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A note on flows towards reflectors

by

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A NOTE ON FLOWS TOWARDS REFLECTORS

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Abstract. A classical problem in geometric optics is to find surfaces that reflect light from a given light source such that a prescribed intensity on a target is realized. We introduce a flow equation for surfaces such that they converge to solutions of this reflector problem both for closed hypersurfaces and for the illumination of prescribed domains.

1. Introduction

The classical reflector problem is to find a hypersurface such that light of a given intensity is reflected at this hypersurface so that a prescribed intensity on a target is realized. A ray of light in direction x is reflected at a hypersurface according to the reflection law to the new direction

$$T(x) = x - 2\langle x, \nu\rangle \nu,$$

where ν is a unit normal to the hypersurface at the point where the ray of light is reflected. In [2] the authors study a light source in $\mathbb{R}^{n+1}$, $n \ge 2$, located at the origin emitting light in all directions with a given smooth positive intensity function $f : S^n \to \mathbb{R}$, defined on the unit sphere. Each ray of light is reflected exactly once at a hypersurface that is star-shaped with respect to the origin. The directions of the reflected light correspond to points on the unit sphere S^n, so the reflection induces a new intensity function. Using elliptic methods it is shown in [2] that for any two intensity functions f and g as above there exists a smooth hypersurface that is star-shaped with respect to the origin such that the intensity function induced by the reflection equals the prescribed function g provided the energy of the emitted and reflected light coincide, i. e.

$$\int\limits_{S^n} f = \int\limits_{S^n} g. \tag{1.1}$$

Moreover, the solution is unique up to dilatations when $T : S^n \to S^n$ is a diffeomorphism. Using indices to denote covariant derivatives on S^n with

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respect to the metric σ_{ij} induced from the standard embedding $S^n \to \mathbb{R}^{n+1}$, this problem is equivalent, see [2], to the partial differential equation

$$\frac{\det\left(u_{ij}+\left(u-\frac{|\nabla u|^2+u^2}{2u}\right)\sigma_{ij}\right)}{\det\left(\frac{|\nabla u|^2+u^2}{2u}\cdot\sigma_{ij}\right)}=\frac{f(x)}{g(T(x))} \tag{1.2}$$

for $u : S^n \to \mathbb{R}_+$ with positive definite matrices as arguments in the determinants. The geometric meaning of this positivity condition is explained in [2]. It means that our hypersurface lies on one side of appropriate parabola that reflect light to one direction. We remark that $|\nabla u|$ is also evaluated using the induced metric of the sphere. The geometrical meaning of u is as follows. For $x \in S^n \subset \mathbb{R}^{n+1}$ we define $\rho : S^n \to \mathbb{R}_+$ such that $\rho(x)\cdot x$ belongs to our hypersurface. Then we have $u(x)=\frac{1}{\rho(x)}$.

We give an alternative proof of the result presented above using a parabolic flow equation. The flow, we are going to use, describes the deformation of reflecting hypersurfaces. These hypersurfaces converge finally to a stationary solution solving Equation (1.2). In general it is difficult to use a parabolic flow equation to obtain solutions to an elliptic problem that admits several solutions. Here it is known that any two solutions differ by a positive multiple. As it seems easier to us to consider a situation in which two solutions differ by an additive constant, we introduce a new function $\varphi : S^n \to \mathbb{R}$ by defining $\varphi(x)=\log u(x)$. It is easy to see that equation (1.2) is equivalent to

$$\frac{\det\left(\varphi_{ij}+\varphi_i\varphi_j+\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)}=\frac{f(x)}{g(T(x))}. \tag{1.3}$$

We wish to investigate a flow that becomes stationary at solutions of the elliptic problem and keeps the argument of the determinant in the numerator positive definite. We choose the following equation

$$\dot\varphi=\Phi\left(\log\left\{\frac{\det\left(\varphi_{ij}+\varphi_i\varphi_j+\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)}\cdot\frac{g(T(x))}{f(x)}\right\}\right) \tag{1.4}$$

with $\Phi : \mathbb{R}\to\mathbb{R}$, $\Phi(0)=0$, $\Phi'>0$ and $\Phi''\le 0$. For a discussion of this ansatz for the flow equation we refer to [5]. Besides the choice $\Phi(t)=t$, another interesting flow is obtained when $\Phi(t)=1-e^{-\lambda t}$, $\lambda>0$, i. e.

$$\dot\varphi=1-\left(\frac{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\varphi_{ij}+\varphi_i\varphi_j+\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}\cdot\frac{f(x)}{g(T(x))}\right)^\lambda.$$

We get the following

Theorem 1.1. *Let f, $g : S^n \to \mathbb{R}_+$ be smooth functions and let $\varphi_0 : S^n \to \mathbb{R}$ be a smooth function such that the argument of the determinant in the numerator in (1.4) is positive definite. Then the evolution equation (1.4) with initial condition $\varphi|_{t=0}=\varphi_0$ has a solution for all positive times, i. e.*

there exists a smooth function $\varphi : S^n \times [0,\infty) \to \mathbb{R}$ satisfying (1.4). The function $\varphi(\cdot,t)$ converges in C^∞ topology to a translating solution φ^∞ as $t \to \infty$, i. e. there exists $v^\infty \in \mathbb{R}$ such that $\varphi^\infty(x,t) = \varphi^\infty(x,0) + v^\infty \cdot t$. Moreover, v^∞ is determined by

$$v^\infty = \Phi\left(\log \int\limits_{S^n} g - \log \int\limits_{S^n} f\right), \tag{1.5}$$

so that we get a solution to the reflector equation (1.3) provided (1.1) holds and the hypersurfaces induced by $\varphi(\cdot,t)$ as described above converge to the reflector we look for as $t \to \infty$.

We remark that our parabolic approach does not only give a constructive method to find reflectors. If (1.1) is violated, a translating solution (at a fixed time) reflects the light such that the intensity of the reflected light equals g up to a constant factor. Note that Theorem 1.1 implies the existence theorem in [2] as $\varphi_0 = c \in \mathbb{R}$ is an admissible initial value.

In the problem considered so far, the light source emitted light in all directions and light should be reflected to all directions. Now we address to a model problem of a reflector that shall only illuminate a prescribed domain. We consider the situation when light is emitted from a domain $\Omega \subset \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ in direction e_{n+1}, where we identify $\mathbb{R}^n$ and $\mathbb{R}^n \times \{0\}$. We assume that a hypersurface, the reflector, is represented as a graph over Ω such that the light is reflected back to a domain $\Omega^* \subset \mathbb{R}^n$.

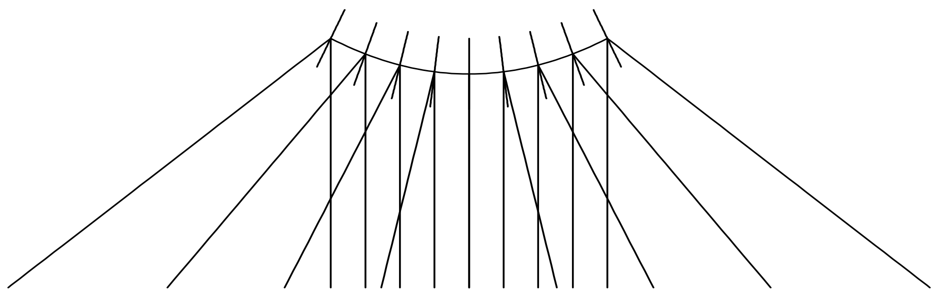

FIGURE 1. Reflection at a surface

This is illustrated in Figure 1. There we see the upwards directed rays of light, the reflecting surface, normals to this surface and finally the reflected rays of light. For simplicity we consider the following simple model. If the domain Ω is small compared to Ω^*, we can neglect the size of the reflector and assume that the reflected light is emitted from a single point – we take $(0,1) \in \mathbb{R}^n \times \mathbb{R}$ – in the direction given by the reflector law. This problem has applications in the design of reflectors for lamps.

Figure 2. Lamp in the court yard of our institute

Figure 2 shows a lamp in the court yard of our institute that illuminates the ground by sending light via a reflector to the ground. Up to now, the reflector consists of four triangles, so it seems desirable to improve the shape used there.

Using a flow ansatz similar as above we show that for any bounded smooth strictly convex domains $\Omega,\ \Omega^* \subset \mathbb{R}^n$ with $0 \in \Omega^*$ and for any smooth functions $f : \overline{\Omega} \to \mathbb{R}_+$, $g : \overline{\Omega^*} \to \mathbb{R}_+$, there exists a hypersurface, represented as graph $u|_\Omega$, such that light emitted with intensity f from Ω is reflected – in our model with small Ω – to Ω^* and the intensity g is realized provided

$$\int\limits_\Omega f = \int\limits_{\Omega^*} g.$$

Indeed, we can solve this problem for a larger class of domains Ω^*, but to describe these domains it is useful to have a technical deviation of the corresponding equations. So we give a description of the admissible class of domains Ω^* and the formulation of the corresponding theorem in Section 4.

In this second part we focus on the geometric description of the situation considered. Then it turns out that we get a second boundary value problem for a Monge-Ampère equation. This equation has been studied before in [6] in the elliptic setting and in a slightly different version in [5], see also the appendix in [4].

It is a further issue to solve the reflector problem with prescribed domains using a model that contains less simplifications.

The paper is organized as follows. In Section 2 we prove a priori estimates and show that a solution to the flow equation (1.4) exists for all time, then we obtain convergence to a translating solution in Section 3. In Section 4 we address to the problem of illuminating domains in $\mathbb{R}^n$ and state the main theorem for this problem.

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2. Longtime existence for closed hypersurfaces

In this section we address to Theorem 1.1. It is known that the initial value problem (1.4), $\varphi|_{t=0} = \varphi_0$, admits a smooth solution for a maximal time interval $[0, T)$. We remark that we get a similar result for $\varphi_0 \in C^{2,\alpha}(S^n)$, $\alpha > 0$, with less regularity at $t = 0$. To prove smooth longtime existence, it suffices to prove that the (spatial) C^2-norm of a smooth solution in a given time interval $[0, t]$ is bounded above by $h(t)$ for any $t > 0$, where $h : \mathbb{R} \to \mathbb{R}$ is a locally bounded function. As we get that the argument of Φ is bounded, we see that our equation is uniformly parabolic. Thus we can apply Corollary 14.9 in [3] and get $C^{2,\alpha}$-estimates for some $\alpha > 0$. Higher regularity follows from Schauder theory. Then it is possible to extend a solution to $[0, \infty)$ due to shorttime existence.

More precisely, we will prove uniform estimates for $\dot{\varphi}$, uniform oscillation estimates for φ and uniform estimates for $D\varphi$ and $D^2\varphi$. Due to the φ-invariance of our problem these estimates imply uniform estimates for all derivatives of φ.

We will use the Einstein summation convention and lift indices with respect to the induced metric on S^n.

We first bound the time derivative of φ.

Lemma 2.1. *Let φ be a smooth solution of our initial value problem. Then we have the estimate*

$$\min\left\{\min_{t=0}\dot\varphi, 0\right\} \le \dot\varphi \le \max\left\{\max_{t=0}\dot\varphi, 0\right\}.$$

Proof. We rewrite the flow equation using

$$\tilde f(x, \nabla\varphi) = \log\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right) - \log g(T(x)) + \log f(x)$$

and

$$w_{ij} = \varphi_{ij} + \varphi_i\varphi_j + \frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij} \equiv \varphi_{ij} + r_{ij}$$

here and in the following. We get

$$\dot\varphi = \Phi\left(\log\det w_{ij} - \tilde f(x, \nabla\varphi)\right). \tag{2.1}$$

For $E := (\dot\varphi)^2$ we obtain the evolution equation

$$\dot E = \Phi' w^{ij} E_{ij} - 2\Phi' w^{ij}\dot\varphi_i\dot\varphi_j + \Phi' w^{ij} r_{ijp_s} E_s - \Phi' \tilde f_{p_i} E_i,$$

where the index p_i indicates derivatives with respect to $\nabla\varphi$ and $\left(w^{ij}\right)$ denotes the inverse of (w_{ij}). The inverse w^{ij} is the only exception to our convention to lift indices with respect to the induced metric on S^n. As $-w^{ij}\dot\varphi_i\dot\varphi_j \le 0$, the maximum principle gives the claimed inequality. More precisely, we see that for some time interval (w^{ij}) remains positive definite. During this time interval we get the claimed inequality. Thus the argument of Φ remains uniformly bounded. From the definition of $\tilde f$, we see that $\tilde f$ is uniformly bounded from below and deduce a uniform lower bound for $\log\det w_{ij}$, thus (w_{ij}) remains positive definite. □

Integrating this estimate we obtain a very rough C^0-estimate

$$|\varphi(x,t)| \le \max|\varphi(x,0)| + t\cdot\max|\dot\varphi(x,0)|.$$

We need a better estimate, that prevents different parts of the hypersurfaces from moving “far apart” from each other. This is contained in the following oscillation estimate

Lemma 2.2. *Let φ be a smooth solution of our initial value problem. Then its oscillation is uniformly bounded during the flow.*

Proof. We rewrite our flow equation as

$$\frac{\det\left(\varphi_{ij} + \varphi_i\varphi_j + \frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}\right)}{\det\left(\frac{1}{2}\left(1+|\nabla\varphi|^2\right)\sigma_{ij}\right)} = \frac{f(x)\cdot e^{\Phi^{-1}(\dot\varphi)}}{g(T(x))}. \tag{2.2}$$

For a fixed time t we consider $\Phi^{-1}(\dot\varphi)$ as a bounded function. Thus we can apply the C^0-estimates of Section 2.1 in [2] and get exactly the claimed oscillation estimate. The C^0-estimate in the cited paper is obtained for normalized surfaces, i. e. the surfaces are rescaled so that the distance of the surface to the origin is equal to 1. Thus these C^0-estimates correspond to oscillation estimates in our setting. □

The following lemma gives C^1-a priori estimates.

Lemma 2.3 (C^1-estimates)**.** *For any function $\varphi \in C^2(S^n)$ with positive definite (w_{ij}) (see the definition in the proof of Lemma 2.1) and bounded oscillation, $|\nabla\varphi|$ is uniformly bounded.*

Proof. The quantity

$$\frac{1}{2}\log|\nabla\varphi|^2+\varphi$$

attains its maximum somewhere on S^n. So we deduce there (we multiply the covariant derivative of the quantity above with φ^i)

$$0=\frac{\varphi^i\varphi_{ij}\varphi^j}{|\nabla\varphi|^2}+|\nabla\varphi|^2.$$

As (w_{ij}) is positive definite, we get in the sense of matrices

$$\varphi_{ij}\geq-\varphi_i\varphi_j-\frac{1}{2}\left(1-|\nabla\varphi|^2\right)\sigma_{ij}$$

and deduce at the maximum point

$$1\geq|\nabla\varphi|^2.$$

Since the oscillation of φ is bounded, we get a uniform bound for $|\nabla\varphi|$ everywhere on S^n. □

Before we estimate the second covariant derivatives of φ we recall formulae for interchanging the order of covariant differentiation for functions on S^n

$$\begin{aligned}\varphi_{ijk}&=\varphi_{kij}+\varphi_j\sigma_{ik}-\varphi_k\sigma_{ij},\\ \varphi_{ijkl}&=\varphi_{klij}+2\varphi_{ij}\sigma_{kl}-2\varphi_{kl}\sigma_{ij}+\varphi_{kj}\sigma_{il}-\varphi_{il}\sigma_{kj}.\end{aligned}$$

Lemma 2.4 (C^2-estimates)**.** *The second covariant derivatives of φ are uniformly bounded during the flow.*

Proof. We use the maximum principle for w_{ij} and compute its evolution equation. We will rewrite

$$\dot{w}_{ij}-\Phi' w^{kl}w_{ijkl}$$

using terms we are able to control. The last two indices of w_{ijkl} denote covariant derivatives on S^n. We use the definition

$$w_{ij}=\varphi_{ij}+r_{ij},$$

differentiate this equation and use it to substitute $\dot{w}_{ij}$ and w_{ijkl}. Next, we differentiate the flow equation (2.1) twice in spatial directions and replace $\dot{\varphi}_{ij}$ using this equation. We rewrite w_{ijkl} in terms of derivatives of φ and

r_{ij} and interchange derivatives of φ_{ijkl}. So the terms containing fourth derivatives of φ drop out.

$$\begin{aligned}\dot w_{ij}-\Phi' w^{kl}w_{ijkl} &= \Phi' w^{kl}(2\varphi_{kl}\sigma_{ij}-2\varphi_{ij}\sigma_{kl}+\varphi_{il}\sigma_{kj}-\varphi_{kj}\sigma_{il})\\ &\quad+r_{ijp_r}\dot\varphi_r-\Phi' w^{kl}r_{ijp_r}\varphi_{rkl}+\Phi' w^{kl}r_{klp_r}\varphi_{rij}\\ &\quad-\Phi' w^{kl}r_{ijp_rp_s}\varphi_{rk}\varphi_{sl}+\Phi' w^{kl}r_{klp_rp_s}\varphi_{ri}\varphi_{sj}\\ &\quad-\Phi' w^{kr}w^{ls}w_{kli}w_{rsj}-\Phi' D_jD_i\tilde f\\ &\quad+\Phi''\left(w^{ab}w_{abi}-D_i\tilde f\right)\left(w^{cd}w_{cdj}-D_j\tilde f\right).\end{aligned}$$

The notation $D.$ indicates that the chain rule has not yet been applied to the respective terms. We interchange both third derivatives of φ and get terms involving w_{ijr} and φ_{klr}. This last term and $\dot\varphi_r$ can be simplified using the differentiated flow equation and the definition of w_{klr}. Two terms with third derivatives of φ cancel. The quantity r_{kl} depends on $(x,\nabla\varphi)$, but its covariant derivatives with respect to the x variable vanish. So we get the evolution equation

$$\begin{aligned}\dot w_{ij}-\Phi' w^{kl}w_{ijkl} &= \Phi' w^{kl}(2\varphi_{kl}\sigma_{ij}-2\varphi_{ij}\sigma_{kl}+\varphi_{il}\sigma_{kj}-\varphi_{kj}\sigma_{il})\\ &\quad-\Phi' r_{ijp_r}D_r\tilde f+\Phi' w^{kl}r_{ijp_r}(r_{klp_s}\varphi_{sr}-\varphi_r\sigma_{kl}+\varphi_l\sigma_{kr})\\ &\quad-\Phi' w^{kl}r_{ijp_rp_s}\varphi_{rk}\varphi_{sl}+\Phi' w^{kl}r_{klp_rp_s}\varphi_{ri}\varphi_{sj}\\ &\quad+\Phi' w^{kl}r_{klp_r}(w_{ijr}-r_{ijp_s}\varphi_{sr}+\varphi_r\sigma_{ij}-\varphi_j\sigma_{ir})\\ &\quad-\Phi' w^{kr}w^{ls}w_{kli}w_{rsj}-\Phi' D_jD_i\tilde f\\ &\quad+\Phi''\left(w^{ab}w_{abi}-D_i\tilde f\right)\left(w^{cd}w_{cdj}-D_j\tilde f\right).\end{aligned}\tag{2.3}$$

Directly from the definitions of r_{kl} and w_{kl} we get

$$\begin{aligned}&-w^{kl}r_{ijp_rp_s}\varphi_{rk}\varphi_{sl}+w^{kl}r_{klp_rp_s}\varphi_{ri}\varphi_{sj}\\ &\quad= w^{kl}\varphi_{lr}\sigma^{rs}\varphi_{sk}\cdot\sigma_{ij}-w^{kl}\sigma_{kl}\cdot w_{ir}\sigma^{rs}w_{sj}\\ &\qquad+w^{kl}\sigma_{kl}\cdot(w_{ir}\sigma^{rs}r_{sj}+w_{jr}\sigma^{rs}r_{si})-w^{kl}\sigma_{kl}\cdot r_{ir}\sigma^{rs}r_{sj}.\end{aligned}$$

The term $-w^{kl}\sigma_{kl}\cdot w_{ir}\sigma^{rs}w_{sj}$ will be very useful for further estimates. We remark that the right-hand side of (2.3) is a tensor with indices i and j and this is also true for both terms on the left-hand side. We multiply the evolution equation with $\xi^i\xi^j$ to be fixed later-on. We will always assume that $\sigma_{ij}\xi^i\xi^j\le c$ and use the notation $w_{11}\equiv w_{ij}\xi^i\xi^j$ with obvious generalizations, but keep in mind that we have contracted the indices, so w_{11} is a scalar function. Due to our a priori estimates, $r_{ij}=w_{ij}-\varphi_{ij}$ is bounded. We use $\operatorname{tr} w^{kl}=w^{kl}\sigma_{kl}$ and get

$$\left|w^{kl}\varphi_{lj}\right|=\left|w^{kl}w_{lj}-w^{kl}(w_{lj}-\varphi_{lj})\right|\le 1+\left|w^{kl}r_{lj}\right|\le c\cdot\left(1+\operatorname{tr} w^{kl}\right).$$

We use the concavity of Φ and obtain the following evolution inequality by interchanging third derivatives in the term containing $\tilde{f}$

$$\begin{aligned}\dot{w}_{11}-\Phi' w^{kl}w_{11kl} \quad\le\quad & \Phi' w^{kl}r_{klp_r}w_{11r}-\Phi' w^{kr}w^{ls}w_{kl1}w_{rs1}\\ & -\Phi' w^{kl}\sigma_{kl}\cdot w_{1r}\sigma^{rs}w_{s1}-\Phi'\tilde{f}_{p_k}w_{11k}\\ & +c\cdot\left(1+\operatorname{tr}w^{kl}+\left|D^2\varphi\right|\cdot\operatorname{tr}w^{kl}+\left|D^2\varphi\right|^2\right).\end{aligned}$$

We also used that Φ' is bounded. Now we consider the function

$$(x,t,\xi)\mapsto w_{ij}\xi^i\xi^j$$

for $x\in S^n$, $t\ge 0$ and $\sigma_{ij}\xi^i\xi^j(x,t)=1$. We assume that restricted to a compact set $(x,t)\in S^n\times[0,T]$, where the flow exists and ξ is as above, this function attains its maximum in (x_0,t_0,ξ_0) with $t_0>0$. According to the parabolic maximum principle we get there

$$\Phi' w_{1r}\sigma^{rs}w_{s1}\cdot\operatorname{tr}w^{kl}\le c\cdot\left(1+\operatorname{tr}w^{kl}+\left|D^2\varphi\right|\cdot\operatorname{tr}w^{kl}+\left|D^2\varphi\right|^2\right).$$

Due to our C^1-estimates for φ, w_{kl} and φ_{kl} coincide up to estimates terms. Φ' is bounded from below by a positive constant. Moreover, as $\log\det w_{kl}$ is bounded, we see that $w_{11}\to\infty$ forces $\operatorname{tr}w^{kl}\to\infty$. We deduce that w_{11} is bounded there and get a time-independent bound for $\left|D^2\varphi\right|$ as long as a smooth solution of (1.4) exists. □

3. Convergence for closed hypersurfaces

Here we complete the proof of Theorem 1.1. The method used in [4] to obtain a translating solution also applies to the case of closed hypersurfaces. Indeed, the proof is a bit simpler in the closed case. For convenience of the reader, we sketch the argument given there. Part of the argument is due to Huisken [1].

For $t_0>0$ fixed we consider

$$w(x,t):=\varphi(x,t+t_0)-\varphi(x,t).$$

Using the mean value theorem we see that w satisfies a parabolic flow equation of the form

$$\dot{w}=a^{ij}w_{ij}+b^iw_i. \tag{3.1}$$

The strong maximum principle shows that the oscillation of w is strictly decreasing during the flow or w is constant. We wish to show that the oscillation does not tend to $\varepsilon>0$. Otherwise we consider for $x_0\in S^n$ fixed and $t_n\to\infty$

$$\varphi(x,t+t_n)-\varphi(x_0,t_n)\quad\text{and}\quad\varphi(x,t+t_0+t_n)-\varphi(x_0,t_0+t_n). \tag{3.2}$$

Due to our a priori estimates we can find a subsequence such that the expressions above converge locally uniformly (in time) in any C^k-norm to a solution of our flow equation for all time. It is easy to see that the difference of the limits solves a parabolic equation similar to (3.1) and has constant oscillation $\varepsilon > 0$. This is excluded by the strong maximum principle. As the oscillation of w tends to zero and w satisfies a parabolic equation of the form (3.1) we see that w tends to some constant as $t \to \infty$. Considering sequences similar to (3.2) we obtain a solution φ^* for all time. One checks that

$$\varphi^*(x, t + t_0) - \varphi^*(x, t) = const.. \tag{3.3}$$

Next, we take an appropriate number, e. g. $t_0 \cdot \sqrt{2}$, instead of t_0 and start with the solution φ^* obtained. Our procedure gives a solution that satisfies (3.3) (with a different constant) also for $t_0 \cdot \sqrt{2}$ instead of t_0, i. e. we obtain a translating solution. Now we compare our original solution with the translating solution and get as above that the oscillation of the difference tends to zero. Smooth convergence to a translating solution is then obtained by using interpolation inequalities.

Thus φ converges smoothly to a translating solution φ^∞ of (1.4) as $t \to \infty$. To check that the velocity v^∞ is as claimed in (1.5) we use the flow equation (1.4) in the form (2.2) for the translating solution φ^∞. We consider this equation as an elliptic equation and obtain from the conservation of energy and the deviation of the ellitic reflector equation, see the appendix in [2],

$$\int\limits_{S^n} e^{\Phi^{-1}(v^\infty)} f = \int\limits_{S^n} g$$

and obtain (1.5) as v^∞ is a constant. This completes the proof of Theorem 1.1.

We wish to remark, that we can enclose our initial function φ from above and from below by the translating solutions obtained. Due to the maximum principle, these translating solutions act as barriers and show that our solutions stay at a finite distance to a translating solution.

4. Illuminating prescribed domains

We start with a deviation of the equation fulfilled by solutions. Therefore we follow light that moves upwards from $(x, 0) \in \mathbb{R}^n \times \mathbb{R}$, $x \in \Omega$, in direction $(0, 1)$. The reflector is described as graph $u|_\Omega$, a unit normal to this hypersurface is given by

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}.$$

The direction of the reflected light is obtained as a function of x as follows

$$x \mapsto (0, 1) - 2\langle (0, 1), \nu\rangle \nu = \frac{\left(2Du, -1 + |Du|^2\right)}{1 + |Du|^2}.$$

Due to our hypotheses that in the simplified model the reflected rays of light start at $(0,1)$, we see that this ray of light meets the “ground”, i. e. the hyperplane $\mathbb{R}^n\times\{0\}$, at $\frac{2Du}{1-|Du|^2}$. Thus we get a map $T:\Omega\to\Omega^*$ such that light from x is reflected to $T(x)$, in a formula

$$T(x)=\frac{2Du}{1-|Du|^2}.$$

It is easy to see, that T is a diffeomorphism onto its image for a smooth strictly convex function u with $|Du|<1$; we will assume this in the following.

Next we derive the equation to be fulfilled by u. We assume that u is a solution to our reflector problem. From the conservation of energy and the transformation formula for integrals we get for open domains $E\subset\Omega$

$$\int\limits_{T(E)} g(y)dy=\int\limits_{E} f(x)dx=\int\limits_{T(E)} f(x)\cdot\frac{1}{\det T_{ij}}dy,$$

where T_{ij} denotes the derivative of the i-th component of T in direction j and $y=T(x)$. Thus we obtain the elliptic equation for the reflecting hypersurface

$$\det T_{ij}=\frac{f(x)}{g(T(x))}.$$

More explicitly, we use the Einstein summation convention and get

$$T_{ij}=\frac{\partial T_i}{\partial x^j}=\frac{2}{\left(1+|Du|^2\right)^2}\left(u_{ij}\left(1-|Du|^2\right)+2u_iu_l\delta^{lk}u_{kj}\right).$$

For the evaluation of the determinant of T_{ij} we may assume without loss of generality that we have chosen coordinates such that $\langle Du,e_1\rangle=|Du|$. (T_{ij}) is then given by

$$\frac{2}{\left(1-|Du|^2\right)^2}\begin{pmatrix} u_{11}\left(1+|Du|^2\right) & u_{12}\left(1-|Du|^2\right) & \cdots & u_{1n}\left(1-|Du|^2\right)\\ u_{12}\left(1+|Du|^2\right) & u_{22}\left(1-|Du|^2\right) & \cdots & u_{2n}\left(1-|Du|^2\right)\\ \vdots & \vdots & \ddots & \vdots\\ u_{1n}\left(1+|Du|^2\right) & u_{2n}\left(1-|Du|^2\right) & \cdots & u_{nn}\left(1-|Du|^2\right)\end{pmatrix},$$

so we see immediately that

$$\det T_{ij}=2^n\cdot\left(1-|Du|^2\right)^{-n-1}\cdot\left(1+|Du|^2\right)\cdot\det D^2u$$

and the reflector equation

$$\det D^2u=\frac{f(x)}{g(T(x))}\cdot 2^{-n}\cdot\frac{\left(1-|Du|^2\right)^{n+1}}{1+|Du|^2}$$

follows. In our approach we consider the flow equation

$$\dot u=\Phi\left(\log\det D^2u-\log\left(\frac{f(x)}{g(T(x))}\cdot 2^{-n}\cdot\frac{\left(1-|Du|^2\right)^{n+1}}{1+|Du|^2}\right)\right)\tag{4.1}$$

with Φ as in (1.4). The inverse map to

$$Du \mapsto \frac{2Du}{1-|Du|^2}$$

is given by

$$\tau : y \mapsto \frac{y}{|y|^2}\left(\sqrt{1+|y|^2}-1\right). \tag{4.2}$$

From the Taylor expansion of the square root at $y=0$ we see that τ extends smoothly to $y=0$. The map τ is a diffeomorphism onto its image, so we can rewrite the boundary condition $T(\Omega)=\Omega^*$ as $Du(\Omega)=\tau(\Omega^*)$. Directly from the estimates in [5] and the appendix in [4] we obtain

Theorem 4.1. *Let Ω, $\Omega^* \subset \mathbb{R}^n$ be smooth bounded domains such that Ω and $\tau(\Omega^*)$ are strictly convex domains where τ is the diffeomorphism introduced in (4.2). Let $u_0 : \overline{\Omega} \to \mathbb{R}$ be a smooth strictly convex function such that $Du_0(\Omega)=\tau(\Omega^*)$. Then there exists a smooth solution $u : \overline{\Omega}\times(0,\infty)$ to Equation (4.1) – with $u(\cdot,t)\to u_0$ in $C^2\left(\overline{\Omega}\right)$ as $t\downarrow 0$ – such that $Du(\Omega)=\tau(\Omega^*)$ or equivalently $T(\Omega)=\Omega^*$ (T is evaluated using $u(\cdot,t)$). $u(\cdot,t)$ converges in the $C^\infty\left(\overline{\Omega}\right)$ topology to a translating solution of (4.1) that moves with speed*

$$\Phi\left(\log\int\limits_{\Omega^*} g - \log\int\limits_{\Omega} f\right).$$

Proof. The existence and convergence to a translating solution follows from the appendix in [4] where we use essentially estimates from [5]. Using the conservation of energy and the transformation formula for integrals as in the deviation of the reflector equation above, we obtain for a translating solution with velocity v^∞

$$\int\limits_{\Omega^*} g(y)dy = \int\limits_{\Omega} e^{\Phi^{-1}(v^\infty)} f(x)dx.$$

Thus we get the formula for v^∞. $\square$

We remark that the maximum principle shows the uniqueness of translating solutions up to additive constants. Again our solution becomes stationary provided the total amount of energy emitted and prescribed on the ground coincide.

At a first glance, the convexity condition for $\tau(\Omega^*)$ seems artificially. As it turns out, however, that our problem corresponds to a second boundary value problem for a Monge-Ampère equation, which can be solved – at least at the moment – in general only for strictly convex domains, we see that our condition for Ω^* is indeed natural. We show in Lemma 4.2 that the convexity condition for $\tau(\Omega^*)$ is fulfilled for a large class of domains.

It remains to prove the assertion of the introduction that this illumination problem can be solved for strictly convex domains Ω^* that contain the origin, i. e. it suffices to prove

Lemma 4.2. *Let $\Omega^* \subset \mathbb{R}^n$ be a convex open set, $0 \in \Omega^*$. For τ as in (4.2), $\tau(\Omega^*)$ is strictly convex.*

Proof. As τ maps each point $x \in \mathbb{R}^n$ to a point $\lambda \cdot x$ where $\lambda = \lambda(|x|)$, we see that it suffices to prove this lemma for $\Omega^* \subset \mathbb{R}^2$. Moreover, as $|x| \mapsto |\tau(x)|$ is a strictly monotone increasing function, we have only to check that τ maps half-planes containing the origin to strictly convex sets. Due to the rotational symmetry it suffices to show that horizontal lines lying "above" the origin are mapped to graphs over part of the horizontal axis, graph u, such that u is a strictly concave positive function. More precisely, we fix $a > 0$ and consider the horizontal line in $\mathbb{R}^2$ parameterized by $\mathbb{R} \ni t \mapsto (t, a)$. The diffeomorphism τ maps this line to

$$t \mapsto (t, a) \cdot \frac{\sqrt{1 + a^2 + t^2} - 1}{a^2 + t^2} \equiv (t, a) \cdot g(t) \equiv (x(t), y(t)).$$

Direct calculations show that

$$\frac{\partial x}{\partial t} = \frac{\left(t^2 - a^2\right) \cdot \left(\sqrt{1 + a^2 + t^2} - 1\right) + a^2 \cdot \left(a^2 + t^2\right)}{\left(a^2 + t^2\right)^2 \sqrt{1 + a^2 + t^2}} > 0.$$

Thus we can use x to parameterize the image. We use the chain rule and obtain

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial t^2} \left(\frac{\partial t}{\partial x}\right)^2 + \frac{\partial y}{\partial t} \frac{\partial^2 t}{\partial x^2} \\ &= \left(\frac{\partial^2 y}{\partial t^2} - \frac{\partial y}{\partial t} \frac{\partial^2 x}{\partial t^2} \frac{\partial t}{\partial x}\right) \cdot \left(\frac{\partial t}{\partial x}\right)^2. \end{aligned}$$

Thus it suffices to show that

$$2 \left(g'(t)\right)^2 > g(t) \cdot g''(t). \tag{4.3}$$

To avoid long calculations we note that

$$g(t) = \frac{1}{1 + f}, \quad \text{where } f(t) := \sqrt{1 + a^2 + t^2}, \quad \text{so } f' = \frac{t}{f}.$$

Now it is easy to obtain (4.3) by direct calculation. Thus our lemma follows. □

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