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Existence and uniqueness of energy weak solutions to boundary and obstacle problems for quasilinear elliptic-parabolic equations

by

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EXISTENCE AND UNIQUENESS OF ENERGY WEAK SOLUTIONS TO BOUNDARY AND OBSTACLE PROBLEMS FOR QUASILINEAR ELLIPTIC-PARABOLIC EQUATIONS

A.V.Ivanov and J.F.Rodrigues

Abstract. We prove existence and uniqueness of weak solutions to initial-boundary value problem and problem with inner obstacle for elliptic-parabolic equations

$$\partial_t b(u) - \operatorname{div}\{|\delta(u)|^{m-2}\delta(u)\} = f(x,t),$$
$$\delta(u) = \nabla u + k(b(u))\vec{e}, \quad |\vec{e}| = 1, m > 1$$

with a monotone nondecreasing function b. These equations arise in the theory of non-Newtonian filtation and mathematical glaciology.

1. Introduction

Let $Q_T = \Omega \times (0,T]$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Consider initial-boundary value problem

$$\mathcal{M}u := \partial_t b(u) - \operatorname{div} a(b(u), \nabla u) = f(x,t) \quad \text{in} \quad Q_T, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad S_T = \partial\Omega \times (0,T], \tag{1.2}$$

$$b(u) = b_0 \quad \text{on} \quad \Omega \times \{t = 0\} \tag{1.3}$$

where $b : \mathbb{R} \to \mathbb{R}$ be monotone nondecreasing and continuous while function a be continuous on $\mathbb{R} \times \mathbb{R}^n$, strictly monotone in $p \in \mathbb{R}^n$ for any $w \in \mathbb{R}$ and satisfies the growth condition

$$|a(w,p)|^{m'} \leq c(|p|^m + \Psi(w) + 1), m > 1, 1/m + 1/m' = 1, \tag{1.4}$$

where $\Psi(w)$ is the Legendre transform of the primitive of b, i.e.,

$$\Psi(w) = \sup_{z \in \mathbb{R}} \{zw - \int_0^z b(\zeta)d\zeta\}. \tag{1.5}$$

In view of (1.5), we have

$$\Psi(b(u)) = ub(u) - \int_0^u b(\zeta)d\zeta. \tag{1.6}$$

Considering problem (1.1)–(1.3) under strict monotonicity condition

$$[a(w,p) - a(w,q)] \cdot (p - q) \geq c|p - q|^m, m > 1 \tag{1.7}$$

Alt and Luckhaus proved existence of solution to problem (1.1)–(1.3) in the natural class of weak solutions with finite energy

$$\sup_{t\in[0,T]}\int_\Omega \Psi(b(u))dx + \iint\limits_{Q_T} |\nabla u|^m dxdt < +\infty. \tag{1.8}$$

Uniqueness of that energy weak solution to problem (1.1)–(1.3) was proved recently by Otto in his paper [2], where along with conditions (1.4) and (1.7) it was used also assumption

$$|a(b(u_1),p) - a(b(u_2),p)|^{m'} \le c|u_1 - u_2|\{|p|^m + \Psi(b(u_1)) + \Psi(b(u_2)) + 1\}. \tag{1.9}$$

Using these conditions Otto proved also important L^1-contraction for problem (1.1)–(1.3).

In reality in the papers [1] and [2] the results cited were obtained for mixt boundary value problem with in general non-homogeneous Dirichlet data and with additional term $g(b(u))$ of the left-hand side to equation (1.1) (under appropriate assumptions on Dirichlet data and function $g(w)$). We shall restrict to consider here problem (1.1)–(1.3) for the sake of simplicity and breavity of a presentation.

In the study of turbulent filtration of a fluid or a gas through porous media ([3]–[5]) as well as of nonstationary saturated — unsaturated flows though porous media ([6]), in the theory of non-Newtonian fluids ([7]) and in the mathematical glaciology ([8]) equations of the type

$$\left.\begin{array}{l}\mathcal{L}u := \partial_t\, b(u) - \operatorname{div}\{|\nabla u + k(b(u))\vec{e}|^{m-2}(\nabla u + k(b(u))\vec{e})\} = f(x,t),\\ m>1, b \quad \text{is nondecreasing and continuous on} \quad \mathbb{R} \quad \text{function},\\ k \quad \text{is continuous on} \quad b(\mathbb{R}), |\vec{e}| = 1\end{array}\right\} \tag{1.10}$$

arise. Unfortunately the results of papers [1] and [2] can not be applied to equation (1.10) in the case $m \in (1,2)$ because the strict monotonicity condition (1.7) is fulfilled for equation (1.10) only if $m \ge 2$. For equation (1.10) with $m \in (1,2)$ the following strict monotonicity condition holds:

$$[a(w,p) - a(w,q)]\cdot(p-q) \ge c|p-q|^2(|p+k(w)\vec{e}|^m + |q+k(w)\vec{e}|^m)^{\frac{m-2}{m}}. \tag{1.11}$$

Extention of results of papers [1] and [2] to equation (1.10) with $m \in (1,2)$ is one of the aims of given paper.

The main aim of this paper is to state our results on existence and uniqueness of energy weak solution to the obstacle problem

$$\left.\begin{array}{l} u \ge \varphi, \mathcal{L}u \ge f, \quad (u-\varphi)(\mathcal{L}u - f) = 0 \quad \text{in} \quad Q_T;\\ u = 0 \ge \varphi \quad \text{on} \quad \Gamma_T;\\ b(u) = b_0 = b(u_0), u_0 \ge \varphi \quad \text{on} \quad \partial\Omega\times\{t=0\}\end{array}\right\} \tag{1.12}$$

where operator $\mathcal{L}$ is defined by (1.10) and the obstacle $\varphi = \varphi(x,t)$ is given on Q_T function.

As for as we know both existence and uniqueness of energy weak solution to problem (1.12) are not established in the case $m \neq 2$ and $b(u) \not\equiv u$. Existence and uniqueness of a strong solution to problem (1.12) in the case $m = 2$ with bounded function b are established in [9]. In paper [9] one can find also some references on the obstacle problems for quasilinear degenerate parabolic equations.

Using the results of papers [1] and [2] as well as the first part of given paper we establish existence of energy weak solution to problem (1.12) assuming that obstacle φ and function b satisfy some compatibility condition (see the statement of Theorem 2.3). In particular existence of weak solution of problem (1.12) is established if function b is strictly monotone on $\mathbb{R}$.

Applying the methods of papers [9], [2] as well as the first part of given paper we prove uniqueness of energy weak solution to problem (1.12) in the case when obstacle φ is independent of time.

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2. Statement of the main results

Further in this paper we always assume that the following conditions are fulfilled.

1) Ω is bounded in $\mathbb{R}^n$, $n \geq 1$.

2) b is monotone nondecreasing and continuous function on $\mathbb{R}$ such that $b(0) = 0$; k is given continuous function on $b(\mathbb{R})$.

3) Function $f = f(x,t)$ satisfies condition

$$f \in L_{m'}(Q_T), \quad 1/m + 1/m' = 1, \quad m > 1.$$

4) Initial function $b_0 = b_0(x)$ is defined by formula

$$\begin{aligned} b_0 = b(u_0) \quad &\text{with some measurable function} \quad u_0 = u_0(x); \\ &b_0 \in L_1(\Omega), \Psi(b_0) \in L_1(\Omega) \quad (\text{see}(1.5)). \end{aligned} \tag{*}$$

It is shown in [1] that function Ψ defined by (1.5) is superlinear in the following sence

$$|w| \leq \delta\Psi(w) + C_\delta \quad \text{for any} \quad \delta > 0.$$

Hence really the first condition in (*) follows from the second one.

Denote

$$V = \overset{\circ}{W}{}^1_m(\Omega)$$

and let V^* is notation of the dual space to Banack space V.

Let the growth condition (1.4) is fulfilled. Function u is a weak solution of problem (1.1)–(1.3) if u satisfies condition (1.8) and

$$\left.\begin{aligned} &\mathcal{M}\{u,\zeta\} := \iint\limits_{Q_T} [(b_0 - b(u))\,\partial_t\,\zeta + a(b(u),\nabla u)\cdot\nabla\zeta] dxdt = \iint\limits_{Q_T} f\zeta dxdt \\ &\text{for all} \quad \zeta \in L_m(0,T;V) \quad \text{with} \quad \partial_t\,\zeta \in L_\infty(Q_T) \quad \text{and} \quad \zeta(T) = 0. \end{aligned}\right\} \tag{2.1}$$

¿From identity (2.1) it follows obviously that the derivative (in the sence of distributions) $\partial_t\, b(u)$ belongs to $W_{m'}(0,T;V^*)$ and

$$\left.\begin{aligned}&\int_0^T \langle \partial_t\, b(u), \zeta\rangle dt + \iint\limits_{Q_T} a(b(u), \nabla u)\cdot \nabla\zeta dxdt = \iint\limits_{Q_T} f\zeta dxdt\\ &\text{for all}\quad \zeta\in L_m(0,T;V)\end{aligned}\right\} \tag{2.2}$$

where pairing $\langle .,.\rangle$ is in V^* and V.

Investigating problem (1.1)–(1.3) Alt and Luckhaus proved the following important statesments.

Proposition 2.1 (Alt–Luckhaus, [1])**.** *Let $u\in L_m(0,T,V)$, $b(u)\in L_\infty(0,T;L_1(\Omega))$, $\partial_t\, b(u)\in L_{m'}(0,T;V^*)$ and for all $\zeta\in L_m(0,T;V)$ with $\partial_t\,\zeta\in L_\infty(Q_T)$ and $\zeta(T)=0$*

$$\int_0^T \langle \partial_t\, b(u), \zeta\rangle dt = \iint_{Q_T} (b_0 - b(u))\, \partial_t\, \zeta dxdt.$$

Then $\Psi(b(u))$ belongs to $L_\infty(0,T;L_1(\Omega))$ and for almost all (a.a.) $\tau\in(0,T]$

$$\int_\Omega [\Psi(b(u(\tau))) - \Psi(b_0)]dx = \int_0^\tau \langle \partial_t\, b(u), u\rangle dt. \tag{2.3}$$

Proposition 2.2 (Alt–Luckhaus, [1])**.** *Suppose that functions u_ε converge weakly in $L_m(0,T;W^1_m(\Omega))$ to u with the estimates*

$$\frac{1}{h}\int_0^{T-h}\int_\Omega [b(u_\varepsilon(t+h) - b(u_\varepsilon(t))][u_\varepsilon(t+h) - u_\varepsilon(t)]dxdt \le c \tag{2.4}$$

and

$$\sup_{t\in[0,T]}\int_\Omega \Psi(b(u_\varepsilon(t)))dx \le c.$$

Then $b(u_\varepsilon)\to b(u)$ in $L_1(Q_T)$ and $\Psi(b(u_\varepsilon))\to\Psi(b(u))$ almost everywhere (a.e.) in Q_T.

Using Proposition 1 and 2 the following existence theorem is proved in [1].

Theorem (Alt–Luckhaus)**.** *Let conditions* (1.4) *and* (1.7) *are fulfilled. Then there exists at least one weak solution of problem* (1.1)–(1.3)*.*

To state the Otto results we have to define at first the following notations.

Function u is a subsolution of problem (1.1)–(1.3) with initial function b_0 (satisfying conditions 4)) if u satisfies (1.8), $u\le 0$ on S_T, and for all nonnegative in Q_T function ζ with the same properties as in (2.1) we have

$$\mathcal{M}\{u,\zeta\} \le \iint\limits_{Q_T} f\zeta dxdt.$$

Function u is a supersolution of problem (1.1)–(1.3) with initial function b_0 (satisfying conditions 4)) if u satisfies (1.8), $u\ge 0$ on S_T and for all nonnegative in Q_T functions ζ like above we have

$$\mathcal{M}\{u,\zeta\} \ge \iint\limits_{Q_T} f\zeta dxdt.$$

¿From the results of [2] the following proposition follows.

Proposition 2.3. *Let conditions* (1.4), (1.7), (1.9) *are fulfilled. Let* u_1 *be a subsolution of problem* (1.1)–(1.3) *with right-hand side* f_1 *and initial function* $b_0^{(1)}$ *while* u_2 *be a supersolution of problem* (1.1)–(1.3) *with right-hand side* f_2 *and initial function* $b_0^{(2)}$ *where* f_i *and* $b_0^{(i)}$ $(i = 1, 2)$ *satisfy conditions* 3) *and* 4) *respectively. Then for any nonnegative function* $\gamma \in C_0^\infty(\mathbb{R}^n \times (-\infty, T))$

$$\begin{aligned}\iint\limits_{Q_T} &\{[(b_0^{(1)} - b_0^{(2)})^+ - (b(u_1) - b(u_2))^+]\,\partial_t\,\gamma \\ &+ \operatorname{sign}^+(u_1 - u_2)(a(b(u_1), \nabla u_1) - a(b(u_2), \nabla u_2))^+ \nabla\gamma\} dxdt \\ &\le \iint\limits_{Q_T} (f_1 - f_2)\operatorname{sign}^+(u_1 - u_2) dxdt. \end{aligned} \tag{2.5}$$

In particular for a.a. $t \in (0, T)$

$$\int_\Omega [b(u_1(t)) - b(u_2(t))]^+ dx \le \int_\Omega (b_0^{(1)} - b_0^{(2)})^+ dx + \iint\limits_{Q_T} (f_1 - f_2)\operatorname{sign}^+(u_1 - u_2) dxdt. \tag{2.6}$$

Theorem (Otto). *Let conditions* (1.4), (1.7), (1.9) *are fulfilled. Then there is at most one weak solution of problem* (1.1)–(1.3).

Consider now a problem

$$\mathcal{L}u = f \quad \text{in} \quad Q_T, \quad u = 0 \quad \text{on} \quad S_T, \quad b(u) = b_0 \quad \text{on} \quad \Omega \times \{t = 0\} \tag{2.7}$$

with operator $\mathcal{L}u$ defined by (1.10) assuming that

$$|k(w)|^m \le c(\Psi(w) + 1) \quad \text{on the set} \quad b(\mathbb{R}), m > 1. \tag{2.8}$$

In view of (2.8) function

$$a(w, p) = |p + k(w)\vec{e}|^{m-2}(p + k(w)\vec{e}) \tag{2.9}$$

satisfies condition (1.4) and hence the definition of weak solution given above can be applied in the case of problem (2.7). Consider the following mapping $\Phi_m(z) : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi_m(z) := |z|^{m-2}z, \quad m > 1. \tag{2.10}$$

It is well known that

$$\left.\begin{aligned} &[\Phi_m(z_1) - \Phi_m(z_2)] \cdot (z_1 - z_2) \ge c|z_1 - z_2|^\varkappa (|z_1|^m + |z_2|^m)^{1-\frac{\varkappa}{m}}, \\ &\varkappa = m \quad \text{if} \quad m \ge 2, \varkappa = 2 \quad \text{if} \quad m \in (1, 2), c = c(n, m). \end{aligned}\right\} \tag{2.11}$$

Then for $a(w, p)$ defined by (2.9) we have

$$\left.\begin{aligned} &[a(w, p) - a(w, q)] \cdot (p - q) \ge c|p - q|^\varkappa (|p + k(u)\vec{e}|^m + |q + k(w)\vec{e}|^m)^{\frac{m-\varkappa}{m}}, \\ &\varkappa = m \quad \text{if} \quad m \ge 2, \varkappa = 2 \quad \text{if} \quad m \in (1, 2), c = c(n, m). \end{aligned}\right\} \tag{2.12}$$

Obviously (2.12) coinsides with (1.7) if $m \geq 2$ and with (1.11) if $m \in (1,2)$.

It is easy to check also directly that for all $w \in \mathbb{R}$, $p \in \mathbb{R}^n$

$$a(w,p) \cdot p \geq \nu_0 |p|^m - c(\Psi(w)+1), \nu_0 > 1, c \geq 0, m > 1. \tag{2.13}$$

Note that inequality (2.13) can be derived also from conditions (1.4), (1.7) for any $m > 1$.

Consider now the following conditions:

$$|k(w_1) - k(w_2)| \leq c|w_1 - w_2|^{1/m} \quad \text{for any} \quad w_1, w_2 \in b(\mathbb{R}) \tag{2.14}$$

and

$$\left.\begin{array}{l} |k(b(u_1)) - k(b(u_2))| \leq c|u_1 - u_2|^{\alpha}(\Psi(b(u_1)) + \Psi(b(u_2)) + 1)^{1/m}, \\ \alpha = 1/m' \quad \text{if} \quad m \geq 2, \alpha = 1/m \quad \text{if} \quad m \in (1,2), u_1, u_2 \in \mathbb{R}. \end{array}\right\} \tag{2.15}$$

The results of given paper allow to state in particular the following theorems.

Theorem 2.1. *Let condition* (2.8) *and* (2.14) *are fulfilled. Then there exists at least one weak solution of problem* (2.7) *for any* $m > 1$.

Theorem 2.2. *Let condition* (2.8) *and* (2.15) *are fulfilled. Then there is at most one weak solution of problem* (2.7) *for any* $m > 1$.

Remark 2.1. In the case $m \geq 2$ Theorem 2.1 and Theorem 2.2 follow from the results of [1] and [2] respectively. Really in the case $m \geq 2$ conditions (2.8), (2.12) imply obviously that conditions (1.4) and (1.7) are fulfilled and hence Theorem of Alt-Luckhaus implies that Theorem 2.1 is valid for $m \geq 2$. To prove that Theorem of Otto implies Theorem 2.2 for $m \geq 2$ it suffices to show that in this case condition (1.9) follows from conditions (2.8), (2.15). Really denote

$$\hat{p}_i := p + G(b(u_i)), \quad G(w) := k(w)\vec{e}, \quad z_i := \Phi_m(\hat{p}_i), i = 1,2.$$

Because $m' \in (1,2)$ for $m \geq 2$ and $\Phi_{m'}(\Phi_m(\hat{p}_i)) = \hat{p}_i$, $i = 1,2$, from (2.11) it follows that

$$\begin{aligned} |\Phi_m(\hat{p}_1) - \Phi_m(\hat{p}_2)|^2 &\leq c[\Phi_{m'}(z_1) - \Phi_{m'}(z_2)] \cdot (z_1 - z_2)(|z_1|^{m'} + |z_2|^{m'})^{\frac{2-m'}{m'}} \\ &\leq c|\hat{p}_1 - \hat{p}_2||\Phi_m(\hat{p}_1) - \Phi_m(\hat{p}_2)|(|\Phi_m(\hat{p}_1)| + |\Phi_m(\hat{p}_2)|)^{2-m'}. \end{aligned}$$

¿From here and (2.8), (2.15) we derive

$$\begin{aligned} &|\Phi_m(\hat{p}_1) - \Phi_m(\hat{p}_2)|^{m'} \\ &\leq c|k(b(u_1)) - k(b(u_2))|^{m'}(|p|^m + \Psi(b(u_1)) + \Psi(b(u_2)) + 1)^{\frac{m-2}{m-1}} \\ &\leq c|u_1 - u_2|(|p|^m + \Psi(b(u_1)) + \Psi(b(u_2)) + 1), \end{aligned}$$

i.e., condition (1.9) is fulfilled.

We are able to prove Theorem 2.1 and Theorem 2.2 in the case $m \in (1,2)$.

State now main results of given paper on existence and uniqueness of weak solution to the obstacle problem (1.12).

We will assume further that (together with conditions 1)–4)) the following conditions are fulfilled.

5) Obstacle $\varphi = \varphi(x,t)$ satisfies assumptions

$$\left.\begin{array}{l} \varphi \in W^{1,0}_m(Q_T), \Psi(b(\varphi)) \in L_\infty(0,T;L_1(\Omega)), \partial_t \varphi \in L_{m'}(Q_T), \\ \varphi \le 0 \quad \text{on} \quad S_T, \varphi \le u_0 \quad \text{and} \quad b(u_0) = b_0 \quad \text{on} \quad \Omega \times \{t=0\}, \\ u_0 \in L_1(\Omega), \Psi(u_0) \in L_1(\Omega) \end{array}\right\} \tag{2.16}$$

and

$$\left.\begin{array}{l} \text{there exists nonnegative function} \quad \zeta \in L_{m'}(Q_T) \quad \text{such that} \\ \displaystyle\iint\limits_{Q_T} [\partial_t b(\varphi)\zeta + a(b(\varphi), \nabla \varphi) \cdot \nabla \zeta] dxdt \le \iint\limits_{Q_T} (f+\xi)\zeta dxdt \\ \text{with} \quad a(w,p) \quad \text{defined in (2.9) for any} \quad \zeta \in L_m(0,T;V). \end{array}\right\} \tag{2.17}$$

Denote

$$K = K(\varphi) := \{v \in L_m(0,T;V) : v \ge \varphi \quad \text{a.e. in} \quad Q_T\}.$$

Function u belonging to $K(\varphi)$ such that $\Psi(b(u)) \in L_\infty(0,T;L(\Omega))$ and $\partial_t(b(u) \in L_{m'}(0,T;V^*)$ is a weak solution of problem (1.12) if u satisfies variational inequality

$$\left.\begin{array}{l} \displaystyle\int_0^T \langle \partial_t b(u), v-u\rangle dt + \iint\limits_{Q_T} a(b(u), \nabla u) \cdot \nabla (v-u) dxdt \ge \iint\limits_{Q_T} f(v-u)dxdt \\ \text{for all} \quad v \in K(\varphi), \end{array}\right\} \tag{2.18}$$

where

$$\left.\begin{array}{l} \displaystyle\int_0^T \langle \partial_t b(u), \zeta\rangle dt = -\iint\limits_{Q_T} b(u)\, \partial_t \zeta dxdt \quad \text{for all} \quad \zeta \in L_m(0,T;V) \\ \text{having a compact support in} \quad \Omega \times (0,T) \quad \text{and such that} \quad \partial_t \zeta \in L_\infty(Q_T), \end{array}\right\}$$

and initial condition

$$\operatorname*{ess\,lim}_{t\to 0} \int_\Omega |b_0 - b(u(t))| dx = 0. \tag{2.19}$$

Consider the following

Condition C. There exist numbers $\delta_0 > 0$ and $\delta_1 > 0$ such that the restriction of function b on the neighborhood $\hat{\Phi} := \bigcup\limits_{\substack{\varphi=\varphi(x,t)\\(x,t)\in Q_T}} [\varphi, \varphi+\delta_0]$ of the range $\varphi(Q_T)$ of the obstacle φ is strictly increasing function with a range containing the neighborhood $\hat{B} := \bigcup\limits_{\substack{\varphi=\varphi(x,t)\\(x,t)\in Q_T}} [b(\varphi), b(\varphi)+\delta_1]$ of the range $b(\varphi(Q_T))$ of function $b \circ \varphi$. Moreover the inverse function to this restriction is uniformly continuous on $\hat{B}$.

The mail results of our investigations can be expressed as the following statements.

Theorem 2.3. *Let conditions* (2.9), (2.14) *and* (2.15) *are fulfilled. Assume that either b is strictly increasing on $\mathbb{R}$ or condition C holds. Then there exists at least one weak solution of problem* (1.12).

Theorem 2.4. *Let conditions* (2.9) *and* (2.15) *are fulfilled. Assume that obstacle φ is independent of time t. Than there is at most one weak solution of problem* (1.12).

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