Definition 4.1. *A generalized k-varifold in an open set Ω of $\mathbb{R}^n$ is a nonnegative Radon measure V in $\Omega \times A_{k,n}$. This class of varifolds will be denoted by $V_k^*(\Omega)$. Here $A_{k,n} = \{A : A$ is a symmetric $n \times n$ matrix such that trace $(A) = k,\ -n\, I_n \leq A \leq I\}$.*

The mass μ_V of a $V \in V_k^*(\Omega)$ is $\pi_\# V$ (the projection of V on Ω), where $\pi : \Omega \times A_{k,n} \to \Omega$ is the projection. Thus $\mu_V(B) = V(B \times A_{k,n})$ for every Borel subset B of Ω.

Definition 4.2. *The first variation, denote by δV of a $V \in V_k^*(\Omega)$, is simply a distribution on Ω, such that*

$$\int_{\Omega \times A_{k,n}} A : \nabla Y(x)\, dV(x, A) = -\int_\Omega Y(x) \cdot \delta\, V(x). \tag{4.1}$$

Here $A : \nabla Y(x) =$ tr $(A \cdot \nabla Y(x))$, $Y \in C_0^\infty(\Omega, \mathbb{R}^n)$.

Remark 4.3. The classically defined varifolds (cf. [SL], [A]) is a subclass of $V_k^*(\Omega)$. There are Radon measures V on $\Omega \times A_{k,n}$ with spt $V \subseteq \Omega \times GL(k,n)$. Here $GL(k,n) = \{A \in A_{k,n}, A^2 = A\}$, i.e., any element $A \in GL(k,n)$ corresponding to an orthogonal projection of $\mathbb{R}^n$ onto a k-dimensional plane.

Example 4.4. Let $u \in W^{1,2}(\Omega, N)$, and let $V_u = \delta_{A_u(x)} \cdot |\nabla u(x)|^2\, dx$. Here $\delta_{A_u(x)}$ is the Dirac measure on $A_{n-2,n}$ centered at

$$A_u(x) = \begin{cases} I_{n-2}\ \dfrac{\nabla u(x) \otimes \nabla u(x)}{|\nabla u(x)|^2} & \text{whenever } \nabla u(x) \neq 0 \\ \begin{pmatrix} I_{n-2} & 0 \\ 0 & 0 \end{pmatrix} & \text{whenever } \nabla u(x) = 0. \end{cases}$$

$\nabla u(x) \otimes \nabla u(x)$ is a $n \times n$ matrix such that its $(i,j)^{\text{th}}$ component is given by $u_{x_i} \circ u_{x_j}$. Then V_u is an element in $V_{n-2}^*(\Omega)$. Moreover, the first variation of V, δV, in this case is given by div $[|\nabla u(x)|^2\, I_n - 2\, \nabla u(x) \otimes \nabla u(x)]$ in the sense of distribution. Suppose u is also weakly harmonic, then it is easy to see that V_u is a stationary generalized varifold in $V_{n-2}^*(\Omega)$, i.e., $\delta\, V \equiv 0$ in the sense of distribution, if and only if, u is a stationary harmonic map.

The following result is recently proved by Ambrosio and Soner.

Theorem [?]. *Let $V \in V_k^*(\Omega)$, and suppose that δV is a Radon measure on Ω. Suppose that*

$$\Theta_\alpha^*(\mu_V, x) > 0 \quad \text{for} \quad \mu_V - a.e.\ x \in \Omega, \tag{4.2}$$

for some $\alpha > 0$. Here $\Theta_\alpha^*(\mu_V, x) = \overline{\lim}_{r\downarrow 0} \frac{\nu_V(B_r(x))}{r^\alpha}$. Let $\{V_x\}_{x\in\Omega}$ be the slicings of V by the projection map $\pi : \Omega \times A_{k,n} \to \Omega$, and let $\bar{A}(x) = \int_{A_{k,n}} A\, dV_x(A)$. Then

(a) if $\alpha < k+2$, then $\bar{A}(x) \geq 0$ μ_V-a.e. $x \in \Omega$;

(b) if $\alpha < k+1$, then $\bar{A}(x) \in GL(k,n)$ μ_V-a.e. $x \in \Omega$, hence $\widetilde{v} = \delta_{A(x)}\, \mu_V$ is a classical varifold;

(c) if $\alpha = k$, then $\widetilde{v}$ is a rectifiable varifold, i.e., $\mu_{\widetilde{v}} = \mu_V$ is supported on a $\mathcal{H}^k$-rectifiable set.

Remark 4.5. Let $\{u_i\}$ be a sequence of stationary harmonic maps from Ω into N such that $u_i \rightharpoonup u$ in $W^{1,2}(\Omega, N)$, weakly and $|\nabla u_i(x)|^2\, dx \rightharpoonup \mu = |\nabla u(x)|^2\, dx + \nu$ as Radon measures, as $i \to \infty$. Then $V_{u_i} \rightharpoonup V$ weakly in $V_{n-2}^*(\Omega)$. Moreover V is also a stationary generalized varifold.

If $u \equiv$ constant, then [L2] shows the assumptions for the case (c) of the above Theorem [?] are valid. Thus one may apply Allard's Theorem (cf. [A]) to deduce that spt $\mu_V \equiv$ spt ν is $\mathcal{H}^{n-2}$-rectifiable. However in [L2] we show that spt (ν) is $\mathcal{H}^{n-2}$-rectifiable even when u is not a constant map. The conclusion of Theorem [L] along with the arguments in [L2] do not imply the following proposition. We note one may apply [?] to any tangent measures of ν.

Proposition 4.6. *Let $\{u_i\}$ be a sequence of stationary harmonic maps from Ω into N such that $u_i \to u$ weakly in $W^{1,2}(\Omega, N)$ and $|\nabla u_i|^2\, dx \rightharpoonup |\nabla u|^2\, dx + \nu$. Then ν is a $\mathcal{H}^{n-2}$-rectifiable measure, and for any $Y \in C_0^1(\Omega, \mathbb{R}^n)$, one has*

$$\begin{aligned} &\int_\Omega \nabla Y(x) : \left[|\nabla u(x)|^2\, I_n - 2\,\nabla u(x) \otimes \nabla u(x)\right] dx + \\ &+ \int_\Omega \nabla Y(x) : \bar{A}(x)\, d\nu(x) = 0 \end{aligned} \tag{4.3}$$

Here $\bar{A}(x)$ can be viewed as $T_\Sigma(x)$, the tangent plane of $\Sigma =$ spt ν at x. (cf. also Theorem 3 of [GMS]). Note that the monotonicity Lemma 2.2. follows also from (4.3) *as in [GMS].*

Now we consider an energy minimizing sequence $\{u_i\} \in C^0(\Omega, N) \cap W^{1,2}(\Omega, N)$ such that $u_i \rightharpoonup u$ weakly in $W^{1,2}(\Omega, N)$ and $|\nabla u_i|^2\, dx \rightharpoonup \mu = |\nabla u|^2\, dx + \nu$.

Lemma 4.7. *The sequence of generalized varifolds $\{V_{u_i}\}$ associated with such energy minimizing sequence $\{u_i\}$, $u_i \in C^0(\Omega, N) \cap W^{1,2}(\Omega, N)$, has the following property $V_{u_i} \rightharpoonup V$ in $V^*_{n-2}(\Omega)$, and V is stationary. In other words, div $[|\nabla u_i|^2 - 2\,\nabla u_i \otimes \nabla u_i] \to 0$ in the sense of distribution in $\mathcal{D}'(\Omega)$.*

Proof. Let $Y(x) \in C^1_0(\Omega, \mathbb{R}^n)$, and consider a family of C^1-diffeomorphisms of Ω, $\phi_t(x)$ $|t| \leq \delta \ll 1$. Here $\phi_t(x) = x + t\,Y(x)$. Since $\{u_i\}$ is an energy minimizing sequence, then

$$\underline{\lim}_{\,i\to\infty} E(u_i(t)) \geq E(u) + \nu(\Omega).$$

Here $u_i(t) = u_i \circ \phi_t^{-1}(x)$, thus $u_i(t) \equiv u_i$ on $\partial\Omega$. We calculate explicitly $E(u_i(t))$:

$$E(u_i(t)) = \int_\Omega \left|\nabla u_i(x)\, D\,\phi_t^{-1}(\phi_t(x))\right|^2 \det(D\,\phi_t^{-1})\, dx. \tag{4.4}$$

since $D\,u_i\; D\,\phi_t^{-1}(\phi_t) = D\,u_i(1 - t\,\nabla Y + O(t^2))$, and $\det D\phi_t = 1 + t$ div $Y + O(t^2)$, as $t \to 0$, we have

$$E(u_i(t)) = \int_\Omega |\nabla u_i|^2\, dx + t \int_\Omega DY : \left[|\nabla u_i|^2 - 2\nabla u_i \otimes \nabla u_i\right] dx + O(t^2). \tag{4.5}$$

We thus conclude for any $t \in (-\delta, \delta)$ small,

$$\underline{\lim}_{\,i\to\infty} \left[t \int_\Omega DY : \left[\nabla u_i\right|^2 - 2\nabla u_i \otimes \nabla u_i\right] dx + O(t^2)\right] \geq 0. \tag{4.6}$$

In particular, we must have

$$\int_\Omega \nabla Y : \left[|\nabla u_i|^2 - 2\nabla u_i \otimes \nabla u_i\right] dx \to 0 \text{ as } i \to \infty,$$

for any $Y \in C^1_0(\Omega, \mathbb{R}^n)$. □

We shall now prove Lemma 3.6 and Lemma 3.8. This will lead to the fact ν is a $\mathcal{H}^{n-2}$-rectifiable measure. Moreover, the formula (4.3) is valid for V in Lemma 4.7. As before, we may apply Theorem [**?**] to any tangent measures of the defect measure at ν-a.e. $x \in$ spt ν.

Proof of Lemma 3.6. As in [L2], we have for ν-a.e. $x \in \Sigma \equiv$ spt ν, and for all r, $0 < r < r_x$, for some $r_x > 0$, there are $(n-2)$ points $x_1 \dots , x_{n-2} \in B_r(x)$ such that

(a) $|x_1| \geq C_0\, r$, dist $(x_j, V_{j-1}) \geq C_0\, r$, for $j = 2, \dots , n-2$, here $C_0 = C_0(n) > 0$, and $V_j =$ span $\{x_1 - x, \dots , x_j - x\}$;

(b) $Ⓗ^{n-2}(\nu, x_j) \geq Ⓗ^{n-2}(\nu, x) - \epsilon(r)$, for $j = 1, \dots n-2$, $\epsilon(r) \to 0^+$ as $r \to 0^+$;

(c) $Ⓗ^{n-2}(\nu, y) \leq Ⓗ^{n-2}(\nu, x) + \epsilon(r)$, for ν-a.e. $y \in B_r(x) \cap$ spt Σ, $R^{2-n}\, \nu(B_R(x)) \leq Ⓗ^{n-2}(\nu, x) + \epsilon(r)$, for all $0 < R < m\, r$ m large;

(d) $r^{2-n} \displaystyle\int_{B_r(x)} |\nabla u|^2 (y)\, dy \leq \epsilon(r)$.

These statements follows from the geometric lemma of [L2], Federer-Ziemer's Theorem and some elementary properties of $Ⓗ^{n-2}(\nu, \cdot)$.

Next, we look at, for sequence $\{r_k\}$, $0 < r_k < r_x$, $r_k \downarrow 0$, a sequence of scaled measures $\{\mu_k\}$ which is obtained from μ by a scaling $\frac{1}{r_k}$ with the center at x. That is $\mu_k = \eta_k \,\#\, \mu$, here $\eta_k : \mathbb{R}^n \to \mathbb{R}^n$, $\eta_k(y) = \frac{1}{r_k}(y - x)$. Then, each μ_k is obtained (as μ) from an energy minimizing sequence $\{u_{k,i}\}_{i=1}^{\infty}$, i.e. $|\nabla u_{k,i}|^2\, dy \rightharpoonup \mu_k$ as $i \to \infty$. Here $u_{k,i}(y) = u_i(x + r_k\, y)$. Note $\mu_k = \eta_k \,\#\, \nu + |\nabla u_k|^2\, dy$, with $u_k(y) = u(x + r_k\, y)$.

We observe from the proof of Lemma 2.2 that

$$\frac{\mu(B_R(x))}{R^{n-2}} - \frac{\mu(B_r(x))}{r^{n-2}} \geq \underline{\lim}_{\,i\to\infty} \int_{B_R \setminus B_r(x)} \frac{1}{\rho^{n-2}} \left| \frac{\partial}{\partial \rho}\, u_i \right|^2 dy. \tag{4.7}$$

We apply (4.7) to the monotonicity Lemma 2.2 for μ_k's with centers of balls at $\frac{x_j^k - x}{r_k}$, $j - 1, \dots , n-2$. Using facts (a), (b), (c), (d) above, one can easily obtain, as in [L2], that

$$\begin{gathered} \int_{B_{C_0/2}(0)} \left| \frac{\partial u_{k,i}}{\partial\, y_j} \right|^2 dy \leq \epsilon_k \to 0 \quad \text{as} \quad k \to \infty \\ \text{for all} \quad i \leq i(k), \quad \text{and} \quad j = 1, \dots , n-2. \end{gathered} \tag{4.8}$$

On the other hand, Lemma 4.7 implies that

$$\text{div } \left[|\nabla u_{k,i}|^2 (y) - 2 \nabla u_{k,i}(y) \otimes \nabla u_{k,i}(y) \right] \to 0 \tag{4.9}$$

in the sense of distribution, as $i \to \infty$.

We decompose $\mathbb{R}^n$ as $\mathbb{R}^{n-2} \times \mathbb{R}^2$, $\mathbb{R}^{n-2} \times \{0\} \equiv \{y \in \mathbb{R}^n : y_{n-1} = y_n = 0\}$. Let $\phi(y_{n-1}, y_n) \in C_0^\infty(B^2_{C_{0/2}})$ and let $\zeta(y_1, \dots, y_{n-1}) \in C_0^\infty(B^{n-2}_{C_{0/4}}(0))$. Then we calculate

$$\frac{\partial}{\partial a_j} \int_{B_{C_{0/2}}(0)} |\nabla u_{k,i}|^2(y)\, \phi(y_{n-1}, y_n)\, \zeta(y_1 - a_1, \dots, y_{n-2} - a_{n-2})\, dy \tag{4.10}$$

for $j = 1, 2, \dots, n-2$. Using (4.8) and (4.9) we obtain the quantity in (4.10) is bounded by $C(\varnothing, \zeta)\, \epsilon_k$ uniformly as $i \to \infty$.

In other words, for $j = 1, 2, \dots, n-2$, one has

$$\left| \frac{\partial}{\partial a_j} \int_{B_{C_{0/2}}(0)} \varnothing \cdot \zeta \, d\mu_k \right| \le 0(\epsilon_k). \tag{4.11}$$

Next we let $k \to \infty$, then $\epsilon_k \to 0^+$, thus the Radon measures $\eta_k = \int_{B_1^2(0)} \varnothing(y_{n-1}, y_n)\, d\mu_k(\cdot, y_{n-1}, y_n)$ has a weak limit η, a Radon measure on $B^{n-2}_{C_{0/2}}(0)$ (by taking subsequences if needed) such that η is a constant multiple of the Lebesgue measure.

In other words , if $\widetilde{\mu}$ is the weak limit of μ_k then the slicings of $\widetilde{\mu}$ by the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-2} \times (0)$ is independent of the $y_T \in B^{n-2}_{C_{0/2}}(0)$. Due to the lower density bounds of $\widetilde{\mu}$ we must have $\widetilde{\mu} = \sum_{j=1}^M c_j\, \delta_{p_j}$, for $p_1, \dots p_n \in B^2_{C_{0/2}}(0)$ and $c_j \ge \epsilon_0$. (cf. also [L2]).

Since $\widetilde{\mu}$ is also a tangent measure of μ at x, we must have $M = 1$ and $p_j = \underline{0}$. This verifies the conclusion of Lemma 3.6. $\square$

Proof of Lemma 3.8. For $\mathcal{H}^{n-2}$-a.e. $x \in E \subset \Sigma$, we have, by the weak tangent plane property Lemma 3.6, that an $(n-2)$ dimension plane $V(r, x)$ depending on x and also r, for $0 < r < r_x$ such that

$$\frac{\nu(B_r(x) - V^\delta(r, x))}{r^{n-2}} \le \epsilon(r) \to 0, \quad \text{as} \quad r \to 0^+,$$

for any given $\delta > 0$.

In particular (as can be easily seen also from the proof of Lemma 3.6) that any tangent measure of η of ν at x is of form $\eta = Ⓗ^{n-1}(\nu, x)\, \mathcal{H}^{n-2}\, LV$ for Some $(n-2)$-dimensional plane V in $\mathbb{R}^n$. Let $r_k \to 0$ be such that $\nu_k \rightharpoonup \eta$ as $k \to \infty$. Here $\nu_k(A) = \frac{\nu(x + \gamma_k a)}{r_k^{n-2}}$. Without loss of the generality, we may also assume the $(n-2)$-dimensional plane V here is simply $\mathbb{R}^{n-2} \times \{0\}$, $x = \underline{0}$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-2} \times \{0\}$ be the orthogonal projection. We then need to show

$$\frac{H^{n-2}(\pi(E \cap B_{r_k}(0)))}{r_k^{n-2}} \to |B_1^{n-2}|, \text{ as } k \to \infty. \tag{4.12}$$

Without loss of the generality (that is also true for $\mathcal{H}^{n-2}$-a.e. $x \in E$) that

$$\frac{H^{n-2}(\Sigma/E \cap B_{r_k}(x))}{r_k^{n-2}} \to 0. \tag{4.13}$$

since $\eta = c\,\mathcal{H}^{n-2} \,L\, \mathbb{R}^{n-2} \times \{0\}$, we have $\eta\left(B_2^{n-2}(0) \times B_2^2(0) \setminus B_\delta^2(0)\right) = 0$, for any $\delta > 0$.

Let $u_{i,k}$ be an energy minimizing sequence in $C^0(B_2^{n-2}(0) \times B_2^2(0), N) \cap W^{1,2}(B_2^{n-2}(0) \times B_2^2(0), N)$ such that $|\nabla u_{i,k}|^2\, dy \rightharpoonup \nu_k + |\nabla u_k|^2\, dy$ as $i \to \infty$ (as Radon measures). Note that $|\nabla u_k|^2\, dy \rightharpoonup 0$ as $k \to \infty$ by our assumption at point x.

Since the lower $\mathcal{H}^{n-2}$ density bounds on ν_k (uniform in k by Theorem 2.3 and Theorem 3.1), we have

$$\lim_k \lim_i \int_{B_2^{n-2}(0)\times(B_2^2(0)\setminus B_\delta^2(0))} |\nabla u_{k,i}|^2\, dy = 0, \quad \text{for any} \quad \delta > 0. \tag{4.14}$$

Moreover, by a diagonal sequence, we may find an energy minimizing sequence v_k (as $u_{i,k}$) such that $|\nabla u_k|^2\, dy \rightharpoonup \eta$ as Radon measures. By the proofs of Theorem 3.1, and Lemma 3.2, we can easily modify such v_k to form a new energy minimizing sequence $C^0\left(B_2^{n-2} \times B_2^2(0)\right)$ such that $\widetilde{v}_k \equiv$ constant on $B_2^{n-2}(0) \times (B_2^2(0) \setminus B_\delta^2(0))$ and that $|\nabla \widetilde{v}_k|^2\, dy \rightharpoonup \eta$, for a given $\delta > 0$. Moreover, $\widetilde{v}_k = v_k$ on $B_2^{n-2}(0) \times B_\delta^2(0)$. We claim, for all $q \in B_2^{n-2}(0)$, the map $\widetilde{v}_k(\cdot, q) : B_2^2(0) \to N$ represents a nontrivial homotopy class α of $\pi_2(N)$. Moreover, $Ⓗ^{n-2}(\nu, x) = \inf\{E(u) : u \in W^{1,2}(\mathbb{S}^2, N)$ and u represents the homotopy class of $\alpha\}$.

Indeed, if α is trivial, then $\widetilde{v}_k(\cdot q)$, for all $q \in B_2^{n-2}(0)$, is homotopy to a constant map, say the boundary value of $\widetilde{v}(\cdot, q)$ on $\partial B_2^2(0)$. Note since $\widetilde{v}_k(\cdot, q) \equiv$ constant on $B_2^2(0) \setminus B_\delta^2(0)$, we may think $\widetilde{v}_k(\cdot, q)$ is a map from $\mathbb{S}^2$ into N. Then we let v_k^* be such that

$$v_k^* \equiv \text{ constant on } B_2^{n-2}(0) \times \left(B_2^2(0) \setminus B_\delta^2(0)\right)$$
$$v_k^* \equiv \text{ constant on } B_{2-\delta}^{n-2} \times B_2^2(0)$$
$$v_k^* \equiv \text{ on } B_2^{n-2}(0) \times \partial B_2^2(0).$$

Moreover $v_k^* \in C^0(B_2^{n-2}(0) \times B_2^2(0))$. On the region $(B_2^{n-2}(0) \setminus B_{2-\delta}^{n-2}) \times B_\delta^2(0)$ we construct a suitable extension of v_k on $\partial B_2^{n-2} \times B_\delta^2(0)$ and v_k^* on the rest of boundaries which is a constant.

For $n = 3$, it is trivial ones imply use the homogeneous degree zero extension of the boundary data on $\partial C_\delta^3 = [2-\delta, 2] \times B_\delta^2(0)$. Such extension is allowed as in our proof of Lemma

2.2. and various other places) because the boundary data is homotopically trivial map from $\mathbb{S}^2 \to N$. The extended map has energy $\leq C\,\delta$. Thus $E(v_k^*) \leq 2C\,\delta$ and $\eta(B_2^{n-2}(0) \times B_2^2(0)) \leq C\,\delta$. Since $\delta > 0$ is arbitrary we obtain a contradiction as $Ⓗ^{n-2}(\nu, x) \geq \epsilon_0$.

For $n \geq 4$, one basically follows the inductive arguments as in proofs of Lemma 3.2 and Lemma 2.4. For example, when $n = 4$, one first choose a suitable δ – net on ∂B_2^{n-2}, *i.e.* a set of points $\{\vec{p}_1, \ldots, \vec{p}_m\}$ such that $|\vec{p}_i - \vec{p}_j| \approx \delta$, $i \neq j$. Then we first do extension on $[2-\delta, 2]\,\vec{p}_i \times B_\delta^2(0)$ as in $n = 3$ case, for $i = 1, \ldots, m$.

Then we start to work first on 4-dimensional cube with boundary given by $C_1 = \widehat{p_1\,p_2} \times [2-\delta, 2] \times B_\delta^2(0)$. If boundary data has trivial topology then we simply do homogeneous zero degree extension to define a map on C_1, otherwise we modify the boundary value on $[2-\delta, 2]\,\vec{p}_2 \times B_\delta^2(0)$ slightly to have this topological condition but does not have to change any energy estimates as in the proof of Lemma 2.6. Then we keep doing till we reach the last cube C_{m-1}. We claim the final cube C_{m-1} has to have boundary value in the trivial topological class. The reason is that the union of all these maps become a map from $B_2^{n-2}(0) \setminus B_{2-\delta}^{n-2}(0) \times B_\delta^2(0)$ into N which has the boundary value in the trivial topological class. And thus we complete the construction of v_k^* for the case $n = 4$ with $E(v_k^*) \leq C\,m\,\delta^2 \leq c\,\delta$.

For $n \geq 5$ one uses the fact for $n-1$. One again first bisect $(B_2^{n-2}(0) \setminus B_{2-\delta}^{n-2}(0)) \times B_\delta^2(0)$ into two cubes, extend the map to two $(n-1)$ dimensional cubes of form $Q_1^{n-3} \times B_\delta^2(0)$, $Q_2^{n-3} \times B_\delta^2(0)$. Then look at two n-dimensional cubes that decompose $B_2^{n-2}(0) \setminus B_{2-\delta}^{n-2}(0)) \times B_\delta^2(0)$, say C_1 and C_2. We have the two possibility again . If the boundary value on ∂C_1 has the trivial topological class (then the same is true for ∂C_2), then since both C_1 and C_2 are bi-Lipschitz equivalent to $B_2^{n-2}(0) \setminus B_{2-\delta}^{n-2}(0) \times B_\delta^2(0)$, we simply do a homogeneous zero degree extension with respect to their centers. The result energy will be again $C\,\delta$ (see the proof of Lemma 3.2). If the boundary value on ∂C_1 is non-trivial topologically then we modify as in the proof of Lemma 3.2 (see also proof of $n = 4$ case above). The conclusion again follows.

This completes the proof of the claim that $\widetilde{v}_k(\cdot, \tau) : B_2^2(0) \to N$ represents a non-trivial homotopy class $\alpha \in \pi_2(N)$. It is then easy to see that as a map from $B_2^2(0) \to N$, $\widetilde{v}_k(\cdot, \epsilon)$ has to have energy $\geq \epsilon_0$ for some $\epsilon_0 > 0$. Note domain dimension is exactly 2.

Note that $\{v_k\}$ can be taken to be $\{u_{i(k),k}\}$ for any suitably large $i(k)$. Thus, by above arguments we see, for all sufficiently large i (say $i \geq i(k)$) one has $u_{i,k}(\cdot, q)$ as a map from $B_2^2(0)$ into N has energy $\geq \epsilon_0$. Since $\int_{B_2^{n-2}(0)\times B_2^2(0)} |\nabla u_k|^2\, dy \to 0$ as $k \to \infty$, we have spt $\nu_k \cap \{q\} \times B_2^2(0) \neq \varnothing$, for all $q \in B_2^{n-2}(0) \setminus B_k$. Here B_k is a subset of $B_2^{n-2}(0)$ such that $L^{n-2}(B_k) \to 0$, as $k \to \infty$. The latter statement follows from the weak- L^1 estimate for $|\nabla u_k|^2$ and the fact that $\int_{B_2^{n-2}(0)\times B_2^2(0)} |\nabla u_k|^2\, dx \to 0$, as $k \to \infty$.

In other words, we have shown that $\mathcal{H}^{n-2}(\pi(\Sigma \cap B_{r_k}(x)))\backslash r_k^{n-2} = \mathcal{H}^{n-2}(\pi(\text{ spt } \nu_k \cap B_1)) \to L^{n-2}(B_1^{n-2}(0))$. Since $\frac{\mathcal{H}^{n-2}(\Sigma\backslash E)\cap B_{r_k}(x))}{r_k^{n-2}} \to 0$ as $k \to \infty$ we have the conclusion of Lemma 3.8. $\square$

Remark 4.8. Similar proofs for results in this section works also for p-energy minimizing case when $p \geq 2$ is an integer. We leave these to readers.

5 Further Remarks

When $p \neq$ integer and $p > 1$. Then the boundary regularity theorem such as Theorem 3 follows easily from the fact that the defect measure $\nu \equiv 0$ (see Theorem 3.9 in Section 3). Indeed, Theorem 3.9, shows that $\nu \equiv 0$ inside M. Then we use constructions in [HL] and the proof of Lemma 2.2 to show the following:

Lemma 5.1. *Suppose ∂M and $g : \partial M \to N$ are of C^2-class, then there are two positive constant r_0 and C_0 depending only on ∂M, g and N such that, for $0 < r < r_0$, the function $\frac{e^{c_0 r}\,[\mu_p(\Omega_r(x)+C_0\, r^n]}{r^{n-p}}$ is monotone nondecreasing. Here $\Omega_r(x) = M \cap B_r(x)$, $x \in \partial M$, $\mu_p = |\nabla u_p|^2\, dx + \nu_p$. Here ν_p is the defect measure which (by Theorem 3.9) has to be supported in ∂M.*

Using this monotonicity lemma we follow the same to the proof of Theorem 3.9 to show $\nu_p \equiv 0$ also on ∂M. We leave this detail to the reader.

Since the defect measure $\nu \equiv 0$ when $p \neq$ integer, we have not only the strong convergence of a minimizing sequence $\{u_i\} \in W_g^{1,p}(M,N) \cap C_g^0(M,N)$ to u_p, but also that such u_p's (with g in a compact c^1-class) form a compact subset of $W_g^{1,p}(M,N)$.

Next as in [HL], we deduce from the proof of Lemma 2.4 that

Lemma 5.2. *For $0 < r < r_{0/2}$, $x \in \partial M$,*

$$\frac{\mu(\Omega_r(x))}{r^{n-p}} \leq \theta\, \frac{\mu(\Omega_{2r}(x))}{(2r)^{n-p}} + C(\theta) \fint_{\Omega_{2r}(x)} |u_p - \overline{u}_p|^p \, dy + C_0\, r,$$

As before $\overline{u}_p = \fint_{\Omega_{2r}(x)} u_p \, dy$, whenever $\frac{\mu(\Omega_{2r}(x))}{(2r)^{n-p}} \leq \epsilon_0(p, N, M)$.

Corollary 5.3. *If $\frac{\mu(\Omega_r(x))}{r^{n-p}} \leq \epsilon_0$, for some ϵ_0, then $u \in C^{1,\alpha}(\Omega_r(x), N) \cap C^{\beta}(\overline{\Omega_r}(x), N)$. Here $x \in \partial M$.*

To show the complete boundary regularity, we do exactly the same deformation as in [HL] to show any tangent map of u_p at a boundary point has to be a constant. Then using the fact that the convergence of scaled sequence to the tangent map of u_p at a boundary point is also strong,, the small energy assumption is then valid everywhere on ∂M. This proves Theorem 3, stated in the Introduction.

Now we describe a few example to illustrate main results presented so far.

Example 5.4 Consider in the Euclidean space $\mathbb{R}^4 = \mathbb{R}^1 \times \mathbb{R}^3$ a tour $M = \mathbb{S}^1 \times \mathbb{S}^2$ which is obtained by rotating $\partial B_1^3(a) \subseteq \{0\} \times \mathbb{R}^3$ around $\mathbb{R}^1$-axis, Here $a = (2, 0, 0) \in \mathbb{R}^3$, with induced metric from $\mathbb{R}^4$.

Let $g : M \to \mathbb{S}^2$ be the map defined by $g(\theta, \psi) = \psi$ for $(\theta, \psi) \in \mathbb{S}^1 \times \mathbb{S}^2$.

Consider an energy minimizing sequence $\{u_i\}$ from M into $N = \mathbb{S}^2$ (standard $\mathbb{S}^2$) such that $\{u_i\} = [g]$, i.e., u_i homotopy to g, for each i, and $u_i \in C^0(M, N)$.

Then $|\nabla_M u_i|^2 \, dx \rightharpoonup 8\pi \, \mathcal{H}^1 \lfloor \Gamma$. Here Γ is the minimal geodesic in M which is homologous to $\partial_p = \mathbb{S}^1 \times \{p\}$, for any $p \in \mathbb{S}^2$.

Example 5.5 Consider in the Euclidean space $\mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$, a solid tour $M = \mathbb{S}^1 \times B_1^2(a)$, $a = (2, 0) \in \mathbb{R}^2$, with induced metric. Let $g : \partial M \to \mathbb{S}^2$ be a constant map. Suppose $u_o : M \to \mathbb{S}^2$ be such a map that $u_o = g$ on ∂M and that when $u(p, \cdot) : B_1^2(a) \to \mathbb{S}^2$, for any $p \in \mathbb{S}^1$, viewed as a map from $\mathbb{S}^2$ into $\mathbb{S}^2$ is of degree 1.

Consider an energy minimizing sequence $\{u_i\}$, $u_i\big|_{\partial M} = g$, $u_i \in C^0(M, \mathbb{S}^2)$ with each u_i homotopy to u_o (relative homotopy). Then $|\nabla_M u_i|^2 \, dx \rightharpoonup 8\pi \, \mathcal{H}^1 \lfloor \Gamma$. Here Γ is the minimal geodesic on $\mathbb{S}^1 \times \partial B_1^2(a)$ which is homologous to $\mathbb{S}^1 \times \{p\}$. $p \in \partial B_1^2(a)$.

Example 5.6 on $\Omega = B_1^+(0) \times [-1,1]$, here $B_1^+(0)$ is a half ball in $\mathbb{R}^2$. One may find an energy minimizing sequence $\{u_i\}$, $u_i \in C^0(\Omega, \mathbb{S}^2)$ such that $u_i \equiv$ constant on $\{(x_1,x_2) : |x_1| \le 1,\, x_2 = 0\} \times [-1,1]$, and such that $|\nabla u_i|^2\, dx \rightharpoonup 8\pi\, \mathcal{H}^1 \lfloor \Gamma_0$, $\Gamma_0 = \{\underline{0}\} \times [-1,1]$, $\underline{0} \in \mathbb{R}^2$.

To end this paper, we would like to make a few remarks concerning further structures of the defect measure ν. For simplicity we discuss again the $p = 2$ case, i.e., the usual energy case.

¿From the proof of Lemma 3.8 in Section 4, we noted that for $\mathcal{H}^{n-2}$-a.e. $x \in$ spt (ν), $Ⓗ^{n-2}(\nu, x)$ is given by

$$\inf[E(u) : u \in W^{1,2}(\mathbb{S}^2, N), [u] = \alpha_x \in \pi_2(N)]$$

for some nontrivial class $\alpha_x \in \pi_2(N)$. Here $[u]$ is the homotopy class represented by u.

¿From a recent work of Duzaar-Kuwert, we see that

$$\circledast \qquad Ⓗ^{n-2}(\nu, x) = \sum_{j=1}^{\ell} E(\phi_j).$$

Here each $\phi_j : \mathbb{S}^2 \to N$ is a smooth harmonic map which is an energy minimizing map from $\mathbb{S}^2$ into N within the topological class $[\phi_j]$. Moreover, these $[\phi_j]$'s is simply a decomposition of $\alpha_x \in \pi_2(N)$ (cf. [DK]) for the details). We define a homotopy class $\alpha \in \pi_2(N)$ to be irreducible if α can be represented by a smooth harmonic map u_α, from $\mathbb{S}^2$ into N, which minimizes energy in its homotopy class, and if $E(u_\alpha)$ cannot be written as $\circledast$ above with each $\phi_j : \mathbb{S}^2 \to N$ has the stated properties, and with $\ell \ge 2$.

It is easy to show that

$$\inf\{E(u) : u \in W^{1,2}(\mathbb{S}^2, N), \quad [u] \ne 0\} \ge \epsilon_1 > 0.$$

Then by a simple induction, any $\phi : \mathbb{S}^2 \to N$ minimizing energy in the class $[\phi]$ can be decomposed into $[\phi_j]$'s, for $j = 1, \dots, \ell$ (for some finite ℓ depending only on $E(\phi)$ and ϵ_1) such that each $[\phi_j]$ is irreducible. (cf. ‘citeDK).

Next, we look at the set

$$\begin{aligned} \Lambda = \{ \quad & E(\phi) : \phi \text{ is an energy minimizing map} \\ & \text{from } \mathbb{S}^2 \text{ into } N \text{ among the class } [\phi], \\ & \text{and } [\phi] \text{ is irreducible } \}. \end{aligned}$$

Then we claim Λ is discrete in R_+.

Indeed, if $\phi : \mathbb{S}^2 \to N$ energy minimizing among class $[\phi]$ with $[\phi]$ irreducible, then $\|\phi\|_{C^2} \le C(E(\phi))$. Suppose not, we would find a sequence $[\phi_j]$ of such maps with $E(\phi_j) \le C_0$ and $\|\nabla\phi_j\|_{L^\infty} \to \infty$ as $j \to \infty$. (Note $\|\phi_j\|_{C^k} \le C(k)$. $[\|\nabla\phi_j\|_{L^\infty} + 1]$ is a well-known fact, for every $k \ge 1$).

By the result of [DK], we would have $\phi_j \rightharpoonup \phi$ in $W^{1,2}(\mathbb{S}^2, N)$ weakly, here $\phi : \mathbb{S}^2 \to N$ is a smooth harmonic map (cf. [SaU]). But $\phi_j \nrightarrow \phi$ strongly in $W^{1,2}(\mathbb{S}^2, N)$, otherwise, $\|\nabla\phi_j\|_{L^\infty}$ would be bounded. Hence we would be able to decompose $[\phi_j]$ as in [DK] and that contradicts to the fact $[\phi_j]$ is irreducible.

Now, for any $E_o > 0$, the set $\Lambda \cap (0, E_o]$ is finite. Indeed, if $[E_j]_{j=1}^\infty$ be such that $E_j \in \Lambda \cap [0, E_o]$. Let $[\phi_j]$ be a corresponding energy minimizing maps from $\mathbb{S}^2$ into N such that $E(\phi_j) = E_j$ and each $[\phi_j]$ is irreducible. Then $\|\phi_j\|_{C^2} \le C(E_o)$. By taking subsequence if needed, we may assume $\phi_j \to \phi$ in $C^{1,\alpha}$-norm. It is obvious ϕ is a harmonic map from $\mathbb{S}^2$ into N with $E(\phi) \le \underline{\lim}_j E(\phi_j)$. Moreover, $[\phi_j] = [\phi]$ for sufficiently large j. Then it is clear that $E(\phi_j) = E_j$ are all equal, by definition, for large j's.

We note that there may be infinitely many such energy minimizing maps ϕ that $E(\phi)$ is a fixed number in Λ. However, if N is, in addition, analytic, then it is also easy to show that any $E_o \in \Lambda$ there are only finitely many connected components of such energy minimizing maps ϕ with $E(\phi) = E_o$. That leads us to the following:

Conjecture 5.7 *Let N be a real analytic, compact Riemannian manifold. Then the energy spectrum*

$$\Lambda = \{E(u) | u : \mathbb{S}^2 \to N \text{ is a harmonic map } \}$$

is a discrete set.

To summarize, we have shown that for $\mathcal{H}^{n-2}$-a.e. $x \in$ spt ν, $\textcircled{H}^{n-2}(\nu, x) = \sum_{j=1}^{\ell} E_j$ with $E_j \in \Lambda$. Since $\textcircled{H}^{n-2}(\nu, x)$ is also bounded by the global energy, we have, therefore, the following:

Theorem 5.8 *The defect measure is the total measure μ_V of an integral rectifiable varifold*

V in the sense that

$$
\begin{aligned}
\nu &\equiv \textcircled{H}^{n-2}(\nu, \cdot)\, \mathcal{H}^{n-2} \lfloor \Sigma \\
&= \sum_{j=1}^{m} \theta_j \, \mathcal{H}^{n-2} \lfloor \Sigma_j .
\end{aligned}
$$

Here Σ_j's are subsets of Σ ($\mathcal{H}^{n-2}$-measurable) and $\theta_j = \sum_{\ell=1}^{\ell_j} \theta_{j\,\ell}$, for some $\theta_{j\,\ell} \in \Lambda$, and $j = 1, \ldots, m$.

We further remark that when $N \approx \mathbb{S}^2$ topologically, then, topological classes simply identified by the degrees of maps from $\mathbb{S}^{@}$ into N. According to plus sign or minus sign, we can even give an orientation of set Σ. In other words, in this case, ν simply comes from μ_T for an integral rectifiable current T. A special case of this last fact is in fact shown in [GMS] when $n = 3$.

Further properties of ν will be discussed in a forthcoming work.

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