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Busemann spaces of Aleksandrov curvature bounded above

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BUSEMANN SPACES OF ALEKSANDROV CURVATURE BOUNDED ABOVE

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Abstract. In this paper we prove the following main result:every Busemann G-space with Aleksandrov curvature locally bounded from above is Riemannian C^0-manifold (with C^1-atlas in which the components of metric tensor are continuous). Previously we find a necessary and sufficient conditions for isometricity of a metric space to (finite- or infinitedimensional) Euclidean space or unit sphere in Euclidean space. Also we prove that for locally compact geodesically complete inner metric space M of Aleksandrov curvature locally bounded from above, the tangent space M_x, defined as O-cone over space of directions to M at any point $x \in M$, is isometric to Gromov tangent cone T_xM, defined as Gromov-Hausdorff limit of scaled space M with the base point x.

1. Introduction and main results.

In this paper we prove the part of results on A.D.Aleksandrov spaces of the curvature $\leq K$ (see [1],[2],[3], [4]) announced earlier in the paper [5]. Let us give necessary definitions and notations.

The distance between two points x, y of a metric space M is denoted by xy. *(Locally) inner* (or *(locally) interior* or *(locally) length*) *metric space* is a metric space in which (locally) any two points x, y can be joined by a path with the length arbitrary close to xy. A path joining a points x, y in M is called *shortest arc* or *segment* (with the ends x, y) if its length is equal to xy; the notation is $[xy]$. *(Locally) geodesic space* is a metric space in which (locally) any two points can be joined by shortest arc. By Cohn-Vossen theorem [10], every complete locally compact inner metric space is a geodesic space. Also we will consider in this paper a metric spaces without local compactness (or even separability) condition.

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A point y lies between a points x, z if $xz = xy + yz$ and $y \neq x, y \neq z$; the notation is (xyz).

An open (closed) ball in M of the radius r and the center x is denoted by $U(x,r)$ (respectively $B(x,r)$); the corresponding sphere is denoted by $S(x,r)$. By S_K we denote the simply connected two-dimensional Riemannian manifold of constant sectional curvature K. For any ordered triple of points (x,y,z) in M, we will denote by $\gamma_k(xyz)$ the angle in the triangle Δ^K in S_K with the sides of the lengths xy, xz and yz, lying opposite to the side with the length xz.

For a point p of a metric space M, we will denote by $\Omega_p(M)$ the space of all directions to M at point p and by $\omega_p(M)$ the subspace of directions to M at the point p, defined by shortest arcs starting at p (see[1], [2] or [4]). The distance between two directions is defined to be the upper angle α between corresponding curves (respectively, shortest arcs, see [1] or [2]). The corresponding 0-cones $M_p := C_0\Omega_p(M)$ and $m_p := C_0\omega_p(M)$ (see [2]) as well as their Hausdorff completions we will call *the tangent spaces to M at point p*.

On the ground of I.G.Nikolaev theorem [13] (or theorem 10.1 in [2]), some additional propositions about (completed) tangent and direction spaces, in particular, the isometricity M_p and Gromov tangent cone T_pM (see [12]) and checking of the inequality (4) in [6], we prove the following generalization of the main result in [6].

Theorem 1.1. *Every Busemann G-space* [9] *with Aleksandrov curvature, locally bounded from above, is a Riemannian C^0-manifold (with C^1-atlas of distance coordinate charts in which the components of metric tensor are continuous).*

Remark 1.2. The author proved in [8] with the help of the paper [14] that for Busemann G-space with curvature bounded below, the components of metric tensor are $C^{1/2}$-functions in distance coordinates. It's unknown for this moment whether this statements is true also under conditions of theorem 1.1 or whether this statement can be even improved in some other coordinates.

2. About (completed) tangent and directions spaces.

The main goal of this section are simple necessary and sufficient conditions for isometricity of completed tangent (directions) space to Aleksandrov space of curvature bounded above at its point to (not necessarily separable) Hilbert (Euclidean) space (or unit sphere in Hilbert (Euclidean) space). For this we prove the following general theorem.

Theorem 2.1. *A metric space W is isometric to unit sphere in an Euclidean or (complete not necessarily separable) Hilbert space H if and only if W possess the following properties:*

1. *W is a $CAT(1)$-space;*
2. *W is complete;*
3. *for any two points $x, y \in W, xy \leq \pi$;*
4. *for any point $x \in W$ there is unique (opposite) point x' such that $xx' = \pi$;*
5. *for any two points $x, y \in W$ with a distance $xy < \pi$ there is a point z such that (xyz).*

Moreover, C_0W is isometric to H if and only if W possess the above properties.

We need at first some lemmas.

Lemma 2.2. *Under conditions (1)-(5) of the theorem 2.1, for any two points $x, y \in W$ with a distance $xy < \pi$, we have (xyx').*

Proof. We will use without any references the conditions (1)-(5) of theorem 2.1. Evidently, the points x, y are joined in W by unique shortest arc $[xy]$. There is a point z such that (xyz) and for every such point $z, xz \leq \pi$; the shortest arc $[yz]$ is also unique and $[xy] \cup [yz] = [xz]$. If $xz < \pi$, we can repeat this process. Since W is complete, every increasing sequence of shortest arcs $[xz_n]$ in W is contained in shortest arc $[xz_\infty]$, where z_∞ is the limit point of z_n and $[xz_\infty]$ is the union of all $[xz_n]$ together with the point z_∞. Using these considerations, we can assume the existence of a maximal (relative to inclusion) shortest arc $[xw]$. Then we must have $xw = \pi$ and $w = x'$, which finishes proof of the lemma. □

Lemma 2.3. *Under conditions (1)-(5) of the theorem 2.1, for every point $p \in W$, the space W is isometric to $C_1(\omega_p W)$ and $\omega_p W$ is isometric to the sphere $S(p, \frac{\pi}{2})$ of radius $\frac{\pi}{2}$ with the center p. Furthemore this sphere $S(p, \frac{\pi}{2})$ is π-convex in the sense that every shortest arc joining a points $x, y \in S(p, \pi/2)$ with a distance $xy < \pi$ is entirely contained in $S(p, \frac{\pi}{2})$.*

Proof. Let p be arbitrary point in W and $pp' = \pi$. By lemma 2.2, $px + xp' = \pi$ for every point $x \in W$. Let $0 < px_i < \pi; i = 1, 2$ for some points $x_i \in W$ and k_i be (unique) shortest arc joining p and p' and passing through x_i. Then k_1, k_2 constitute a digon with vertices p, p' and angles γ, γ' at p, p' respectively. Let suppose at first that both angles γ, γ' are less than π. We get the inequality $\gamma \leq \gamma'$ if consider k_1, k_2 as shortest arcs with origin p and use 1-concavity condition (see

[3]). We get the inequality $\gamma' \leq \gamma$ by interchanging the points p, p', so $\gamma' = \gamma$. Let suppose now that $\gamma = \pi$ and consider k_1, k_2 as shortest arcs with origin p, parametrized by arclength. Then $k_1(\frac{\pi}{2})k_2(\frac{\pi}{2}) = \pi$ by 1-concavity and $(k_1(\frac{\pi}{2})p'k_2(\frac{\pi}{2}))$ by lemma 2.2. Hence $\gamma' = \pi$ also. So in all cases we proved that $\gamma' = \gamma$. It follows from this also that the digon, considered above, is isometric to corresponding digon in S_1 with angles γ. Then W is isometric to 1-cone $C_1(\omega_p W)$ over $\omega_p W$ (see [3]). Hence $\omega_p W$ is isometric to the sphere $S(p, \frac{\pi}{2})$, where a direction $l \in \omega_p W$, defined by a shortest arc k with origin p, corresponds to the point $k(\frac{\pi}{2}) \in S(p, \frac{\pi}{2})$. We proved the first statement of lemma. Now the second statement follows from general fact that every geodesic space M is naturally isometric to the π-convex subspace in $C_1 M$ (see [7]) and from the isometric correspondence between $\omega_p W$ and $S(p, \frac{\pi}{2})$ indicated above. □

Proof. of theorem 2.1. Evidently, the conditions $(1) - (5)$ are necessary.We will prove now that these conditions are also sufficient.

It's possible that W contains only two elements: x and x'. Then W is isometric to 0-dimensional sphere of radius 1 while $C_1 W$ is isometric to 1-dimensional Euclidean space. So we can assume that W has more than two elements.

If $xy < \pi$ for $x, y \in W$ then by lemma 2.2 $[xy] \cup [yx'] = [xx']$, $[xy'] \cup [y'x'] = [xx']$, $[yx] \cup [xy'] = [yy']$ and $[yx'] \cup [x'y'] = [yy']$. This means that the union $S_{x,y} := [xy] \cup [yx'] \cup [x'y'] \cup [y'x]$ is isometric to 1-dimensional sphere and $C_1 S_{x,y}$ is isometric to 2-dimensional Euclidean space. Hence the proof is finished if $W = S_{x,y}$. In the opposite case $S_{x,y}$ is π- convex subset in W (uniquely defined by x, y) and $C_1 S_{x,y}$ is convex subset in $C_1 W$. Here the π-convexity of $S_{x,y}$ means that any two points in $S_{x,y}$ with distance less than π, can be joined by (unique) shortest arc, lying in $S_{x,y}$. Now the statements above garantee that one can define correctly and uniquely the sum of any two elements in $C_1 W$ by parallelogramm rule. We would prove that $C_0 W$ is Euclidean or Hilbert vector space and hence that W is unit sphere in this space if we will prove that any three noncomplanar elements x, y, z in W are contained in unique $\pi-$convex subset $S_{x,y,z}$ (in W) which is isometric to S_1. The word "noncomplanar" means that no point of x, y, z lies between other two and all these points are pairwise different.

Now, if x, y, z are any noncomplanar points in W, we denote by k_y (respectively k_z) the unique shortest arc parametrized by arclength which contains the points x, y, x' (respectively, x, z, x'). Then the set $S := S_{k_y(\frac{\pi}{2}), k_z(\frac{\pi}{2})}$ is contained in $S(x, \frac{\pi}{2})$ by lemma 2.3. Now it follows from lemma 2.3 that we can define the above set $S_{x,y,z}$ as the union of all shortest arcs $[xw] \cup [wx']$, where $w \in S$. $\square$

We get immediately from theorems in [13] and 2.1 above the following

Corollary 2.4. *The completed tangent space $\overline{M}_p$ at a point p of a space M of curvature $\leq K$ is isometric to Euclidean space or (complete not necessarily separable) Hilbert space H if and only if $W := \overline{\omega_p M}$ possess the properties (4), (5) from theorem 2.1. Under this W is isometric to unit sphere in H.*

Proposition 2.5. *The conditions of corollary 2.4 are satisfied if for every element $l \in \omega_p M$ there is an element $l' \in \omega_p M$ such that $\alpha(l, l') = \pi$ and*

$$\alpha(l, l') + \alpha(l', n) + \alpha(n, l) = 2\pi \tag{2.1}$$

for any element $n \in \omega_p M$. Here α denotes the angle between two directions.

Proof. It follows from equality (2.1) that if also $\alpha(l, l'') = \pi$, then $\alpha(l', l'') = 0$ and l', l'' define one and the same element of $\omega_p M$.

Clearly the equality (2.1) is satisfied also for any $n \in \overline{\omega_p M}$. We state also that all conditions of the proposition are satisfied in $\overline{\omega_p M}$. Indeed, let suppose that $l_k \in \omega_p M$ and $l_k \to l \in \overline{\omega_p M}$. By condition there is (unique) element $l'_k \in \omega_p M$ such that $\alpha(l_k, l'_k) = \pi$. It follows from (2.1) and triangle inequality for α that

$$\pi = \alpha(l_k, l'_k) = \alpha(l_k, l_s) + \alpha(l_s, l'_k),$$

$$\pi = \alpha(l_s, l'_s) = \alpha(l_s, l'_k) + \alpha(l'_k, l'_s).$$

Hence $\alpha(l'_k, l'_s) = \alpha(l_k, l_s)$ and the sequence l'_k is fundamental. Then $l'_k \to l' \in \overline{\omega_p M}$ and by continuity

$$\pi = \alpha(l_k, l'_k) \to \alpha(l, l'),$$

what is required.

As earlier for $\omega_p M$, it follows now that the condition (4) from theorem 2.1 is satisfied for $W := \overline{\omega_p M}$. Since the conditions of the proposition are satisfied for W, the condition (5) from theorem 2.1 is satisfied for $x = l, y = m, z = l'$. $\square$

The condition (4) of theorem 2.1 for $W := \omega_p M$, where M is Aleksandrov space with curvature bounded above, splits into two parts which follow from geometric properties of shortest arcs in M stated in the next two propositions.

Proposition 2.6. *A (locally geodesic) space M of curvature locally bounded from above has the property of local extendability of shortest arcs (see* [3]*) if and only if for every point $p \in M$ and every element $l \in \omega_p M$ there is an element $l' \in \omega_p M$ such that $\alpha(l', l) = \pi$.*

Proof. Necessity is evident.Let us prove sufficiency. Let suppose that K-concavity condition is satisfied in a neighborhood $U(q)$ and l is a shortest arc in $U(q)$, joining a different points q_1, q_2. By condition, there is a shortest arc l' in $U(q)$ with origin q_1 such that $\alpha(l', l) = \pi$. Then $\gamma_K(xq_1x')$, where $x \in l, x' \in l'$, is nondecreasing function on q_1x, q_1x' because of K-concavity condition. Hence $l' \cup l$ is shortest arc. Similarly, the shortest arc $l' \cup l$ can be extended to a shortest arc beyond the point q_2. □

Corollary 2.7. *A locally geodesic space M of curvature locally bounded from above has the property of local extendability of shortest arcs if and only if it is geodesically complete.*

Proposition 2.8. *A (locally geodesic) space M of curvature locally bounded from above has no bifurcating shortest arcs (see* [3]*) only if for every point $p \in M$ and every element $l \in \omega_p M$ there is at most one element $l' \in \omega_p M$ such that $\alpha(l', l) = \pi$.*

Proof. Let suppose that

$$\alpha(l, l') = \pi = \alpha(l, l'')$$

for a shortest arcs $l, l', l'' \in \omega_p M$. Then $l' \cup l, l'' \cup l$ are shortest arcs (near the point p) by proposition 2.6. Then $l' = l''$ near the point p because M has no bifurcating shortest arcs. This means that $l' = l'' \in \omega_p M$. □

The following proposition is evident.

Proposition 2.9. *The completed tangent space $\overline{M_p}$ to a (locally geodesic) space M of curvature locally bounded from above at a point $p \in M$ has the global property of extendability of shortest arcs if and only if $W := \overline{\omega_p M}$ satisfies the condition (5) of theorem 2.1.*

Remark 2.10. It follows from [1] or [2] that $\overline{\omega_p M} = \Omega_p M = \omega_p M$ if M has a compact neighborhood of p, has the curvature locally bounded from above and every shortest arc $[px]$ can be extended to a shortest arc $[py]$ of some fixed length $r > 0$ which is independent on $[px]$. In particular, this is true for any finite-dimensional manifold with inner metric of curvature locally bounded from above (see [1] or [2]). Later in this section we will consider primarily a spaces with conditions mentioned in the first sentence of this remark. Our next aim is the proof of isometricity of tangent space M_p and Gromov tangent cone T_pM under these conditions. The author proved this result in his Dr.Sci. dissertation earlier but did not publish it.

Let suppose that a closed ball $B(p,r), r > 0$, in metric space (M,ρ) with properties as in previous remark is compact, has globally the property of K-concavity and is convex. For the last property see [1] or [2]. We define the family of semimetrics $\rho_\lambda, 0 < \lambda \leq 1$, on $B(p,r)$ as follows (see also [2]). Every point $x \in B(p,r)$ is joined with point p by unique shortest arc $[px]$ (see [1] or [2]).Let λx be the point on $[px]$ such that $p(\lambda x) = \lambda(px)$. For any points $x, y \in B(p,r)$ we define $\rho_\lambda(x,y) := \frac{\rho(\lambda x, \lambda y)}{\lambda}$.

Proposition 2.11. *The family of semimetrics ρ_λ converges under $\lambda \to 0$ uniformly on $B(p,r) \times B(p,r)$ to some continuous semimetric ρ_0. Under this the metric space $(\beta(r), \overline{\rho_0})$, induced on $B(p,r)$ by semimetric ρ_0, is isometric to the closed ball of radius r in M_p with the center at its origin.*

We need the following lemma.

Lemma 2.12. *Under conditions of proposition 2.11, we have inequality*

$$\rho_\lambda \leq (1 + CKr^2), K > 0,$$

where C is positive constant depending only on K and r. Moreover, $C \to \frac{1}{2}$ if $K \to 0$ and

$$\rho_\lambda \leq \rho, K \leq 0.$$

Proof. The case $K \leq 0$ is evident. Let K be a positive number.

Let a, b be a positive fixed constants and $c(\phi)$ be the length of the third side in euclidean triangle with the sides a, b and angle $0 \leq \phi \leq \pi$ between them. By cosine theorem,

$$c(\phi)^2 = a^2 + b^2 - 2ab \cos \phi.$$

Differentiating by ϕ, we get

$$c'(\phi) = ab\frac{\sin \phi}{c} = b \sin \phi_a \leq b.$$

We used here the sine theorem for angle ϕ_a which is opposite to side a. So we have for $\phi_2 \geq \phi_1$,

$$c(\phi_2) - c(\phi_1) \leq b(\phi_2 - \phi_1). \tag{2.2}$$

Let now

$$T = pAB, T^K = p^K A^K B^K$$

be respectively a triangle in $B(p,r)$ with sides

$$pA = a, pB = b, AB = c(\phi_1)$$

and corresponding comparison triangle in S_K, i.e.

$$p^K A^K = a, p^K B^K = b, A^K B^K = AB.$$

Let A^K_λ, B^K_λ be the points on the sides $p^K A^K, p^K B^K$ of triangle T^K with conditions

$$p^K A^K_\lambda = \lambda a, p^K B^K_\lambda = \lambda b.$$

At the end let ϕ_2 be the angle of euclidean triangle with sides λa, λb, $A^K_\lambda B^K_\lambda$, opposite to the third side. One get from definition of semimetric ρ_λ and K-concavity property the inequality

$$\rho_\lambda \leq c(\phi_2).$$

Thus we only need prove the inequality

$$c(\phi_2) \leq (1 + CKr^2)c(\phi_1). \tag{2.3}$$

It follows from 0-convexity of metric on S_K (see [1]) and Gauss-Bonnet theorem that

$$\phi_2 - \phi_1 \leq KS(T^K) \leq KC_0 S(T^0), \tag{2.4}$$

where C_0 depends only on K and r, and converges to 1, if $K \to 0$. Now from (2.2) and (2.4) it follows the inequality

$$c(\phi_2) - c(\phi_1) \leq bKC_0 S(T^0) \leq \frac{1}{2} KC_0 bac(\phi_1).$$

Then we get for $C := \frac{1}{2}C_0$,

$$c(\phi_2) \leq (1 + CKab)c(\phi_1) \leq (1 + CKr^2)c(\phi_1),$$

as required. □

Proof. of proposition 2.11. It follows from remark 2.10 that $\overline{\omega_p M} = \Omega_p M = \omega_p M$, hence $\overline{M_p} = C_0\overline{\omega_p M} = C_0 \omega_p M = M_p$. The last statement of proposition and pointwise convergence $\rho_\lambda \to \rho_0$ on $B(p,r) \times B(p,r)$ follows easily from existence of angle between any two shortest arcs with origin p (see [2] or [3]) and definition of M_p. We only need to prove that the convergence $\rho_\lambda \to \rho_0$ is uniform on $B(p,r) \times B(p,r)$. It follows from lemma 2.12 that the family of semimetrics $\rho_\lambda, 0 < \lambda \leq 1$, on compact space $B(p,r) \times B(p,r)$ is equicontinuous. Now it follows from Ascoli-Arzela theorem that the convergence above is uniform. □

Corollary 2.13. *The map*

$$f_0 : (B(p,r), \rho) \to (\beta(r), \overline{\rho_0}), \tag{2.5}$$

induced by identity map of the ball $B(p,r)$, is Lipshitz map of closed ball $B(p,r)$ in M onto closed ball $(\beta(r), \overline{\rho_0})$ of radius r in M_p with the center at its origin.

Theorem 2.14. *Under conditions mentioned just before the proposition 2.11, there exists the Gromov tangent cone T_pM and this cone is isometric to M_p.*

Proof. By proposition 2.11, metric space $(\beta(r)_\lambda, \overline{\rho_\lambda})$, induced by semimetric ρ_λ, is isometric to closed ball of radius r with the center p in the space $(M, \frac{\rho}{\lambda})$. Let us define continuous functions on $B(p,r)$ by formulas

$$f_\lambda(b)(y) := \rho_\lambda(y,b) - \rho_\lambda(y,p),$$

$$f_0(b)(y) := \rho_0(y,b) - \rho_0(y,p),$$

where $b, y \in B(p,r)$. Let $C(B(p,r))$ be the Banach space of continuous functions on the compact $B(p,r)$ with Chebyshev norm

$$||f|| := \max\{|f(x)|, x \in B(p,r)\}$$

and distance

$$\eta(f,g) := ||f-g||.$$

The images

$$f_\lambda(B(p,r)), f_0(B(p,r)) \subset C(B(p,r))$$

of the mappings

$$f_\lambda : b \in B(p,r) \to f_\lambda(b), f_0 : b \in B(p,r) \to f_0(b)$$

are isometric to the spaces

$$(\beta(r)_\lambda, \overline{\rho_\lambda}), (\beta(r)_0, \overline{\rho_0}).$$

Then the Gromov-Hausdorff distance between these metric spaces admits the following upper evaluation

$$dist_{GH}((\beta(r)_\lambda, \overline{\rho_\lambda}), (\beta(r)_0, \overline{\rho_0})) \leq$$

$$H_{C(B(p,r))}(f_\lambda(B(p,r)), f_0(B(p,r))) \leq$$

$$\sup\{|\rho_\lambda(y,b) - \rho_0(y,b)|, y, b \in B(p,r)\},$$

what converges on the ground of proposition 2.11 to zero, if $\lambda \to 0$.Here $H_{C(B(p,r))}$ denotes the Hausdorff distance between a subsets in $C(B(p,r))$.

This means that there exists the Gromov tangent cone T_pM and the closed balls of radius r in M_p and T_pM with the centers at corresponding points (vertices) are isometric. Evidently, the spaces M_p and T_pM admit the homotheties with any coefficient $\mu, \mu > 0$, and the centers at vertices. Furthemore these spaces are finitely compact (every closed bounded subset is compact) on the ground of corollary 2.13. Then the spaces M_p and T_pM are isometric.The theorem is proved. $\square$

3. Busemann G-spaces of curvature bounded above.

A Busemann space [9] can be described as locally compact complete inner metric space without bifurcating shortest arcs and with the property of local extendability of shortest arcs.

The following proposition immediately follows from the corollary 2.7.

Proposition 3.1. *Under presence of (Aleksandrov) curvature locally bounded from above, one can change the last condition in the above description of Busemann G-space by (weaker) condition of geodesic completeness.*

Theorem 3.2. *Every Busemann G-space* (M, ρ) *with curvature locally bounded from above is topological manifold of some finite dimension* n*. Moreover, for every point* $p \in M$*, the tangent space* $M_p = T_pM$ *is isometric to* n*-dimensional Euclidean space and direction space* $\omega_pM = \overline{\omega_pM}$ *is isometric to* $(n-1)$*-dimensional unit sphere (in this Euclidean space).*

Proof. By condition M has the property of local extendability of shortest arcs. Then it follows from remark 2.10 that $\overline{\omega_pM} = \Omega_pM = \omega_pM$, hence $\overline{M_p} = C_0\overline{\omega_pM} = C_0\omega_pM = M_p$. We will prove at first the last statement. Since M is locally compact, in the light of corollaries 2.13, 2.4, propositions 2.6, 2.8, 2.9 and theorem 2.14, we must prove for this only the global property of extendability of shortest arcs in M_p for every $p \in M$. (The coincidence of n for different p will follow from other statements in the proof.)

We may suppose that every shortest arc $[xy]$ with $x, y \in B(p, r/4)$ can be extended to (unique) shortest arc $[xy_1]$ where

$$(xyy_1), xy_1 = 2xy. \tag{3.1}$$

and r satisfies the conditions mentioned before the proposition 2.11. The same condition is satisfied in the space $(M, \frac{\rho}{\lambda})$ when $r_\lambda = \frac{r}{\lambda}$, $0 < \lambda \leq 1$. By definition,

$$T_pM = GH \lim_{\lambda \to 0} (M, \frac{\rho}{\lambda}),$$

where the abbreviation $GH \lim$ means Gromov-Hausdorff limit (see [12]) and p is the basis point.

As a corollary of the propositions 2.6 and 2.8, any two shortest arcs of length r with origin p coincide if the angle between them is zero. Also all shortest arcs $[px]$ are unique if $x \in B(p, r)$. Thus the Lipshitz map f_0 from the corollary 2.13, which maps the compact ball $B(p, r)$ onto closed ball of radius r in tangent space M_p with the center O (which contains p as its representative) is a bijection and hence a homeomorphism. Also it follows from this that all semimetrics $\rho_\lambda, 0 \leq \lambda \leq 1$, are really a metrics.

Let x_0, y_0 be any points in M_p and

$$\rho_0(x_0, O), \rho_0(y_0, O), \rho_0(x_0, y_0) < s.$$

Let assume that $\lambda < \frac{r}{4s}$ and Gromov-Hausdorff distance (see [12])

$$dist_{GH}((B(p, 4\lambda s), \frac{\rho}{\lambda}), B_{T_pM}(O, 4s)) < \varepsilon.$$

Then there are a points $x, y \in B(p, \lambda s)$ and $y_1 \in B(p, 4\lambda s)$ such that the condition (3.1) is satisfied and for some point $y_0^\varepsilon \in B_{T_pM}(O, 4s)$ the maximum of the following numbers

$$|(\frac{\rho}{\lambda})(x, y) - x_0y_0|, |(\frac{\rho}{\lambda})(x, y_1) - x_0y_0^\varepsilon|,$$

$$|(\frac{\rho}{\lambda})(y, y_1) - y_0y_0^\varepsilon|$$

is less than 3ε.

Hence

$$x_0y_0 + y_0y_0^\varepsilon \leq x_0y_0^\varepsilon + 9\varepsilon, |x_0y_0 - y_0y_0^\varepsilon| < 6\varepsilon. \tag{3.2}$$

Using finite compactness of T_pM, one can choose a sequence of points $y_0^{\varepsilon_k}$, where $\varepsilon_k \to 0$, converging to a point $y_{0,1} \in T_pM$. On the ground of (3.2),

$$x_0y_0 + y_0y_{0,1} = x_0y_{0,1}, x_0y_0 = y_0y_{0,1}.$$

So shortest arcs in M_p are globally extandable, as required.

Hence the mentioned map f_0 maps closed (or open) ball $B(p, r)$ (or $U(p, r)$) homeomorphically onto closed (open) $r-$ball in euclidean space of some finite dimension n (because $B(p, r)$ is compact). The space M is connected. Thus it follows from above results and Brower theorem on invariance of dimension of a region that n one and the same for all p.Hence M is a topological n-dimensional manifold. □

The proof of theorem 1.1 needs some preliminary statements.

Proposition 3.3. *If in R_K (see [2]) a shortest arcs l_k, m_k with an origins p_k converge (in induced Hausdorff distance) to a shortest arcs l, m with origin p when $k \to \infty$, then*

$$\alpha(l, m) \geq \limsup \alpha(l_k, m_k).$$

Proof. Let x, y be arbitrary points on shortest arcs l, m different from p. Then on shortest arcs l_k, m_k there exist a points x_k, y_k different from p_k such that $x_k \to x$ and $y_k \to y$ (evidently, $p_k \to p$). Clearly, $\gamma_K(x_k p_k y_k) \to \gamma_K(xpy)$. Under this $\alpha(l_k, m_k) \leq \gamma_K(x_k p_k y_k)$. Thus

$$\gamma_K(xpy) \geq \limsup \alpha(l_k, m_k).$$

Then under conditions $x, y \to p$ and $k \to \infty$,

$$\alpha(l, m) = \lim \gamma_K(xpy) \geq \limsup \alpha(l_k, m_k),$$

what is required.

□

Proposition 3.4. *In a region R_K of Busemann G-space under the same conditions as in proposition 3.3, we have*

$$\alpha(l, m) = \lim_{k \to \infty} \alpha(l_k, m_k).$$

Proof. By propositions 2.6, 2.8, the shortest arcs l, l_k has unique opposite shortest arcs l', l'_k with origins p, p_k. Then $l'_k \to l'$, if we suppose for definiteness that all shortest arcs have length $r > 0$. In opposite case one could choose a subsequence k_s such that $l'_{k_s} \to l''$, where $s \to \infty$ and $l'' \neq l'$. Then it would follow from proposition 3.3 that $\alpha(l, l'') = \pi$, which is impossible by proposition 2.8.

It follows from proposition 3.3 that

$$\alpha(l', m) \geq \limsup \alpha(l'_k, m_k). \tag{3.3}$$

On the ground of theorem 3.2,

$$\alpha(l', m) = \pi - \alpha(l, m), \alpha(l'_k, m_k) = \pi - \alpha(l_k, m_k). \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\alpha(l, m) = \pi - \alpha(l', m) \leq \pi - \limsup \alpha(l'_k, m_k) =$$

$$\pi - (\pi - \liminf \alpha(l_k, m_k)) = \liminf \alpha(l_k, m_k).$$

This inequality together with proposition 3.3 gives the required statement. □

Proposition 3.5. *Let $B(p, r)$ be a convex region R_K in Busemann G-space, $px = r$. Then for every $\varepsilon > 0$ there is a number $r_1, 0 < r_1 < r$, such that for any two different points x_1, x_2 with condition $px_j < r_1; j = 1, 2$,*

$$|\alpha([x_1 x], [x_1 x_2]) - \gamma_K(x x_1 x_2)| < \varepsilon.$$

Proof. We will have by K-concavity for corresponding r_1, x_1, x_2 the inequality

$$\alpha([x_1 x], [x_1 x_2]) < \gamma_K(x x_1 x_2) + \varepsilon.$$

Let suppose that proposition is false. Then for any number $r_k := \frac{1}{k}$, there are a points $x_1^k \neq x_2^k$ such that $px_j^k < \frac{1}{k}; j = 1, 2$, while

$$\alpha([x_1^k x], [x_1^k x_2^k]) \leq \gamma_K(x x_1^k x_2^k) - \varepsilon. \tag{3.5}$$

The shortest arcs $[x_1^k x_2^k]$ can be extended to a shortest arcs $[x_1^k y^k]$ of fixed positive length.Let choose a subsequence (which we will denote simply by y^k) of the sequence y^k , converging to a point y. The shortest arcs $[x_1^k x], [x_1^k y_k]$ converge respectively to the shortest arcs $[px], [py]$ (see [1] or [2]). It follows from proposition 3.4, K-concavity of metric and (3.5) that for any point $x_2 \neq p$ on $[py]$,

$$\alpha([px], [py]) \leq \gamma_K(x p x_2) - \varepsilon. \tag{3.6}$$

But from the existence of the angle in strong sense (see [1] or [2]) in R_K we get the equality

$$\alpha([px], [py]) = \lim \gamma_K(x p x_2),$$

where $x_2 \to p$. This contradicts to inequality (3.6). $\square$

Proof. of theorem 1.1. We can prove this theorem in the same way as the theorem 5 in [8]. We have only note that the inequality (4) in [6], which we need for introducing the distance coordinate system, follows immediately from proposition 3.5. The components of metric tensor in distance coordinates are continuous by proposition 3.4. The theorem is proved. $\square$

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