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Abstract

We obtain a Bernstein theorem for special Lagrangian graphs in $\mathbb{C}^n = \mathbb{R}^{2n}$ for arbitrary n only assuming bounded slope but no quantitative restriction.

1 Introduction

Let M be the graph in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ of a smooth map $f : \Omega \to \mathbb{R}^n$, with $\Omega \subseteq \mathbb{R}^n$ an open domain. M is a Lagrangian submanifold of $\mathbb{C}^n$ if and only if the matrix $\left(\frac{\partial f^i}{\partial x^j}\right)$ is symmetric. In particular, in that case if Ω is simply connected, then there exists a function $F : \Omega \to \mathbb{R}$ with

$$\nabla F = f.$$

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A Lagrangian submanifold of $\mathbb{C}^n$ is called special if it is a minimal submanifold at the same time. In the above situation, the graph of ∇F is a special Lagrangian submanifold of $\mathbb{C}^n$ if and only if for some constant ϑ,

$$\text{Im}\ (\det(e^{i\vartheta}(I + i\ \text{Hess}\ (F))) = 0 \tag{1.1}$$

(Im = imaginary part, I = identity matrix, Hess $F = (\frac{\partial^2 F}{\partial x^i \partial x^j})$).

Special Lagrangian calibration constitute an example of a calibrated geometry in the sense of Harvey and Lawson [11]. More recently, Strominger-Yau-Zaslow [18] established a conjectural relation of fibrations by special Lagrangian tori with mirror symmetry. However, in general, some of these tori are singular. More generally, understanding such fibrations systematically means understanding the moduli space of special Lagrangian tori, and for that purpose, one needs to study the possible singularities of special Lagrangian submanifolds in flat space. By asymptotic expansions at singularities, the so-called blow-ups, the study of singular special Lagrangian spaces is reduced to the study of special Lagrangian cones.

In the theory of minimal submanifolds, there exists a close link between the rigidity of minimal cones and Bernstein type theorems, saying that - under suitable boundedness assumptions - entire minimal graphs are necessarily planar. Of course, the known Bernstein type theorems for entire minimal graphs, in particular [12] and [13], apply here. Those results seem to be close to optimal already. It turns out, however, that under the Lagrangian condition, one may prove still stronger such results. This is the content of the present paper.

Returning to (1.1), the Bernstein question then is whether, or more precisely, under which conditions, an entire solution has to be a quadratic polynomial.

Fu [7] showed that for $n = 2$, any solution defined on all of $\mathbb{R}^2$ is harmonic or a quadratic polynomial. Before stating our results on this question, however, let us briefly observe that the equation (1.1) is similar to the Monge-Ampère equation

$$\det\left(\frac{\partial^2 F}{\partial x^i \partial x^j}\right) = 1 \tag{1.2}$$

that naturally arises in affine differential geometry. There, one is interested in convex solutions, and Calabi [3] showed that, for $n \leq 5$, any convex solution of (1.2) that is defined on all of $\mathbb{R}^n$ has to be a quadratic polynomial. Pogorolev subsequently extended this result to all n ([15], [19]).

In that direction, we have

Theorem 1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined on the whole $\mathbb{R}^n$. Assume that the graph of ∇F is a special Lagrangian submanifold M in $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$, namely F satisfies equation (1.1). If*

(i) F is convex;

(ii) there is a constant $\beta < \infty$ such that

$$\Delta_F \leq \beta, \tag{1.3}$$

where

$$\Delta_F = \{\det(I + (Hess\ (F))^2\}^{\frac{1}{2}}, \tag{1.4}$$

then F is a quadratic polynomial and M is an affine n-plane.

For the proof of this theorem, we shall use the same strategy as in our previous paper [13], dealing with minimal graphs in general.

As M is a minimal submanifold of Euclidean space, by the theorem of Ruh-Vilms [16], its Gauss map is harmonic. The Gauss map takes its values in the Grassmannian $G_{n,n}$ of n-planes in $2n$-space. In order to show that M is affine linear, we need to show that the Gauss map is constant. The strategy of Hildebrandt-Jost-Widman [12] then was to show that the image of the Gauss map is contained in some geodesically convex ball, and to show a Liouville type theorem to the extent that any such harmonic map with values in such a ball is constant. The method works optimally if we look at harmonic maps with values in a space of constant sectional curvature, i.e. a sphere. In the case of higher codimension k, the Grassmannian $G_{n,k}$, however, does not have the same sectional curvature in all directions anymore. The strategy of Jost-Xin [13] then was to exploit the Grassmannian geometry more carefully and to construct other geodesically convex sets for which such a Liouville type theorem for harmonic maps still holds. This led to a considerable strengthening of the Bernstein type theorems for minimal graphs of higher codimension. Still, however, those results do not yet imply the preceding theorem. We need to exploit the fact that M is not only minimal, but also Lagrangian. In other words, its Gauss map takes its values in a certain subspace of $G_{n,n}$, namely the Lagrangian Grassmannian LG_n of Lagrangian linear subspaces of $\mathbb{R}^{2n}$. LG_n is a totally geodesic subspace of $G_{n,n}$, and so the Gauss map of M as a map into LG_n is still harmonic. We can now exploit the geometry of LG_n to construct suitable geodesically convex subsets in that space and deduce a corresponding Liouville type theorem. In that way, we shall show that the Gauss map of M is constant under the conditions stated in Theorem 1, and so M is planar.

For proving that M is flat another possible approach is to study its tangent cone $C\tilde{M}$ at infinity. In our situation $C\tilde{M}$ is a special Lagrangian cone. Its link is a compact minimal Legendrian submanifold in S^{2n-1}. Thus, we prove Theorem 2. It is interesting in its own right.

Theorem 2. *Let M be a simple (in the sense of [12]) or compact minimal Legendrian submanifold in S^{2n-1}. Suppose that there are a fixed n-plane P_0 and some $\delta > 0$, such that*

$$\langle P, P_0 \rangle \geq \delta \tag{1.5}$$

holds for all normal n-planes P of M in S^{2n-1}. Then M is contained in a totally geodesic subsphere of S^{2n-1}.

Although in general, any minimal 2-sphere in a sphere S^m is totally geodesic (see [2]), there exist higher dimensional minimal submanifolds of S^m that are not totally geodesic and by [14], we know that the analogue of Theorem 2 does not hold for minimal submanifolds of spheres for arbitrarily small values of δ. In fact, there even exist nontrivial minimal Legendrian S^3's in S^7 [4], and already for $n = 2$, one finds infinitely many different minimal Legendrian two-dimensional Legendrian tori in S^5 [10]. Thus, condition (1.5) cannot be dropped even in the Legendrian case.

By using Theorem 2 we can remove the convexity condition for the function F in Theorem 1. This is in fact our main result:

Theorem 3. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function on the whole $\mathbb{R}^n$. The graph of ∇F defines a special Lagrangian submanifold M in $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$. In other words, F satisfies equation (1.1). If there is a constant $\beta < \infty$ that satisfies (1.3) and (1.4), then F is a quadratic polynomial and M is flat.*

The celebrated theorem of Bernstein says that the only entire minimal graphs in Euclidean 3-space are planes. This result has been partially generalized to higher codimension. If $f : \mathbb{R}^k \to \mathbb{R}^n$ is an entire solution of the minimal surface system with bounded gradient, then f is linear for $k = 2$ by a theorem of Osserman-Chern and for $k = 3$ by a result of Fischer-Colbrie [8]. For larger k, however, there exist counterexamples of Lawson-Osserman. By way of contrast, our Theorem 3 shows that minimal Lagrangian graphs with bounded gradient are always planar.

2 Geometry of Lagrangian Grassmannian manifolds

Let $\mathbb{R}^{m+n}$ be an $(m+n)$-dimensional Euclidean space. The set of all oriented n-subspaces (called n-planes) constitutes the Grassmannian manifold $G_{n,m}$, which is the irreducible symmetric space $SO(m+n)/SO(m) \times SO(n)$.

Let $\{e_\alpha, e_{n+i}\}$ be a local orthonormal frame field in $\mathbb{R}^{m+n}$, where $i, j, ... = 1, ..., m; \alpha, \beta, ... = 1, ..., n; a, b, ... = 1, ..., m+n$ (say, $n \le m$). Let $\{\omega_\alpha, \omega_{n+i}\}$ be its dual frame field so that the Euclidean metric is

$$g = \sum_\alpha \omega_\alpha^2 + \sum_i \omega_{n+i}^2.$$

The Levi-Civita connection forms ω_{ab} of $\mathbb{R}^{m+n}$ are uniquely determined by the equation

$$\begin{aligned} d\omega_a &= \omega_{ab} \wedge \omega_b, \\ \omega_{ab} + \omega_{ba} &= 0. \end{aligned} \tag{2.1}$$

The canonical Riemannian metric on $G_{n,m}$ can be defined by

$$ds^2 = \sum_{\alpha,i} \omega_{\alpha\, n+i}^2. \tag{2.2}$$

From (2.1) and (2.2) it is easily seen that the curvature tensor of $G_{n,m}$ is

$$\begin{aligned} R_{\alpha i\, \beta j\, \gamma k\, \delta l} = \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{ik}\delta_{jl} + \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{kl} \\ -\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{il}\delta_{kj} - \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{ij}\delta_{kl} \end{aligned} \tag{2.3}$$

in a local orthonormal frame field $\{e_{\alpha i}\}$, which is dual to $\{\omega_{\alpha\, n+1}\}$.

Let P_0 be an oriented n-plane in $\mathbb{R}^{m+n}$. We represent it by n vectors e_α, which are complemented by m vectors e_{n+i}, such that $\{e_\alpha, e_{n+i}\}$ form an orthonormal base of $\mathbb{R}^{m+n}$. Then we can span the n-planes P in a neighborhood $\mathbb{U}$ of P_0 by n vectors f_α:

$$f_\alpha = e_\alpha + z_{\alpha z} e_{e+i},$$

where $(z_{\alpha i})$ are the local coordinates of P in $\mathbb{U}$. The metric (2.2) on $G_{n,m}$ in those local coordinates can be described as

$$ds^2 = tr((I_n + ZZ^T)^{-1} dZ (I_m + Z^T Z)^{-1} dZ^T) \tag{2.4}$$

where $Z = (z_{\alpha i})$ is an $(n \times m)$-matrix and I_n (res. I_m) denotes the $(n \times n)$-identity (res. $m \times m$) matrix.

Now we consider the case $\mathbb{R}^{2n} = \mathbb{C}^n$ which has the usual complex structure J. For any $u = (x, y) = (x_1, ...x_n; y_1, ...y_n)$ in $\mathbb{R}^{2n}$

$$Ju = (-y_1, ..., -y_n; x_1, ..., x_n).$$

For an n-plane $\zeta \subset \mathbb{R}^{2n}$ if any $u \in \zeta$ satisfies

$$\langle u, Ju \rangle = 0,$$

then ζ is called a Lagrangian plane. The Lagrangian planes yield the Lagrangian Grassmannian manifold. It is the symmetric space $U(n)/SO(n)$.

For the Grassmannian $G_{n,n}$ its local coordinates are described by $(n\times n)$-matrices. For any $A \in LG_n \subset G_{n,n}$, let $u = (x, xA)$ and $\tilde{u} = (\tilde{x}, \tilde{x}A)$ be two vectors in A. By definition $\langle u, J\tilde{u}\rangle = 0$ and so we have

$$A = A^T$$

From (2.4) it is easy to see that the transpose is an isometry of $G_{n,n}$. Hence the fix point set LG_n is a totally geodesic submanifold of $G_{n,n}$. By the Gauss equation the Riemannian curvature tensor of LG_n is also defined by (2.3).

Let $\dot{\gamma} = x_{\alpha i}e_{\alpha i}$ be a unit tangent vector at P_0, where $\{e_{\alpha i}\}$ is a local orthonormal frame field. By an action of $SO(n)$

$$x_{\alpha i} = \lambda_\alpha \delta_{\alpha i},$$

there $\sum\limits_\alpha \lambda_\alpha^2 = 1$. In our previous paper [13] we have computed the eigenvalues of the Hessian of the distance function from a fixed print P_0 at the direction $\dot{\gamma} = (x_{\alpha i}) = (\lambda_\alpha \delta_{\alpha i})$. Considering the present situation, when $m = n$ and the eigenvectors are symmetric matrices, the eigenvalues are as follows:

$$\begin{aligned} (\lambda_\alpha - \lambda_\beta)\cot(\lambda_\alpha - \lambda_\beta)r &\quad \text{with multiplicity } 1 \\ \frac{1}{r} &\quad \text{with multiplicity } n-1 \end{aligned} \tag{2.5}$$

where r is the distance from P_0, for sufficiently small r.

The geodesic from P_0 at $(x_{\alpha i}) = (\lambda_\alpha \delta_{\alpha i})$ in the local coordinates neighborhood U is (see [20])

$$(z_{\alpha i}(t)) = \begin{pmatrix} \tan(\lambda_1 t) & & 0 \\ & \ddots & \\ 0 & & \tan(\lambda_n t) \end{pmatrix} \tag{2.6}$$

where t is the arc length parameter and $0 \le t < \frac{\pi}{2|\lambda_n|}$ with $|\lambda_n| = \max(|\lambda_1|, ..., |\lambda_n|)$.

3 Gauss map

Let M be an n-dimensional oriented submanifold in $\mathbb{R}^{m+n}$. Choose an orthonormal frame field $\{e_1, ..., e_{m+n}\}$ in $\mathbb{R}^{m+n}$ such that the $e_\alpha' s$ are tangent to M. Let $\{\omega_1, ..., \omega_{m+n}\}$ be its coframe field. Then, the structure equations of $\mathbb{R}^{m+n}$ along M are as follows.

$$\omega_{n+i} = 0,$$

$$d\omega_\alpha = \omega_{\alpha\beta} \wedge \omega_\beta,\ \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0,$$

$$\omega_{n+i\,\alpha} = h_{i\alpha\beta}\omega_\beta,$$

$$d\omega_{ij} = \omega_{i\kappa} \wedge \omega_{\kappa j} + \omega_{i\alpha} \wedge \omega_{\alpha j},$$

where the $h_{i\alpha\beta}$, the coefficients of the second fundamental form of M in $\mathbb{R}^{m+n}$, are symmetric in α and β. Let 0 be the origin of $\mathbb{R}^{m+n}$. Let $SO(m+n)$ be the manifold consisting of all the orthonormal frames $(0; e_\alpha, e_{n+i})$. Let $P = \{(x; e_1, ..., e_n); x \in M, e_\alpha \in T_xM\}$ be the principal bundle of orthonormal tangent frames over $M, Q = \{(x; e_{n+1}, ..., e_{m+n}); x \in M, e_{n+i} \in N_xM\}$ be the principal bundle of orthonormal normal frames over M, then $\bar{\pi} : P \otimes Q \to M$ is the projection with fiber $SO(m) \times SO(n), i : P \otimes Q \hookrightarrow SO(m+n)$ is the natural inclusion.

We define the generalized Gauss map $\gamma : M \to \mathbf{G}_{n,m}$ by

$$\gamma(x) = T_xM \in \mathbf{G}_{n,m}$$

via the parallel translation in $\mathbb{R}^{m+n}$ for $\forall x \in M$. Thus, the following commutative diagram holds

$$\begin{array}{ccc} P \otimes Q & \xrightarrow{i} & SO(m+n) \\ {\scriptstyle\bar{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\gamma} & \mathbf{G}_{n,m} \end{array}$$

Using the above diagram, we have

$$\gamma^*\omega_{n+i\,\alpha} = h_{i\alpha\beta}\omega_\beta. \tag{3.1}$$

Now, we assume that M is a Lagrangian submanifold in $\mathbb{R}^{2n}$. The image of the Gauss map $\gamma : M \to \mathbf{G}_{n,m}$ then lies in its Lagrangian Grassmannian LG_n. We then have $e_\alpha \in TM$ and $Je_\alpha \in NM$.

Furthermore,

$$h_{i\alpha\beta} = \langle \nabla_{e_\alpha} e_\beta, Je_i\rangle = -\langle \nabla_{e_\alpha} Je_i, e_\beta\rangle = \langle \nabla_{e_\alpha} e_i, Je_\beta\rangle = h_{\beta\alpha i}.$$

Thus, the $h_{i\alpha\beta}$ are symmetric in all their indices. (3.1) can also be written as its dual form

$$\gamma_* e_\beta = h_{i\alpha\beta} e_{\alpha i}. \tag{3.2}$$

For each $e_\beta, (h_{i\alpha\beta})$ is a symmetric matrix.

4 Minimal Legendrian submanifolds in the sphere and minimal Lagrangian cones

In the sphere $S^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ there is a standard contact structure. Let X be the position vector field of the sphere and η be the dual form of JX in S^{2n-1}, where J is the complex structure of $\mathbb{C}^n = \mathbb{R}^{2n}$. It is easily seen that

$$d\eta = 2\omega, \tag{4.1}$$

where ω is the Kähler form of $\mathbb{C}^n$. Therefore,

$$\eta \wedge (d\eta)^{n-1} \neq 0 \tag{4.2}$$

everywhere and η is a constant form in S^{2n-1}. The maximal dimensional integral submanifolds of the distribution

$$\eta = 0 \tag{4.3}$$

are $(n-1)$-dimensional and are called Legendrian submanifolds in S^{2n-1}.

Now, let us consider the cone CM over M. CM is the image under the map $M \times [0, \infty)$ into $\mathbb{R}^{2n}$ defined by $(x, t) \to tx$, where $x \in M, t \in [0, \infty)$. CM has a singularity at $t = 0$. The associated truncated cone CM_ε is the image of $M \times [\varepsilon, \infty)$ under the same map, where ε is any positive number.

We have (see [17])

Proposition 4.1. *CM_ε is minimal submanifold in $\mathbb{R}^{2n}$ if and only if M is a minimal submanifold in S^{2n-1}.*

For a fixed point $x \in M$ choose a local orthonormal frame field $\{e_s\}$ $(s = 1, ..., n-1)$ near x in M with $\nabla_{e_s} e_t|_x = 0$.

By parallel translating along rays from the origin, we obtain a local vector field E_s in CM. Obviously, $E_s = \frac{1}{r} e_s$, where r is the distance from the origin. Thus, $\{E_s, \tau\}$ is a frame field in CM, where $\tau = \frac{\partial}{\partial r}$ is the unit tangent vector along rays. Obviously $\nabla_\tau \tau = 0$.

In the case of M being Legendrian

$$\eta(e_s) = 0,$$

and

$$\begin{aligned} d\eta(e_s, e_t) &= (\nabla_{e_s}\eta)e_t - (\nabla_{e_t}\eta)e_s \\ &= \nabla_{e_s}\eta(e_t) - \nabla_{e_t}\eta(e_s) - \eta([e_s, e_t]) = 0. \end{aligned}$$

From (4.1) it follows that

$$\omega(E_s, E_t) = \frac{1}{r^2}\omega(e_s, e_t) = 0. \tag{4.4}$$

Obviously

$$\omega(E_s, \tau) = \langle E_s, J\tau\rangle = \frac{1}{r}\eta(e_s) = 0 \tag{4.5}$$

(4.4) and (4.5) mean that CM is a Lagrangian submanifold in $\mathbb{R}^{2n}$ if and only if M is a Legendrian submanifold in S^{2n-1}.

Now, let us compute the coefficients of the second fundamental form of CM in $\mathbb{R}^{2n}$.

We have a local orthonormal frame field $\{E_s, \tau, JE_s, J\tau\}$ in $\mathbb{R}^{2n}$ along CM, where $\{E_s, \tau\}$ is a local orthonormal frame field in CM.

Note

$$\nabla_{E_s}\tau = \nabla_{E_s}\frac{X}{r} = \frac{1}{r}E_s,$$

where X denotes the position vector of the concernd point.

Then

$$\langle\nabla_{E_s}E_t, \tau\rangle = -\langle E_s, \nabla_{E_t}\tau\rangle = -\frac{1}{r}\delta_{st},$$

and

$$\begin{aligned}
\frac{d}{dr}\langle\nabla_{E_s}E_t, E_u\rangle &= \langle\nabla_\tau\nabla_{E_s}E_t, E_u\rangle\\
&= \langle\nabla_{E_s}\nabla_\tau E_t, E_u\rangle + \langle\nabla_{[\tau,E_s]}E_t, E_u\rangle\\
&= -\frac{1}{r}\langle\nabla_{E_s}E_t, E_u\rangle,\\
\frac{d}{dr}\langle\nabla_{E_s}E_t, JE_u\rangle &= -\frac{1}{r}\langle\nabla_{E_s}E_t, JE_u\rangle.
\end{aligned}$$

Integrating them gives

$$\langle\nabla_{E_s}E_t, E_u\rangle = \frac{C_{ust}}{r}$$

and

$$\langle\nabla_{E_s}E_t, JE_u\rangle = \frac{D_{ust}}{r},$$

where C_{ust}, D_{ust} are constants along the ray. They can be determined by the conditions at $r = 1$ as follows

$$C_{ust} = 0 \quad , \quad D_{ust} = h_{ust},$$

where h_{ust} are the coefficients of the second fundamental form of M in S^{2n-1} in the Je_u directions. We also have

$$\langle\nabla_{E_s}E_t, J\tau\rangle = -\langle E_t, \nabla_{E_s}J\tau\rangle = -\frac{1}{r}\langle E_t, JE_s\rangle = 0.$$

Thus, we obtain the coefficients of the second fundamental form CM in $\mathbb{R}^{2n}$ as follows. In the JE_u directions

$$B_{uij} = \begin{pmatrix} \frac{h_{uij}}{r} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.6}$$

and in the $J\tau$ direction

$$B_{nij} = 0 \tag{4.7}$$

From (3.2), (4.5) and (4.6) we know that the Gauss map of the cone CM has rank $n-1$ at most. We summarize the results of this section as

Proposition 4.2. *Let M be an $(n-1)$-dimensional submanifold in S^{2n-1}. It is minimal and Legendrian if and only if the cone CM over M is a minimal Lagrangian submanifold in $\mathbb{R}^{2n}$. Furthermore, the Gauss map $\gamma : CM \to LG_n$ has rank $n-1$ at most.*

5 Harmonic Maps

Let (M,g) and (N,h) be Riemannian manifolds with metric tensors g and h, respectively. Harmonic maps are described as critical points of the following energy functional

$$E(f) = \frac{1}{2}\int_M e(f) * 1, \tag{5.1}$$

where $e(f)$ stands for the energy density. The Euler-Lagrange equation of the energy functional is

$$\tau(f) = 0, \tag{5.2}$$

where $\tau(f)$ is the tension field. In local coordinates

$$e(f) = g^{if}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}h_{\beta\gamma}, \tag{5.3}$$

$$\tau(f) = (\Delta_M f^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j})\frac{\partial}{\partial y^\alpha}, \tag{5.4}$$

where $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbols of the target manifold N. For more details on harmonic maps consult [5].

A Riemannian manifold M is said to be simple, if it can be described by coordinates x on $\mathbb{R}^n$ with a metric

$$ds^2 = g_{ij}dx^i dx^j, \tag{5.5}$$

for which there exist positive numbers λ and μ such that

$$\lambda|\xi|^2 \leq g_{ij}\xi^i\xi^j \leq \mu|\xi|^2 \tag{5.6}$$

for all x and ξ in $\mathbb{R}^n$. In other words, M is topologically $\mathbb{R}^n$ with a metric for which the associated Laplace operator is uniformly elliptic on $\mathbb{R}^n$.

Hildebrandt-Jost-Widman proved a Liouville-type theorem for harmonic maps in [12]:

Theorem 5.1. *Let f be a harmonic map from a simple or compact Riemannian manifold M into a complete Riemannian manifold N, the sectional curvature of which is bounded above by a constand $\kappa \geq 0$. Denote by $B_R(Q)$ a geodesic ball in N with radius $R < \frac{\pi}{2\sqrt{\kappa}}$ which does not meet the cut locus of its center Q. Assume also that the range $f(M)$ of the map f is contained in $B_R(Q)$. Then f is a constand map.*

Remark. In the case where $B_R(Q)$ is replaced by another geodesically convex neighborhood, the iteration technique in [12] is still applicable and the result remains true (for example, a general version of that iteration technique that directly applies here has been given in [9]).

By using the composition formula for the tension field, one easily verifies that the composition of a harmonic map $f: M \to N$ with a convex function $\phi : f(M) \to \mathbb{R}$ is a subharmonic function on M. The maximum principle then implies

Proposition 5.2. *Let M be a compact manifold without boundary, $f : M \to N$ a harmonic map with $f(M) \subset V \subset N$. Assume that there exists a strictly convex function on V. Then f is a constant map.*

Let $M \to \mathbb{R}^{m+n}$ be an n-dimensional oriented submanifold in Euclidean space. We have the relation between the property of the submanifold and the harmonicity of its Gauss map in [16].

Theorem 5.3. *Let M be a submanifold in $\mathbb{R}^{m+n}$. Then the mean curvature vector of M is parallel if and only if its Gauss map is a harmonic map.*

Let $M \to S^{m+n} \hookrightarrow \mathbb{R}^{m+n-1}$ be an m-dimensional submanifold in the sphere. For any $x \in M$, by parallel translation in $\mathbb{R}^{m+n+1}$, the normal space N_xM of M in S^{m+n} is moved to the origin of $\mathbb{R}^{m+n+1}$. We then obtain an n-subspace in $\mathbb{R}^{m+n+1}$. Thus, the so-called normal Gauss map $\gamma : M \to G_{n,m+1}$ has been defined. There is a natural isometry η between $G_{n,m+1}$ and $G_{m+1,n}$ which maps any n-subspace into its orthogonal complementary $(m+1)$-subspace. The map $\eta^* = \eta\circ\gamma$ maps any point $x \in M$ into an $(m+1)$-subspace spanned by T_xM and the position vector of x. From Theorem 5.3 and Proposition 4.1 it follows that

Proposition 5.4. *M is a minimal m-dimensional submanifold in the sphere S^{m+n} if and only if its normal Gauss map $\gamma : M \to G_{n,m+1}$ is a harmonic map.*

6 Proofs of the theorems

Proof of Theorem 1

Since M is a graph in $\mathbb{R}^{2n}$ defined by ∇F, the induced metric g on M is

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta,$$

where

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \frac{\partial^2 F}{\partial x^\alpha \partial x^\gamma} \frac{\partial^2 F}{\partial x^\beta \partial x^\gamma}.$$

It is obvious that the eigenvalues of the matrix $(g_{\alpha\beta})$ at each point are ≥ 1. The condition (1.3) implies that the eigenvalues of the matrix $(g_{\alpha\beta})$ are $\leq \beta^2$.

The condition (5.6) is satisfied and M is a simple Riemannian manifold.

Let $\{e_\alpha, e_{n+\beta}\}$ be the standard orthonormal base of $\mathbb{R}^{2n}$. Choose P_0 as an n-plane spanned by $e_1 \wedge \ldots \wedge e_n$. At each point in M its image n-plane P under the Gauss map is spanned by

$$f_\alpha = e_\alpha + \frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} e_{n+\beta},$$

which lies in the Lagrangian Grassmannian manifold LG_n.

Suppose the eigenvalues of Hess (F) at each point x are $\mu_\alpha(x)$ which are positive by the convexity of the function F. The condition (1.3) means

$$\prod_\alpha (1 + \mu_\alpha^2) \leq \beta.$$

Hence,

$$\mu_\alpha \leq \sqrt{\beta^2 - 1} \tag{6.1}$$

Define in the normal polar coordinates of P_0 in $LG(n)$

$$\tilde{B}_{LG}(P_0) = \{(X, t); X = (\lambda_\alpha \delta_{\alpha i}), \lambda_\alpha \geq 0; 0 \leq t \leq t_x = \tan^{-1} \sqrt{\beta^2 - 1}\}.$$

Two points P_0 and P can be joined by a unique geodesic $P(t)$ spanned by

$$\tilde{f}_\alpha(t) = e_\alpha + z_{\alpha\beta}(t) e_{n+\beta},$$

where

$$z_{\alpha\beta}(t) = \begin{pmatrix} \tan(\lambda_1 t) & & 0 \\ & \ddots & \\ 0 & & \tan(\lambda_n t) \end{pmatrix}.$$

Therefore, the image under the Gauss map γ of M lies in $\tilde{B}_{LG}(P_0)$.

On the other hand, from (2.5) we see that when $\lambda_\alpha \geq 0$ the square of the distance function r^2 from P_0 is a strictly convex smooth function in $\tilde{B}_{LG}(P_0)$. Furthermore, it is a geodesically convex set.

Now, we have the Gauss map $\gamma : M \to \tilde{B}_{LG}(P_0) \subset LG_n$ which is harmonic by Thoerem 5.3. Hence the conclusion follows by using Theorem 5.1.

Remark. If the graph of ∇F is a submanifold with parallel mean curvature instead of a minimal submanifold the Theorem remains true as well.

Proof of Theorem 2

From Proposition 5.4 we know that the normal Gauss map $\gamma : M \to G_{n,n}$ is harmonic. Let η be the isometry in $G_{n,n}$ which maps any n-plane into its orthogonal complementary n-plane. Hence $\eta \circ \gamma$ is also harmonic. On the other hand, from the discussion in § 4 it follows that

$$\eta \circ \gamma(M) = \gamma'(CM_\varepsilon), \tag{6.2}$$

where $\gamma' : CM_\varepsilon \to G_{n,n}$ is the Gauss map. Now let $\eta(P_0)$ be spanned by n vectors e_α, which are complemented by n vectors e_{n+i}. The condition (1.5) ensures that for all normal n-planes P of M in $S^{2n-1}, \eta(P)$ lies in the coordinate neighborhood U of $\eta(P_0)$ and $\eta(P)$ is spanned by n vectors f_α:

$$f_\alpha = e_\alpha + z_{\alpha i} e_{n+i},$$

where $(z_{\alpha i})$ are local coordinates of $\eta(P)$ in U. Noting (6.2), $\eta(P)$ lies in LG_n and $(z_{\alpha i})$ is a symmetric matrix. the geodesic from $\eta(P_0)$ at $(X_{\alpha i}) = (\lambda_\alpha \delta_{\alpha i})$ in U is described by (2.6).

Noting Proposition 4.2 there exists α_0 such that $\lambda_{\alpha_0} = 0$. Then by actions of $SO(n)$ we can achieve that at most one of $\lambda'_\alpha s$ is negative. Then by one more action of $SO(n)$ all the $\lambda'_\alpha s$ are nonnegative.

Take any point $\eta(p)$ in $\eta \circ \gamma(M) = \gamma'(CM_\varepsilon)$. Draw a geodesic $P(t)$ from $\eta(P_0)$ to $\eta(P)$. Let $P(t)$ be spanned by

$$f_\alpha = e_\alpha + z_{\alpha i} e_{n+i}$$

where $z_{\alpha i}$ is defined by (2.6). Let

$$\tilde{f}_1 = \cos(\lambda_1 t) f_1, ..., \tilde{f}_n = \cos(\lambda_n t) f_n.$$

Those $\tilde{f}_1, ..., \tilde{f}_n$ are orthonormal. Therefore,

$$\langle \eta(P_0), P(t) \rangle = \prod_{\alpha=1}^{n} \cos(\lambda_\alpha t),$$

where $\lambda_\alpha \geq 0$ and $\Sigma\lambda_\alpha^2 = 1$. From (1.5) it follows that

$$t \leq \frac{\cos^{-1}\delta}{\max\limits_{\alpha}(\lambda_\alpha)}. \tag{6.3}$$

Define in the normal polar coordinates around $\eta(P_0)$

$$\tilde{B}_{LG}(\eta(P_0)) = \{(X,t); X = (\lambda_\alpha\delta_{\alpha i}), 0 \leq t \leq t_X = \frac{\cos^{-1}\delta}{\max\limits_{\alpha}(\lambda_\alpha)}\}. \tag{6.4}$$

From (2.6) we see that $\tilde{B}_{LG}(\eta(P_0))$ lies inside the cut locus of $\eta(P_0)$. We also know from (2.5) that the square of the distance function r^2 from $\eta(P_0)$ is a strictly convex smooth function in $\tilde{B}_{LG}(\eta(P_0))$. By a similar argument as for $B_G(P_0)$ in our previous paper [13] it can be shown that $\tilde{B}_{LG}(P_0)$ is a geodesically convex set.

We thus have a harmonic map $\eta \circ \gamma$ from M into a geodesically convex set $\tilde{B}_{LG}(\eta(P_0))$. By using Theorem 5.1 we conclude that $\eta \circ \gamma$ is a constant map, and then so is the map γ. This completes the proof.

Proof of Theorem 3

Let us consider the tangent cone of M at ∞ as Fleming in [6]. Take the intersection of M with the ball of radius t and contract by $\frac{1}{t}$ to get a family of minimal submanifolds in the unit ball with submanifolds of S^{2n-1} as boundaries. More precisely, we define a sequence

$$F^t = \frac{1}{t^2}F(tx).$$

For each t

$$\frac{\partial^2 F^t}{\partial x^\alpha \partial x^\beta} = \frac{\partial^2 F}{\partial u^\alpha \partial u^\beta},$$

where $u^\alpha = tx^\alpha$. It turns out F^t satisfies the same conditions as F. Moreover, there is a subsequence $t_j \to \infty$ such that

$$\lim_{t_j \to \infty} F^t(x) = \tilde{F}(x).$$

$\tilde{F}$ satisfies (1.1), (1.2) and (1.3) and the graph $\nabla\tilde{F}$ is a special Lagrangian cone $C\tilde{M}$ whose link is a compact minimal Legendrian submanifold $\tilde{M}$.

Let $\{e_\alpha, e_{n+\beta}\}$ be the standard orthonormal base of $\mathbb{R}^{2n}$. Choose P_0 as an n-plane spanned by $e_1 \wedge ... \wedge e_n$. At each point of $C\tilde{M}$ its image n-plane P under the Gauss map is spanned by

$$f_\alpha = e_\alpha + \frac{\partial^2 \tilde{F}}{\partial x^\alpha \partial x^\beta} e_{n+\beta}$$

It follows that

$$|f_1 \wedge ... \wedge f_n|^2 = \det\left(\delta_{\alpha\beta} + \frac{\partial^2 \tilde{F}}{\partial x^\alpha \partial x^\gamma}\frac{\partial^2 \tilde{F}}{\partial x^\beta \partial x^\gamma}\right)$$

and

$$\Delta_f = |f_1 \wedge ... \wedge f_n|.$$

The n-plane P is also spanned by

$$P_\alpha = \Delta_f^{-\frac{1}{n}} f_\alpha,$$

moreover,

$$|p_1 \wedge ... \wedge p_n| = 1.$$

We then have

$$\begin{aligned}\langle P, P_0\rangle &= \det(\langle e_\alpha, P_\beta\rangle)\\ &= \Delta_f^{-1} \geq \beta^{-1}.\end{aligned}$$

Let η be the isometry in $G_{n,n}$ that maps any n-plane into its orthogonal complementary n-plane. We thus have

$$\langle \eta P, \eta P_0\rangle \geq \beta^{-1}.$$

By the discussion in § 4 we know that ηP is just the normal n-plane of $\tilde{M}$ in S^{2n-1}. Then Theorem 2 tells us that $\tilde{M}$ is a totally geodesic sphere S^{n-1} in S^{2n-1} and therefore, $C\tilde{M}$ is an n-plane in $\mathbb{R}^{2n}$. Allard's result [1] then implies that the original special Lagrangian submanifold M is an affine n-plane and F is a quadratic polynomial.

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