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Almost complex algebraic curvature tensors in dimension 4

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Abstract

We give a classification of almost complex algebraic curvature tensors in spaces of signature $(0,4)$ and $(2,2)$ which are almost complex spacelike IP. We describe the components of these curvature tensors with respect to the decomposition of the space of all algebraic curvature tensors into irreducible $SO(4)$- (or $SO(2,2)$-) representations.

1 Introduction

In differential geometry to each semi-Riemannian manifold one assigns the Riemannian curvature tensor R. This tensor field carries geometric information about the manifold. We are interested in the following general question. Assume that the Riemannian curvature tensor satifies certain algebraic conditions. Which geometric consequences do we obtain for the manifold? In particular we consider conditions regarding the eigenvalue structure of the skew-symmetric curvature operator which is associated with the Riemannian curvature tensor. This was first done by Ivanova and Stanilov in [IS 95]. The curvature operator is defined as follows. Denote by $\mathrm{Gr}_2^+(TM)$ the Grassmannian of non-degenerate oriented 2-planes in TM and define

$$R: \quad \mathrm{Gr}_2^+(TM) \longrightarrow \mathrm{Hom}(TM)$$

by $R(\pi) = R(X,Y)$, where X, Y is an oriented orthonormal basis of π. Our general aim is to study manifolds for which in a certain sense the Jordan form of $R_x(\pi)$ does not depend on π for all $x \in M$. More exactly, let $(V, \langle\cdot\,,\cdot\rangle)$ be a pseudo-Euclidean vector space of signature (p,q) and let $\mathcal{C}V \subset \mathrm{Sym}^2{\bigwedge}^2 V$ be the space of algebraic curvature tensors on V. Then $R \in \mathcal{C}V$ is called a *spacelike Jordan IP curvature tensor* (or an *IP curvature tensor* in case $p=0$) if the Jordan form of the complexification of $R(\cdot)$ is constant on the Grassmannian of all oriented spacelike 2-planes. A semi-Riemannian metric on a manifold M is called *spacelike Jordan IP* (or *IP* in the Riemannian case) if its curvature tensor is a spacelike Jordan IP curvature tensor (or an IP curvature tensor) in each point of M. Such curvature tensors and metrics were first classified by Ivanov and Petrova in spaces of signature $(0,4)$, see [IP 98]. For signature $(0,n)$, $n \geq 5$, $n \neq 7$ Gilkey, Leahy, Sadowsky obtained a classification in [G 99], [GLS 99]. Furthermore, there are some partial results for higher signatures due to Gilkey and Zhang, see [GZ 00], [Z 00]. Here we will consider a slightly different notion of an IP curvature tensor which is adapted to Hermitian vector spaces and almost Hermitian manifolds. Let $(V, J, \langle\cdot\,,\cdot\rangle)$ be a Hermitian vector space. An algebraic curvature tensor R is said to be *almost complex* if $J \circ R(\pi) = R(\pi) \circ J$ for every non-degenerate complex line π. An almost complex algebraic curvature tensor R is said to be *almost complex spacelike IP* if the Jordan form of $R(\,\cdot\,)$ is constant on the Grassmannian of spacelike complex lines. In [GI 00] Gilkey and Ivanova construct many examples of such curvature tensors using Clifford matrices. The aim of this paper is to give a classification of almost complex algebraic curvature tensors in spaces of signature $(0,4)$ and $(2,2)$ which are spacelike IP. We describe the components of these curvature tensors with respect to the decomposition of the space of all algebraic curvature tensors into irreducible $SO(4)$- (or $SO(2,2)$-) representations.

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2 Classification results

Let $(V, \langle\cdot\,,\cdot\rangle, J)$ be a 4-dimensional Hermitian vector space of signature $(0,4)$ or $(2,2)$. Let $\omega := \langle\,\cdot\,, J\cdot\,\rangle$ be the Kähler form and let the orientation of V be given by the form $\omega \wedge \omega$. The space $\bigwedge^2 V$ of alternating 2-forms decomposes into $\bigwedge^2_+ V \oplus \bigwedge^2_- V$, where $\bigwedge^2_+ V$ and $\bigwedge^2_- V$ are the eigenspaces of the Hodge operator $* : \bigwedge^2 V \longrightarrow \bigwedge^2 V$ with eigenvalues 1 and -1. The inner product $\langle\cdot\,,\cdot\rangle$ induces an inner product on $\bigwedge^2 V$. Restricted to $\bigwedge^2_\pm V$ this inner product has signature $(0,3)$ in the definite and signature $(2,1)$ in the split case. The complex structure J maps $\bigwedge^2_+ V$ to $\bigwedge^2_+ V$ and we denote by $U_+ \subset \bigwedge^2_+ V$ and $U_- \subset \bigwedge^2_+ V$ the eigenspaces of J on $\bigwedge^2_+ V$ with eigenvalues 1 and -1. Then $\dim U_+ = 1$ and $\dim U_- = 2$.

The space $\mathcal{C}V$ of algebraic curvature tensors decomposes as a $SO(V, \langle\cdot\,,\cdot\rangle)$-representation into the following irreducible components,

$$\mathcal{C}V = \mathcal{U}V \oplus \mathcal{W}^+V \oplus \mathcal{W}^-V \oplus \mathcal{Z}V\,,$$

where

$$\begin{aligned}
\mathcal{U}V &= \mathbb{R}\,\mathrm{Id}_{\bigwedge^2 \mathrm{V}} \\
\mathcal{W}^+V &= \{R \in \mathrm{Sym}^2_0 \textstyle\bigwedge^2 V \mid *R = R* = R\} \cong \mathrm{Sym}^2_0 \textstyle\bigwedge^2_+ V \\
\mathcal{W}^-V &= \{R \in \mathrm{Sym}^2_0 \textstyle\bigwedge^2 V \mid *R = R* = -R\} \cong \mathrm{Sym}^2_0 \textstyle\bigwedge^2_- V \\
\mathcal{Z}V &= \{R \in \mathrm{Sym}^2 \textstyle\bigwedge^2 V \mid *R = -R*\} \cong \mathrm{Hom}(\textstyle\bigwedge^2_- V, \textstyle\bigwedge^2_+ V)\,.
\end{aligned}$$

Here we identify Sym^2 with self-adjoint maps and Sym^2_0 with traceless selfadjoint maps. For $R \in \mathcal{C}V$ we denote by $\tau \mathrm{Id}_{\bigwedge^2 \mathrm{V}}$, W^+, W^-, and Z the components of R with respect to this decomposition. The number τ is related to the scalar curvature s of R by $\tau = 12s$.

Proposition 2.1 *Let $(V, \langle\cdot\,,\cdot\rangle, J)$ be a 4-dimensional Hermitian vector space of signature $(2,2)$ or $(0,4)$ and $R \in \mathcal{C}V$ an algebraic curvature tensor. Then $R = \tau \mathrm{Id}_{\bigwedge^2 \mathrm{V}} + \mathrm{W}^+ + \mathrm{W}^- + \mathrm{Z}$ is almost complex if and only if $Z(\bigwedge^2_- V) \subset U_+$ and $W^+(U_\pm) \subset U_\pm$.*

Theorem 2.1 *Let $(V, \langle\cdot,\cdot\rangle, J)$ be a 4-dimensional Hermitian vector space of signature $(2,2)$ or $(0,4)$ and $R \in \mathcal{C}V$ an algebraic curvature tensor. Then $R = \tau \mathrm{Id}_{\bigwedge^2 \mathrm{V}} + \mathrm{W}^+ + \mathrm{W}^- + \mathrm{Z}$ is almost complex spacelike IP if and only if the following conditions are satisfied,*

(i) $Z = 0$, i.e. R is Einstein,

(ii) $\overline{W}^- := \tau\mathrm{Id}_{\bigwedge^2_- \mathrm{V}} + \mathrm{W}^-$ satisfies $(\overline{W}^-)^2 = \lambda \mathrm{Id}_{\bigwedge^2_- \mathrm{V}}$, $\lambda \in \mathbb{R}$,

(iii) $W^+(U_+) \subset U_+$, $W^+(U_-) \subset U_-$.

In case of signature $(0,4)$ the curvature operator of an almost complex IP curvature tensor is diagonalizable (as a complex map). Its eigenvalues are equal if and only if $\lambda = 0$ which is equivalent to $\overline{W}^- = 0$. In signature $(2,2)$ a spacelike complex IP curvature tensor has a diagonalizable curvature operator if and only if $\lambda \neq 0$ or $\overline{W}^- = 0$. The eigenvalues are different if and only if $\lambda \neq 0$.

3 Proof of the theorem

First we want to reformulate the property of the curvature tensor to be almost complex. P. B. Gilkey proved in [GI 00] that a curvature tensor R is almost complex if and only if

$$R(X, Y, JU, JW) = R(JX, JY, U, W) \tag{1}$$

holds for all $X, Y, U, W \in V$.

Proof of Proposition 2.1. We consider R and J as maps from $\bigwedge^2 V$ into $\bigwedge^2 V$. Then (1) is equivalent to $R \circ J = J \circ R$. Hence, R preserves the eigenspaces of J, which are $U_+ \oplus \bigwedge^2_- V$ and U_-. The proposition is only a reformulation of this fact using the components W^+, W^- and Z of R. □

In the following we will make use of the fact that the eigenvalues of a complex $k \times k$-matrix A are determined by $\mathrm{Tr}\, A, \mathrm{Tr}\, A^2, \ldots, \mathrm{Tr}\, A^k$. We recall the following well-known fact.

Lemma 3.1 *If R is almost complex spacelike IP, then $\langle X, X\rangle^{-1} R(X, JX)$ has the same eigenvalues for all non-isotropic vectors X.*

Proof. Let $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ the complex bilinear extension of $\langle\cdot,\cdot\rangle$. By (1) the maps $R(X, JX)$ and $R(X, JY) + R(Y, JX)$ are complex for all X, $Y \in V$. Hence, for $Z = X + iY \in V^{\mathbb{C}}$ we can define a complex endomorphism $R^{\mathbb{C}}(Z, JZ)$ on V by

$$R^{\mathbb{C}}(Z, JZ) := R(X, JX) - R(Y, JY) + i(R(X, JY) + R(Y, JX)) .$$

Using the trace of this complex endomorphism we define

$$f_1(Z) = \langle Z, Z\rangle_{\mathbb{C}}^{-1} \,\mathrm{Tr}\, R^{\mathbb{C}}(Z, JZ)$$

and

$$f_2(Z) = \langle Z, Z \rangle_{\mathbb{C}}^{-2} \operatorname{Tr} R^{\mathbb{C}}(Z, JZ)^2,$$

for $Z \in V^{\mathbb{C}}$. Both functions are holomorphic on the open connected set

$$\Omega = \{Z \in V^{\mathbb{C}} \mid \langle Z, Z \rangle_{\mathbb{C}} \neq 0\} \subset V^{\mathbb{C}}$$

and they determine the eigenvalues of $\langle Z, Z \rangle_{\mathbb{C}}^{-1} R(Z, JZ)$. Since they are constant on the set of all spacelike vectors in V, they have to be constant on Ω. In particular, they are constant on the set of non-isotropic vectors in V. $\square$

Proof of Theorem 2.1. In the first part of the proof we will concentrate on signature $(0,4)$, the calculations in the $(2,2)$-case are essentially the same. We fix an orthonormal basis e_1, $e_2 = Je_1$, e_3, $e_4 = Je_3$ and define

$$\begin{aligned} f_1^{\pm} &= e_1 \wedge e_2 \pm e_3 \wedge e_4 \\ f_2^{\pm} &= e_1 \wedge e_3 \mp e_2 \wedge e_4 \\ f_3^{\pm} &= e_1 \wedge e_4 \pm e_2 \wedge e_3 \,. \end{aligned}$$

Then f_1^+, f_2^+, f_3^+ is a basis of $\bigwedge_+^2 V$ and f_1^-, f_2^-, f_3^- is a basis of $\bigwedge_-^2 V$. Furthermore, f_1^+ spans U_+. Now let R be an almost complex curvature tensor. By (1) the linear maps $R(e_1, e_2)$, $R(e_3, e_4)$, $R(e_1, e_3) + R(e_2, e_4)$, $R(e_1, e_4) - R(e_2, e_3) : V \longrightarrow V$ commute with J and taking into account the symmetry properties of the curvature tensor we may assume

$$\begin{gathered} A := R(e_1, e_2) = \begin{pmatrix} ia & -b+ic \\ b+ic & id \end{pmatrix} \qquad B := R(e_3, e_4) = \begin{pmatrix} id & -e+if \\ e+if & ig \end{pmatrix} \\ C := R(e_1, e_3) + R(e_2, e_4) = \begin{pmatrix} 2ib & -h+ij \\ h+ij & 2ie \end{pmatrix} \\ D := R(e_1, e_4) - R(e_2, e_3) = \begin{pmatrix} 2ic & -j+il \\ j+il & 2if \end{pmatrix} \end{gathered} \tag{2}$$

with respect to the basis e_1, e_3 of the complex vector space (V, J). Conversely, if (2) holds, than

$$\begin{aligned} \mathcal{R}(u, v, x, y) &:= R(ue_1 + ve_2 + xe_3 + ye_4, J(ue_1 + ve_2 + xe_3 + ye_4)) \\ &= (u^2 + v^2)A + (x^2 + y^2)B + (-uy + vx)C + (ux + vy)D \end{aligned} \tag{3}$$

commutes with J and, thus, R is almost complex.

By Proposition 2.1 an almost complex curvature tensor maps $U_+ \oplus \bigwedge_-^2 V$ into $U_+ \oplus \bigwedge_-^2 V$ and U_- into U_-. Hence, condition *(iii)* is automatically satisfied. Therefore we consider only $R|_{U_+ \oplus \bigwedge_-^2 V}$. With respect to the basis $f_1^+, f_1^-, f_2^-, f_3^-$ of $U_+ \oplus \bigwedge_-^2 V$ the map $R|_{U_+ \oplus \bigwedge_-^2 V}$ is given by

$$R|_{U_+ \oplus \bigwedge_-^2 V} = \begin{pmatrix} (a+g+2d)/2 & (a-g)/2 & b+e & c+f \\ (a-g)/2 & (a+g-2d)/2 & b-e & c-f \\ b+e & b-e & h & j \\ c+f & c-f & j & l \end{pmatrix} .$$

Using this a simple calculation shows, that for an almost complex curvature tensor conditions *(i)* and *(ii)* are satisfied if and only if

(i') $\operatorname{Tr} A = \operatorname{Tr} B,\ \operatorname{Tr} C = \operatorname{Tr} D = 0$

(ii') $\operatorname{Tr} AC = \operatorname{Tr} AD = \operatorname{Tr} CD = 0,\ \operatorname{Tr}(A-B)^2 = \operatorname{Tr} C^2 = \operatorname{Tr} D^2$.

Now let R be an almost complex spacelike IP curvature tensor. Then the eigenvalues of $\mathcal{R}(u,v,x,y)$ and, hence, the functions $\operatorname{Tr}\mathcal{R}(u,v,x,y)$ and $\operatorname{Tr}\mathcal{R}(u,v,x,y)^2$ must be constant for all $u,v,x,y \in \mathbb{R}$ with $u^2+v^2+x^2+y^2 = 1$. In particular, we have $\operatorname{Tr} A = \operatorname{Tr} B$ and $\operatorname{Tr} A^2 = \operatorname{Tr} B^2$. Furthermore, the trace of

$$\mathcal{R}(\cos t, 0, \sin t, 0) = \cos^2 tA + \sin^2 tB + \cos t \sin tD$$

is constant for all $t \in \mathbb{R}$. Hence, $\operatorname{Tr} D = 0$. Similarly, $\operatorname{Tr} C = 0$. Now consider

$$\begin{aligned}(\mathcal{R}(\cos t, 0, \sin t, 0))^2 &= \cos^4 t\, A^2 + \sin^4 t\, B^2 + \cos^2 t \sin^2 t\,(AB + BA + D^2)\\ &\quad + \cos^3 t \sin t\,(AD + DA) + \cos t \sin^3 t(BD + DB)\\ &= A^2 + t(AD + DA) + t^2(-2A^2 + AB + BA + D^2) + \mathcal{O}(t^3)\,.\end{aligned}$$

The trace of this map must be constant for all $t \in \mathbb{R}$. Using $\operatorname{Tr} AB = \operatorname{Tr} BA$ and $\operatorname{Tr} A^2 = \operatorname{Tr} B^2$ we obtain

$$\operatorname{Tr}(\mathcal{R}(\cos t, 0, \sin t, 0))^2 = \operatorname{Tr} A^2 + 2t \operatorname{Tr} AD + t^2(-(A-B)^2 + D^2) + \mathcal{O}(t^3)$$

and, hence, $\operatorname{Tr} AD = 0$, $\operatorname{Tr}(A-B)^2 = \operatorname{Tr} D^2$. Similarly, $\operatorname{Tr} AC = 0$, $\operatorname{Tr}(A-B)^2 = \operatorname{Tr} C^2$. It remains to prove $\operatorname{Tr} CD = 0$. For this let us consider

$$\begin{aligned}&(\mathcal{R}(\cos t, 0, \tfrac{1}{\sqrt{2}}\sin t, \tfrac{1}{\sqrt{2}}\sin t))^2 = \cos^4 t\, A^2 + \sin^4 t\, B^2\\ &\quad + \cos^2 t \sin^2 t\,(AB + BA + \tfrac{1}{2}(D-C)^2) + \tfrac{1}{\sqrt{2}} \cos t \sin^3 t(B(D-C) + (D-C)B)\\ &\quad + \tfrac{1}{\sqrt{2}} \cos^3 t \sin t\,(A(D-C) + (D-C)A)\\ &= A^2 + \tfrac{1}{\sqrt{2}} t(A(D-C) + (D-C)A) + t^2(-2A^2 + AB + BA + \tfrac{1}{2}(D-C)^2)\\ &\quad + \mathcal{O}(t^3)\,.\end{aligned}$$

Since also for this family of operators the trace must be constant we get

$$\begin{aligned}0 &= \operatorname{Tr}(-2A^2 + AB + BA + \tfrac{1}{2}(D-C)^2)\\ &= \operatorname{Tr}(-(A-B)^2 + \tfrac{1}{2}D^2 + \tfrac{1}{2}C^2) + \operatorname{Tr} CD\,.\end{aligned}$$

Using $\operatorname{Tr}(A-B)^2 = \operatorname{Tr} C^2 = \operatorname{Tr} D^2$ we obtain $\operatorname{Tr} CD = 0$. Consequently, *(i')* and *(ii')* and therefore *(i)* and *(ii)* are satisfied.

Conversely, let R be a curvature tensor which satisfies *(i)*, *(ii)* and *(iii)*. Conditions *(i)* and *(iii)* imply that R is almost complex. Hence, *(i)* and *(ii)* imply *(i')* and *(ii')* and it remains to prove that an almost complex curvature tensor which satisfies *(i')* and *(ii')* is spacelike IP. First we derive some further equations from *(i')* and *(ii')*. Condition *(i')* implies

$$a = g,\ b = -e,\ c = -f$$

and, thus,

$$\begin{aligned} \operatorname{Tr} A^2 &= \operatorname{Tr} B^2 && (4)\\ \operatorname{Tr} BC &= -\operatorname{Tr} AC = 0 && (5)\\ \operatorname{Tr} BD &= -\operatorname{Tr} AD = 0\,. && (6) \end{aligned}$$

Let u, v, x, y be real numbers such that $u^2+v^2+x^2+y^2=1$. By (3) and Condition *(i')*

$$\operatorname{Tr} \mathcal{R}(u,v,x,y) = (u^2+v^2+x^2+y^2) \operatorname{Tr} A = \operatorname{Tr} A = \text{ const}\,.$$

According to *(ii')*, (5), (6) we have

$$\begin{aligned} \operatorname{Tr}(\mathcal{R}(u,v,x,y))^2 &= (u^2+v^2)^2 \operatorname{Tr} A^2 + (x^2+y^2)^2 \operatorname{Tr} B^2 + (-uy+vx)^2 \operatorname{Tr} C^2\\ &\quad +(ux+vy)^2 \operatorname{Tr} D^2 + (u^2+v^2)(x^2+y^2) \operatorname{Tr}(AB+BA)\,. \end{aligned}$$

Now using

$$\operatorname{Tr}(AB+BA) = -\operatorname{Tr}(A-B)^2 + \operatorname{Tr} A^2 + \operatorname{Tr} B^2 = -\operatorname{Tr}(A-B)^2 + 2\operatorname{Tr} A^2$$

(ii') and (4) yield

$$\begin{aligned} \operatorname{Tr}(\mathcal{R}(u,v,x,y))^2 &= ((u^2+v^2)^2 + (x^2+y^2)^2 + 2(u^2+v^2)(x^2+y^2)) \operatorname{Tr} A^2\\ &\quad +((-uy+vx)^2 + (ux+vy)^2 - (u^2+v^2)(x^2+y^2)) \operatorname{Tr} C^2\\ &= (u^2+v^2+x^2+y^2)^2 \operatorname{Tr} A^2 = \operatorname{Tr} A^2 = \text{ const}\,. \end{aligned}$$

Consequently, the eigenvalues of $\mathcal{R}(u,v,x,y)$ are constant for all u, v, x, $y \in \mathbb{R}$ such that $u^2+v^2+x^2+y^2=1$. These eigenvalues are equal if and only if those of A are equal. This is the case if and only if $(a-d)^2+4b^2+4c^2=0$. On the other hand, $\lambda = (a-d)^2+4b^2+4c^2$. The last statement of the theorem is clear in the Riemannian case.

We finish the proof with a few remarks concerning the split case. According to Lemma 3.1 we can proceed as in the Riemannian case up to some changes of signs. Now we obtain that the curvature operator is almost complex and the eigenvalues are constant if and only if

$$\begin{aligned} R(e_1,e_2) &= \begin{pmatrix} -ia & b-ic \\ b+ic & id \end{pmatrix} \qquad R(e_3,e_4) = \begin{pmatrix} -id & b-ic \\ b+ic & ia \end{pmatrix}\\ R(e_1,e_3)+R(e_2,e_4) &= \begin{pmatrix} -2ib & h-ij \\ h+ij & 2ib \end{pmatrix} \qquad (7)\\ R(e_1,e_4)-R(e_2,e_3) &= \begin{pmatrix} -2ic & j-il \\ j+il & 2ic \end{pmatrix} \end{aligned}$$

with respect to any orthonormal basis e_1, $e_2 = Je_1$, e_3, $e_4 = Je_3$ with timelike e_1 and e_2,

$$\overline{W}^- = \begin{pmatrix} a+d & 2b & 2c \\ -2b & -h & -j \\ -2c & -j & -l \end{pmatrix}$$

with respect to the basis $f_1^- = e_1 \wedge e_2 + e_3 \wedge e_4$, $f_2^- = e_1 \wedge e_3 + e_2 \wedge e_4$, $f_3^- = e_1 \wedge e_4 - e_2 \wedge e_3$ of $\bigwedge_-^2 V$, and

$$(\overline{W}^-)^2 = \lambda \, \mathrm{Id}_{\bigwedge_-^2 V} .$$

The eigenvalues are different if and only if $\lambda \neq 0$. If $\overline{W}^- = 0$ then the curvature operator is diagonalizable. Now we consider the case $\lambda = 0$ and $\overline{W}^- \neq 0$. Recall that the inner product induced on $\bigwedge_-^2 V$ has signature $(2,1)$. The first column vector of $\overline{W}^-$ does not vanish, since $\overline{W}^- \neq 0$ implies $a+d \neq 0$. Furthermore, the column vectors are all isotropic and pairwise orthogonal, hence they must be multiples of the first one. More exactly, we have

$$j = \frac{4bc}{a+d}, \quad l = \frac{4c^2}{a+d}, \quad h = \frac{4b^2}{a+d} . \tag{8}$$

In order to show that R is almost complex spacelike IP in this case we will prove that $R(X, JX)$ is not diagonalizable for all spacelike vectors X. Since the eigenvalues are equal we have to show that $\mathcal{R}(u,v,x,y)$ is not diagonal for all u,v,x,y such that $-u^2 - v^2 + x^2 + y^2 = 1$. Assume that it is diagonal. Then

$$(u^2+v^2+x^2+y^2)(b+ic) + (-uy+vx)(h+ij) + (ux+vy)(j+il) = 0 .$$

By (8) we obtain

$$\begin{aligned} (u^2+v^2+x^2+y^2)b + (-uy-vx)\frac{4b^2}{a+d} + (ux+vy)\frac{4bc}{a+d} &= 0 \\ (u^2+v^2+x^2+y^2)c + (-uy-vx)\frac{4bc}{a+d} + (ux+vy)\frac{4c^2}{a+d} &= 0 . \end{aligned}$$

Since $b^2 + c^2 \neq 0$ this implies

$$(a+d)(u^2+v^2+x^2+y^2) + 4b(-uy+vx) + 4c(ux+vy) = 0$$

which is equivalent to

$$\left\langle \begin{pmatrix} a+d \\ -2b \\ -2c \end{pmatrix}, \begin{pmatrix} u^2+v^2+x^2+y^2 \\ 2(-uy+vx) \\ 2(ux+vy) \end{pmatrix} \right\rangle = 0 .$$

The first vector of the l.h.s. is isotropic since it equals the first column vector of $\overline{W}^-$. The second one is spacelike because of

$$(u^2+v^2+x^2+y^2)^2 - 4(-uy+vx)^2 - 4(ux+vy)^2 = (u^2+v^2-x^2-y^2)^2 = 1 .$$

This is a contradiction to the fact that the orthogonal complement of a spacelike vector with respect to an inner product of signature $(2,1)$ is timelike.

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