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Electromagnetic Resolution of Curvature and Gravitational Instantons

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Summary. —We study the electromagnetic resolution of the Riemann curvature on a space-time manifold M with metric g into its electric and magnetic parts relative to a unit timelike vector with respect to g. There exists a duality transformation between the active and passive electric parts which leaves invariant a subclass of field equations which correspond to the generalized gravitational instanton equations. We also discuss various geometric formulations of the equations of gravitational instantons and their generalization which includes as a special case the classical vacuum Einstein equations. Some special solutions and their physical significance are also considered.

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1. – Introduction

It is well known since the beginning of this century that the electric and magnetic fields can be combined in a natural way to obtain the electromagnetic field tensor over the Minkowski space and that Maxwell's equations can be expressed as differential equations for the components of this tensor. Conversely, the electric and magnetic parts of the field tensor can be recovered once a coordinate system is fixed. However, the fact that the electromagnetic field tensor can be regarded as the curvature 2-form (or a gauge field) of a connection (or a gauge potential) in a principal bundle with gauge group $U(1)$ and that Maxwell's equations can be expressed as gauge field equations became clear only recently. This identification of curvature and gauge field extends to bundles over arbitrary Riemannian manifolds. The study of the space of solutions of gauge field equations has led to new results and a better understanding of the topology and geometry of low dimensional manifolds. For background material on geometry and topology with applications to gauge theory see, for example, Marathe and Martucci [20] and [21]. It is natural to ask about the significance of electric and magnetic parts of an arbitrary gauge field regarded as the curvature of a gauge potential. There does not seem to be any discussion of this aspect of gauge fields in the literature. However, a decomposition of the Weyl and the Riemann curvature of the Levi-Civita connection on a space-time manifold is known (see, for example [3] and references therein). In section 2 we consider the decomposition of the full Riemann curvature of a space-time manifold $(M,\ g)$ into its electric and magnetic parts relative to a unit timelike vector with respect to g. We shall refer to this decomposition as the electromagnetic resolution of the curvature. In [10], Dadhich has defined a duality transformation which exchanges the active and passive electric parts and leaves the antisymmetric magnetic part unchanged. This duality transformation implies interchanging the Ricci and the Einstein tensors. Duality transformation leaves the Einstein vacuum equations invariant but changes the sign of the Weyl tensor [11]. Duality transformation would then imply that the mass M of the Schwarzschild solution goes over to $-M$. Furthermore, it turns out that the Einstein vacuum equations can be modified so that the modified equations are not duality invariant and yet admit the same solution while the dual set of equations admit a solution distinct from the Schwarzschild solution. Similarly we can construct space-times dual to the Reissner-Nordstrom, NUT [24] and the Kerr solutions [12]. The dual space-times describe the original space-times with global monopole charge and global texture. The duality so defined is thus intimately related to the topological defects.

We show that the gravitational instanton Lagrangian R^2 and the instanton equations are invariant under the duality transformation. The main aim of this paper is to study the role played by the duality transformation in the case of gravitational instantons with or without their coupling to matter fields. In section 3 we consider a gravitational instanton coupled to classical fields. The resulting field equations include the usual gravitational equations with source term and with or without the cosmological constant. Its relation to duality is considered in section 4. In the concluding section 5, we mention some open problems.

2. – Electromagnetic Resolution of Curvature

Let $(M,\ g)$ be a space-time manifold with metric g of signature $(+,\ -,\ -,\ -)$. We identify the Riemann curvature with the symmetric operator it induces on $\Lambda^2_x(M)$ for each $x \in M$ and denote the Hodge star operator on $\Lambda^2_x(M)$ by J. Let u_a be a unit timelike vector. We define the electromagnetic resolution of the Riemann curvature relative to the unit timelike vector u_a, as follows :

$$E_{ac} = R_{abcd}u^b u^d, \quad \tilde{E}_{ac} = (JRJ)_{abcd}u^b u^d \tag{1}$$

$$H_{ac} = (JR)_{abcd}u^b u^d = H_{(ac)} + H_{[ac]}, \quad \tilde{H}_{(ac)} = (RJ)_{abcd}u^b u^d \tag{2}$$

where

$$H_{(ac)} = (JC)_{abcd}u^b u^d \text{ and } H_{[ac]} = \frac{1}{2}\eta_{abce}R^e_d u^b u^d. \tag{3}$$

are the symmetric and the skew symmetric parts of the tensor H_{ac}. Here C_{abcd} is the Weyl conformal curvature and η_{abcd} is the 4-dimensional volume element. The pair of tensors $(E_{ac}, \tilde{E}_{ac})$ constitutes the electric part of curvature. The tensor E_{ac} is called the active part and the tensor $\tilde{E}_{ac}$ is called the passive part [10]. The tensor H_{ac} is called the magnetic part of the curvature ($\tilde{H}_{ac} = H_{ca}$ and hence is dropped from further consideration). We write $E = E^a_a$, $H = H^a_a$ and $\tilde{E} = \tilde{E}^a_a$. The active and the passive electric parts are symmetric tensors. The electric and the magnetic parts have the following additional properties.

$$E_{ab}u^b = 0,\ E = R_{bd}u^b u^d \tag{4a}$$

$$\tilde{E}_{ab}u^b = 0,\ \tilde{E} = (R_{bd} - \frac{1}{2}Rg_{bd})u^b u^d \tag{4b}$$

$$H_{ab}u^b = 0,\ H = 0 \tag{4c}$$

We note that the electric and magnetic parts are orthogonal to the timelike vector u_a and thus behave (at least locally) as 3-tensors. The electric parts are symmetric and account for 12 components while the magnetic part is trace-free and accounts for 8 components of the Riemann curvature. Thus the entire gravitational field is decomposed into electromagnetic parts. The resolution is not unique and depends on the choice of a unit timelike vector. Using the above decomposition, Dadhich [10] has defined the duality transformation as follows:

$$E_{ab} \longleftrightarrow \tilde{E}_{ab},\ H_{[ab]} \longleftrightarrow H_{[ab]},\ H_{(ab)} \longleftrightarrow -H_{(ab)}. \tag{5}$$

Thus the duality transformation interchanges the active and passive electric parts and changes the sign of the symmetric magnetic part while leaving the antisymmetric magnetic part unchanged. From equation (4) it is clear that the duality transformation would map the Ricci tensor into the Einstein tensor and vice-versa. This is because contraction of Riemann is Ricci while that of its double dual is Einstein. The Ricci tensor can be written in terms of the electric and magnetic parts as follows.

$$R_{ab} = E_{ab} + \tilde{E}_{ab} + (E + \tilde{E})u_a u_b - \tilde{E}g_{ab} + \frac{1}{2}H^{mn}u^c\left(\eta_{acmn}u_b + \eta_{bcmn}u_a\right) \tag{6}$$

The scalar curvature R is given by

$$R = 2(E - \tilde{E}) \tag{7}$$

In section 4 we show that the vacuum Einstein equations with the cosmological constant are equivalent to the equations

$$E_a^b + \tilde{E}_a^b = 0 \text{ and } H_{[ab]} = 0$$

This together with equation (7) imply that the vacuum equation, $R_{ab} = 0$ is in general equivalent to

$$E_{ab} + \tilde{E}_{ab} = 0, \; H_{[ab]} = 0 \text{ and } E = 0 \;\; (\text{or equivalently}) \;\; \tilde{E} = 0. \tag{8}$$

Thus the vacuum equation (8) is symmetric in E_{ab} and $\tilde{E}_{ab}$ and hence is invariant under the duality transformation (5).

For obtaining the Schwarzschild solution, we follow [10] and consider the spherically symmetric metric,

$$ds^2 = C^2(r,t)dt^2 - A^2(r,t)dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{9}$$

The natural choice for the resolving vector is, in this case, pointing along the t-line which is hypersurface orthogonal. From equation (8), $H_{[ab]} = 0$ and $E_2^2 + \tilde{E}_2^2 = 0$ lead to $AC = 1$ (for this, no boundary condition of asymptotic flatness is needed). Now $\tilde{E} = 0$ gives $A = (1 - 2M/r)^{-1/2}$, which is the Schwarzschild solution. Note that we did not need to use the remaining equation $E_1^1 + \tilde{E}_1^1 = 0$, which is hence free and is implied by the rest. Without affecting the Schwarzschild solution, we can introduce some distribution in the 1-direction.

We consider the following equation which is not invariant under the duality transformation

$$H_{[ab]} = 0 = \tilde{E}, \; E_{ab} + \tilde{E}_{ab} = \lambda w_a w_b \tag{10}$$

where λ is a scalar and $w_a = \dot{u}_a/|\dot{u}_a|$ is a spacelike unit vector parallel to the acceleration. It is clear that it will also admit the Schwarzschild solution as the general solution, and it determines $\lambda = 0$. For spherical symmetry the above alternate equation (10) also characterizes vacuum, because the Schwarzschild solution is unique. Now applying the duality transformation (5) to the above equation (10) we obtain the equation

$$H_{[ab]} = 0 = E, \; E_{ab} + \tilde{E}_{ab} = \lambda w_a w_b. \tag{11}$$

Again we shall have $AC = 1$ and $E = 0$ will then yield $C = (1 - 2k - 2M/r)^{1/2}$ and $\lambda = 2k/r^2$. Thus the general solution of equation (11) for the metric (9) is given by

$$C = A^{-1} = (1 - 2k - \frac{2M}{r})^{1/2}. \tag{12}$$

This is the Barriola-Vilenkin solution [2] for the Schwarzschild particle with global monopole charge. This has non-zero stresses given by

$$T_0^0 = T_1^1 = \frac{2k}{r^2}. \tag{13}$$

A global monopole is described by a triplet of scalars,

$$\psi^a(r) = \eta f(r) x^a / r, \text{ where } x^a x^a = r^2,$$

Through the usual Lagrangian it generates the energy-momentum distribution at large distance from the core, precisey of the form given above in (13) . Like the Schwarzschild solution the monopole solution (12) is also the unique solution of equation (11).

On the other hand, flat space-time could also be characterized in an alternative form by

$$\tilde{E}_{ab} = 0 = H_{[ab]}, E_{ab} = \lambda w_a w_b \tag{14}$$

leading to $C = A = 1$, and implying $\lambda = 0$. Its dual will be

$$E_{ab} = 0 = H_{[ab]}, \tilde{E}_{ab} = \lambda w_a w_b \tag{15}$$

yielding the general solution,

$$C' = A' = 0 \Longrightarrow C = 1, A = const. = (1-2k)^{-1/2} \tag{16}$$

which is non-flat and represents a global monopole of zero mass, as it follows from the solution (12) when $M = 0$.

In general the duality transformation for non-vacuum space-time maps a Lorentzian space-time into another Lorentzian space-time with Ricci and Einstein tensors interchanged. The vacuum equation is however invariant. Note that under the duality transformation the Weyl tensor does change sign and this implies that the mass, angular momentum and the NUT parameter change sign [11]. In the case of the Kerr and the NUT solutions, H_{ac} changes sign. We can however have distinct solutions only when the vacuum equation is suitably modified, without disturbing the original vacuum solution, so that it is not duality-invariant. This would happen when there exists a free equation which plays no role in determining the original vacuum solution. Then it is possible to write the alternate equation which though duality non-invariant, gives the same vacuum solution. (Note that as in equation (10), the λ-distribution is introduced along the direction of the acceleration vector, which is radial for spherical symmetry). Now the dual equation would in general give a solution different from the original solution. This is how we can construct solutions dual to vacuum or flat space-time.

Finally, the duality transformation essentially means interchange of roles of the Newtonian (active part) and the relativistic (passive part) aspects of the field. The two aspects are bound together by the Einstein field equations. As argued and demonstrated above duality seems to indicate "minimal" departure in physical features from the original space-time, and hence the covariant dual-flatness condition could be considered as defining "minimally" curved space-time [10].

The most general duality-invariant expression consisting of the Ricci and the metric is $R^a_b - (\frac{R}{4} - \Lambda) g^a_b$. The corresponding vacuum equation is

$$R^a_b - (\frac{R}{4} - \Lambda) g^a_b = 0.$$

By taking traces of both sides of the above equation we get $\Lambda = 0$ and hence it reduces to the equation for gravitational instanton

$$R^a_b - \frac{R}{4} g^a_b = 0. \tag{17}$$

We note that, over a 4-dimensional, Riemannian manifold (M, g), the gravitational instantons are critical points of the quadratic, Riemannian functional or action $\mathcal{A}_2(g)$ defined by

$$\mathcal{A}_2(g) = \int_M R^2 dv_g.$$

The action $\mathcal{A}_2(g)$ is conformally invariant. From equation (7) it follows that it is also invariant under the duality transformation (5). A generalization of this action is considered in section 5. Furthermore, the standard **Hilbert-Einstein action**

$$\mathcal{A}_1(g) = \int_M R dv_g$$

also leads to the instanton equations when the variation of the action is restricted to metrics of volume 1.

There are several differences between the Riemannian functionals used in theories of gravitation and the Yang-Mills functional used to study gauge field theories. The most important difference is that the Riemannian functionals are dependent on the bundle of frames of M or its reductions, while the Yang-Mills functional can be defined on any principal bundle over M. However, we have the following interesting theorem [1].

Theorem 2.1. Let (M, g) be a compact, 4-dimensional, Riemannian manifold. Let $\Lambda^2_+(M)$ denote the bundle of self-dual 2-forms on M with induced metric G_+. Then the Levi-Civita connection λ_g on M satisfies the gravitational instanton equations if and only if the Levi-Civita connection λ_{G_+} on $\Lambda^2_+(M)$ satisfies the Yang-Mills instanton equations.

The instanton action and the corresponding field equations are duality-invariant. They are also conformally invariant as well. As a matter of fact conformal invariance singles out the R^2-instanton action. That means the conformal invariance includes the duality invariance, while the duality invariance of the Palatini action with the condition that the resulting equation be valid for all values of R would lead to the conformal invariance [22]. The simplest and well-known instanton solution is the de Sitter space-time. Here the duality only leads to the anti-de Sitter space-time.

Coupling matter and other fields to gravitational instanton leads naturally to the notion of generalized gravitational instantons. It was introduced in [16]. We discuss this in the next section and then consider the relation between duality and instantons.

3. – Generalized Gravitational Instantons

We begin with a summary of the results obtained in earlier papers [17, 18, 19], which are relevant to the present paper. Let M be a space-time manifold. The space $\Lambda^2_x(M)$ of 2-forms at $x \in M$ is a real six-dimensional vector space. The Hodge star operator J on $\Lambda^2_x(M)$ defines a complex structure on it (i.e. $J^2 = -I$ is the identity transformation of $\Lambda_x^2(M)$). Using the complex structure $J, \Lambda_x^2(M)$ can be made into a complex three-dimensional vector space. The space of complex linear transformations of this complex vector space can be identified with the subspace of real transformations S of the real vector space defined by $\{S \mid SJ = JS\}$.

We now define a tensor of curvature type.

Definition 3.1. Let C be a tensor of type $(4,0)$ on M. We can regard C as a quadrilinear mapping (pointwise) so that for each $x \in M$, C_x can be identified with a multilinear map

$$C_x : T_x^*(M) \times T_x^*(M) \times T_x^*(M) \times T_x^*(M) \to \mathbf{R},$$

where $\mathbf{R}$ is the real number field. We say that the tensor C is of curvature type if C_x satisfies the algebraic properties of the Riemann-Christoffel curvature tensor, i.e. if C_x satisfies the following conditions for each $x \in M$ and for all $\alpha, \beta, \gamma, \delta \in T_x^*(M)$:

1. $C_x(\alpha, \beta, \gamma, \delta) = -C_x(\beta, \alpha, \gamma, \delta)$;
2. $C_x(\alpha, \beta, \gamma, \delta) = -C_x(\alpha, \beta, \delta, \gamma)$;
3. $C_x(\alpha, \beta, \gamma, \delta) + C_x(\alpha, \gamma, \delta, \beta) + C_x(\alpha, \delta, \beta, \gamma) = 0$.

From the above definition it follows that a tensor C of curvature type also satisfies the following condition:

$$C_x(\alpha, \beta, \gamma, \delta) = C_x(\gamma, \delta, \alpha, \beta), \ \forall x \in M.$$

We observe that, if C is of curvature type, then C_x can be identified with a self-adjoint linear transformation of $\Lambda_x^2(M)$ for each $x \in M$, where $\Lambda_x^2(M)$ is the space of second order differential forms at x. We now define the curvature product of two symmetric tensors of type $(2,0)$ on M. It was introduced in [16] and used in [18] to obtain a geometric formulation of Einstein's equations.

Definition 3.2. Let g and T be two symmetric tensors of type $(2,0)$ on M. The **curvature product** *of g and T, denoted by $g \times_c T$, is a tensor of type $(4,0)$ defined by*

$$\begin{aligned}(g \times_c T)_x(\alpha, \beta, \gamma, \delta) := \ & \tfrac{1}{2}\big[g(\alpha, \gamma)T(\beta, \delta) + g(\beta, \delta)T(\alpha, \gamma) \\ & -g(\alpha, \delta)T(\beta, \gamma) - g(\beta, \gamma)T(\alpha, \delta)\big],\end{aligned}$$

for all $x \in M$ and $\alpha, \beta, \gamma, \delta \in T_x^*(M)$.

In the following proposition we collect together some important properties of the curvature product and tensors of curvature type.

Proposition 3.1. Let g and T be two symmetric tensors of type $(2,0)$ on M and let C be a tensor of curvature type on M. Then we have the following:

1. $g \times_c T = T \times_c g$.
2. *$g \times_c T$ is a tensor of curvature type.*
3. *Let G denote the tensor $g \times_c g$. Then G_x induces a pseudo-inner product on $\Lambda_x^2(M), \forall x \in M$.*

4. *C_x induces a symmetric, linear transformation of $\Lambda^2_x(M), \forall x \in M$.*

When g is the fundamental tensor of M and T is the energy-momentum tensor, we call $g \times_c T (= T \times_c g)$ the interaction tensor between the field and the source (energy-momentum tensor). It plays an essential role in definition of the new field equations given here. We denote the Hodge star operator on $\Lambda^2_x(M)$ by J_x. The fact that M is a Lorentz 4-manifold implies that J_x defines a complex structure on $\Lambda^2_x(M)$, $\forall x \in M$. Using this complex structure we can give a natural structure of a complex vector space to $\Lambda^2_x(M)$. We now define the gravitational tensor W_g, of curvature type, which includes the source term.

Definition 3.3. Let M be a space-time manifold with fundamental tensor g and let T be a symmetric tensor of type $(2,0)$ on M. Then the **gravitational tensor** *W_g is defined by*

$$W_g := R + g \times_c T, \tag{18}$$

where R is the Riemann-Christoffel curvature tensor of type $(4,0)$.

We are now in a position to give a geometric formulation of the generalized field equations of gravitation.

Theorem 3.2. Let W_g denote the gravitational tensor defined by (18) with source tensor T. We also denote by W_g the linear transformation of $\Lambda^2_x(M)$ induced by W_g. Then the following statements are equivalent:

1. *g satisfies the generalized field equations of gravitation*

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu}. \tag{19}$$

2. *W_g commutes with J, i.e.*

$$[W_g \ , J] = 0. \tag{20}$$

3. *W_g induces a complex linear transformation of the complex vector space $\Lambda^2_x(M)$ for each $x \in M$.*

4. *Let G denote the inner product on $\Lambda^2_x(M)$ induced by g. Define the submanifolds D_+ and D_- of $\Lambda^2_x(M)$ by*

$$D_\pm = \{v \in \Lambda^2_x(M) | G(v,v) = \pm 1 \ \textit{and} \ v \wedge v = 0\}. \tag{21}$$

Define a real valued function f on $D_+ \cup D_-$, called the **gravitational sectional curvature**, *by*

$$f(v) = s(v) \times G(W_g v, v), \tag{22}$$

where $s(v) = +1$, if $v \in D_+$ and $s(v) = -1$, if $v \in D_-$. Then

$$f = f \circ J. \tag{23}$$

We shall call the triple (M, g, T) a **generalized gravitational field** if any one of the conditions of the above theorem is satisfied. Generalized gravitational field equations were introduced by Marathe in [16]. In the special case when the energy-momentum tensor T is zero the results of the above theorem were obtained in [29, 31]. Solutions of Marathe's generalized gravitational field equations which are not solutions of Einstein's equations have been discussed in [5].

We note that the above theorem and the last condition in proposition 3.1 can be used to discuss the Petrov classification of gravitational fields (see Petrov [25]). We observe that the generalized field equations of gravitation contain Einstein's equations with or without the cosmological constant as special cases. Solutions of the source-free generalized field equations are called **gravitational instantons**. If the base manifold is Riemannian, then gravitational instantons correspond to Einstein spaces. A detailed discussion of the structure of Einstein spaces and their moduli spaces may be found in Besse [4].

If one wishes to retain the coupling constant, say k, in Einstein's equations one need only introduce it in the definition of W as follows:

$$W = R + kg \times_c T. \tag{24}$$

The equations (19) are then replaced by

$$R^{ij} - \frac{1}{4}Rg^{ij} = -k(T^{ij} - \frac{1}{4}Tg^{ij}). \tag{25}$$

In particular by taking $R = kT$, in equations (25) we obtain

$$R^{ij} - \frac{1}{2}Rg^{ij} = -kT^{ij} \tag{26}$$

which are Einstein's equations with the coupling constant k.

We observe that the equation (25) does not lead to any relation between the scalar curvature and the trace of the source tensor, since both sides of equation (25) are trace-free. Taking divergence of both sides of equation (25) and using the Bianchi identities we obtain the generalized conservation condition

$$\nabla_i T^{ij} - \frac{1}{k}g^{ij}\Phi_i = 0, \tag{27}$$

where ∇_i is the covariant derivative with respect to the vector $\frac{\partial}{\partial x^i}$,

$$\Phi = \frac{1}{4}(kT - R) \tag{28}$$

and $\Phi_i = \frac{\partial}{\partial x^i}\Phi$. Using the function Φ defined by equation (28), the field equations can be written as

$$R^{ij} - \frac{1}{2}Rg^{ij} - \Phi g^{ij} = -kT^{ij}. \tag{29}$$

In this form the new field equations appear as Einstein's field equations with the cosmological constant replaced by the function Φ, which we may call the cosmological function.

The cosmological function is intimately connected with the classical conservation condition expressing the divergence-free nature of the energy-momentum tensor as is shown by the following proposition.

Proposition 3.3. The energy-momentum tensor satisfies the classical conservation condition

$$\nabla_i T^{ij} = 0 \tag{30}$$

if and only if the cosmological function Φ is a constant. Moreover, in this case the generalized field equations reduce to Einstein's field equations with cosmological constant.

Equation (30) is called the differential or local law of conservation of energy and momentum. To obtain global conservation laws one needs to integrate equations (30). However, such integration is possible only if the space-time manifold admits Killing vectors. In general, a manifold does not admit any Killing vectors. Thus equations (30) have no clear physical meaning. An interesting discussion of this point is given in Sachs and Wu [28]. We note that, if the energy-momentum tensor is non-zero but is localized in the sense that it is negligible away from a given region, then the scalar curvature acts as a measure of the cosmological constant.

4. – Duality and Instanton Equations

By setting the energy-momentum tensor to zero in Theorem 3.2 we obtain various characterizations of the usual gravitational instanton. Thus equation (19) reduces to the instanton equation (17) and equation (20) becomes

$$[R\ ,\ J] = 0, \text{ i.e. } RJ - JR = 0 \text{ which is equivalent to } R + JRJ = 0. \tag{31}$$

Using equations (6) and (7) we obtain

$$R^b_a - \frac{1}{4}Rg^b_a = E^b_a + \tilde{E}^b_a + (E + \tilde{E})(u_a u^b - \frac{1}{2}g^b_a) + \frac{1}{2}H^{mn}u^c(\eta_{acmn}u^b + \eta^b_{cmn}u_a) \tag{32}$$

Equations (1) and (31) imply that

$$E^b_a + \tilde{E}^b_a = 0 \text{ and hence } E + \tilde{E} = 0. \tag{33}$$

Equations (2) and (31) imply that

$$H_{[ab]} = 0 \tag{34}$$

Thus the instanton equations are equivalent to the pair of equations (33) and (34). We note that the instanton equations are equivalent to the vacuum Einstein equations with the cosmological term. These well known results are special cases of statements 1 and 2 of Theorem 3.2 and are characteristic of Einstein spaces [18]. Equations (33) and (34) contain as a special case the vacuum Einstein equations (8) when E or $\tilde{E}$ is zero and are, therefore, duality invariant. On the other hand Einstein's equations with non-zero energy-momentum tensor are not duality invariant. Thus to include matter fields we now consider the generalized instanton equation (25) with the coupling constant $k = 1$. Let us consider the case of perfect fluid. In this case the energy-momentum tensor is given by

$$T_a^b = (\rho + p)u_a u^b - p g_a^b. \tag{35}$$

Its trace T is then given by

$$T = (\rho + p) - 4p = \rho - 3p. \tag{36}$$

Thus equation (25) becomes

$$R_a^b - \frac{1}{4}R g_a^b = -(\rho + p)(u_a u^b - \frac{1}{4}g_a^b). \tag{37}$$

Note that equation (37) reduces to the instanton equation if $\rho + p = 0$. We do not need ρ and p to vanish separately.

In general, equation (25) can give only the three differences $T_0^0 - T_1^1, T_0^0 - T_2^2, T_0^0 - T_3^3$ between the components of the energy-momentum tensor. For a perfect fluid this will determine $\rho + p$ and an equation of state $\rho = \rho(p)$ will be necessary to find ρ and p separately. For example, the condition of pressure isotropy will require

$$E_1^1 + \tilde{E}_1^1 = E_2^2 + \tilde{E}_2^2 = E_3^3 + \tilde{E}_3^3. \tag{38}$$

In equation (37), if we put $\bar{\rho} = \rho + p$ and $\bar{\rho} - R = -4\Lambda$ (a constant), then it reduces to Einstein's equation with the fluid distribution replaced by dust with density $\bar{\rho} = \rho + p$ and Λ acts as the cosmological constant. We now obtain a solution of equation (37) in the special case of the static spherically symmetric metric

$$ds^2 = C^2(r)dt^2 - A^2(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{39}$$

The natural choice for the resolving vector in this case is along the t-direction. A direct calculation leads to the following two equations for the three unknowns A, C and $\bar{\rho}$.

$$\frac{C''}{C} - \frac{A'C'}{AC} + \frac{A^2 - 1}{r^2} = \frac{1}{r}\left(\frac{A'}{A} + \frac{C'}{C}\right). \tag{40}$$

$$\frac{A'}{A} + \frac{C'}{C} = -\frac{1}{2}rA^2\bar{\rho}. \tag{41}$$

Given $\bar{\rho}$ as a function of r, we can eliminate C from equations (40) and (41) to obtain the following non-linear differential equation for A:

$$\frac{A''}{A} - 3\frac{A'^2}{A^2} - \frac{1}{2}rAA'\bar{\rho} - \frac{1}{4}r^2A^4\bar{\rho}^2 - \frac{A^2 - 1}{r^2} + \frac{1}{2}rA^2\bar{\rho}' = 0. \tag{42}$$

A particular solution is given by

$$\bar{\rho} = constant, \quad A^{-2} = 1 + \frac{1}{2}r^2\bar{\rho} \text{ and } C = 1. \tag{43}$$

Here the constant $\bar{\rho}$ remains undetermined and we get

$$R = 3\bar{\rho} \text{ and hence } \bar{\rho} - R = -2\bar{\rho}. \tag{44}$$

The particular solution given by (43) corresponds to dust of uniform density $\bar{\rho}$ with the cosmological constant $\Lambda = \frac{1}{2}\bar{\rho}$. We note that for this solution $E_{ab} = 0$ and hence the gravitational field is entirely supported by the passive electric part of the curvature. This is a remarkable property of this space-time which is an Einstein space. The constant parameter of the Einstein metric remains undetermined. Thus in the generalized equations a perfect fluid with the equation of state $\rho + p = constant$ corresponds to a uniform density dust with a cosmological constant.

5. – Some Open problems

It is intriguing and remarkable that the duality is intimately related to the topological defects; i.e. it simply incorporates a topological defect (global monopole) into the original field. Thus duality transformation seems to be a topological defect generating process. In this context it is important to note that this process generates zero gravitational charge ($T^0_0 - T^a_a = 0$), and hence it would continue to share the basic physical properties at the Newtonian level with the original vacuum or flat space-time. This raises a couple of interesting questions: i) Is the topological defect producing property general or is it specific only to spherical and axial symmetries? ii) Would a space-time with topological defect be always dual to some vacuum (including electrovac with cosmological term) or flat space-time, i.e. is it always obtainable as a solution of equations dual to the vacuum or flat space-time equations? This is the case for the Schwarzschild, Reissner Nordstrom, Kerr and the NUT solutions [10, 11, 12, 24]

We now discuss a natural generalization of the instanton equations (by considering the metric and connection as independent variables) that usually goes under the name of the Palatini principle. Note that the connection is not on an arbitrary principal bundle but on the bundle of frames $L(M)$ of (M, g).

In [14] the Palatini principle has been used to study the field equations obtained from the variation of the action

$$S(\Gamma, g) := \int_M f(\rho)\sqrt{g}\; d^n x \tag{45}$$

where (M, g) is an n-dimensional pseudo-Riemannian manifold with metric g and Γ is a torsion-free connection on $L(M)$. The function f is taken as an analytic function of one variable ρ which is the scalar curvature determined by the pair (Γ, g). If we require the Lagrangian function f to be a non-constant function, then it follows that the pair (Γ, g) satisfy the equations

$$\rho_{(\mu\nu)} - \frac{1}{n}\rho g_{\mu\nu} = 0 \tag{46}$$

These equations must be considered together with the consistency condition

$$f'(\rho)\rho - \frac{n}{2}f(\rho) = 0 \tag{47}$$

We note that the consistency condition is identically satisfied by the conformally invariant Lagrangian

$$f(\rho) = \rho^{\frac{n}{2}} \ .$$

In particular, when $n = 4$ and Γ is the Levi-Civita connection we obtain the well known quadratic Lagrangian $f(R) = R^2$ considered in section 2. The equations (46) contain as a very special case the vacuum Einstein equations but are much more general. The Palatini principle has been used with quadratic Lagrangians f as well as for other Lagrangians in [15, 26, 30]. Following the definition given in [22], we call the pair (Γ, g) an Einstein pair if it satisfies the equations (46). The equations (46) can be generalized further as follows:

$$\rho_{(\mu\nu)} = \phi g_{\mu\nu} \tag{48}$$

where ϕ is a real scalar field on M. The equations (48) generalize the gravitational instanton equations which in turn generalize the vacuum Einstein equations.

Thus the universal equations obtained from the non-constant Lagrangian $f(\rho)$ by the Palatini principle are not the Einstein equations but the generalized gravitational instanton equations [22]. The equations (46) and (48) as well as equations (25) can be regarded as deformations of the usual gravitational instanton equations. A discussion of some properties and solutions of these equations is given in [23]. It is important to note that the tensor appearing in the equations (46) is not the usual Einstein tensor but is the trace-free part of the Ricci tensor $\rho_{\mu\nu}$. Definition of the electromagnetic resolution is not applicable in this case, since the Riemann tensor cannot be considered as a 2-form with values in the space of 2-forms. It would be interesting to find a general definition of the electromagnetic resolution and then define a generalized duality transformation. Work on these problems is in progress.

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REFERENCES

[1] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. *Proc. Roy. Soc. Lond.* A, **362** (1978) 425 .
[2] M. Barriola and A. Vilenkin *Phys. Rev. Lett.*, **63** (1989) 341.
[3] M.A.G. Bonilla and J.M.M. Senovilla *Gen. Relativ. Gravit.*, **29** (1997) 91.
[4] A. Besse. *Einstein Manifolds.* (Springer-Verlag, Berlin) 1986.
[5] D. Canarutto. *Il Nuovo Cimento* B, **75** (1983) 134 .
[6] N. Dadhich *Gen. Relativ. Gravit.*, **28** (1996) 1455.
[7] N. Dadhich *Pramana*, **49** (1997) 417.
[8] N. Dadhich and K. Narayan, *Gen. Relativ. Gravit.*, **30** (1998) 1133.

[9] N. Dadhich, K. Narayan and U.A. Yajnik, *Pramana*, **50** (1998) 307.
[10] N. Dadhich, On Electrogravity duality in general relativity, Received Honorable mention in 1998 Gravity Research Foundation Essay, gr-qc/9805068 Preprint, 1998.
[11] N. Dadhich, On Electromagnetics of gravitation, to be submitted, 1998.
[12] N. Dadhich and L.K. Patel, Spacetime dual to the Kerr solution, to be submitted, 1998.
[13] N. Dadhich, L.K. Patel and R. Tikekar *Class. Quantum Grav. Lett.*, **15** (1998) L27.
[14] M. Ferraris, M. Francaviglia, and I. Volovich. *Class. Quantum Grav.*, **11** (1994) 1505.
[15] P. Higgs. *Il Nuovo Cimento*, **11** (1959) 816.
[16] K. B. Marathe. PhD thesis, University of Rochester, 1971.
[17] K. B. Marathe. *Rend. Mat. (Roma)*, **6** (1972) 439.
[18] K. B. Marathe. *J. Math. Phys.*, **14** (1973) 228 .
[19] K. B. Marathe. *Physica* A, **114** (1982) 143.
[20] K. B. Marathe and G. Martucci. *The Mathematical Foundations of Gauge Theories.* Studies in Mathematical Physics, vol. 5 (North-Holland, Amsterdam) 1992.
[21] K. B. Marathe, G. Martucci, and M. Francaviglia. *Seminario di Matematica dell'Università di Bari*, **262** (1995) 1.
[22] K.B. Marathe and G. Martucci. *Il Nuovo Cimento* B, **113** (1998) 1175.
[23] K.B. Marathe and G. Martucci, Deformation of gravitational instantons, to be submitted.
[24] M. Nouri-Zonoz, N. Dadhich and D. Lynden-Bell, Spacetime dual to the NUT space, submitted for publication, 1998.
[25] A. Z. Petrov. *Einstein Spaces.* (Pergamon Press, New York) 1969.
[26] J. R. Ray. *Il Nuovo Cim.* B, **25** (1975) 707.
[27] A.K. Raychaudhuri *Phys.Rev.*, **90** (1955) 1123.
[28] R. K. Sachs and H. Wu. *General Relativity for Mathematicians.* (SpringerVerlag, Berlin) 1977.
[29] I. M. Singer and J. Thorpe. in *Global Analysis, Papers in honor of K. Kodaira*, (Princeton Uni. Press, Princeton) 1969, pp. 355–365.
[30] G. Stephenson. *Il Nuovo Cimento*, **9** (1958) 263.
[31] J. A. Thorpe. *J. Math. Phys.*, **10** (1969) 1.