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for shells from three dimensional
nonlinear elasticity by
Gamma-convergence**

by

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Derivation of nonlinear bending theory for shells from three dimensional nonlinear elasticity by Gamma-convergence

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Abstract

We show that the nonlinear bending theory of shells arises as a Γ -limit of three dimensional nonlinear elasticity.

Résumé Nous montrons que la théorie non linéaire des coques émerge comme Γ -limite de la théorie de l'élasticité tridimensionnelle.

Version française abrégée. Dans cette note nous dérivons la théorie des coques non linéaires comme Γ -limite de la théorie d'élasticité tridimensionnelles non linéaire.

Le problème tridimensionnel. Soit M une surface orientable dans \mathbb{R}^3 de la classe C^2 . Soit μ la normale de M et soit $M_h := \{x + s\mu(x) : x \in M, s \in (-h/2, h/2)\}$, pour $h > 0$ suffisamment petit. Nous supposons, pour simplicité d'exposition, que M est donné par une seule application $\psi : \Omega' \subset \mathbb{R}^2 \rightarrow M$ où Ω' est un ouvert borné lipschitzien. Donc M_h est l'image de $\Omega := \Omega' \times (-1/2, 1/2)$ sur l'application $\psi^{(h)}(z', z_3) = \psi(z') + hz_3\eta(z')$, où $\eta = \mu \circ \psi$.

L'énergie élastique d'une application $u : M_h \rightarrow \mathbb{R}^3$ est

$$E^{(h)}(u) = \int_{M_h} W(\nabla u) dx.$$

Ici W est une fonction Borelienne, de classe C^2 dans un voisinage de $SO(3)$, qui satisfait (11) et (12) ci-dessous.

Le problème bidimensionnel. Considerons la classe suivante d'applications isométriques de M

$$\mathcal{A} = \{u \in W^{2,2}(M; \mathbb{R}^3) : (\nabla_{\tan} u)^T (\nabla_{\tan} u) = I \text{ p.p. dans } M\}. \quad (1)$$

Ici la dérivée tangentielle $\nabla_{\tan} u(x)$ est une application de $T_x M$ à \mathbb{R}^3 . La quantité importante est *l'application relative de Weingarten* $S_{M,u}$ qui mesure la différence entre les secondes formes fondamentales de M et de $N = u(M)$. Soit ν la normale de N et soit $Y = u \circ \psi : \Omega' \rightarrow N$. Nous définissons $S_{M,u}$ par

$$S_{M,u}(x) \nabla \psi = R^T(x) \nabla'(\nu \circ Y) - \nabla'(\mu \circ \psi), \quad (2)$$

où $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ et $R(x) = \nabla_{\tan} u(x)$. Les caractérisations équivalentes sont données par (15) ou (16) ci-dessous.

Nous considérons les formes quadratiques $Q_3(G) := \partial^2 W / \partial F^2(I)(G, G)$, où I est l'identité, et $Q_2(x, G) := \min_{a \in \mathbb{R}^3} (G + a \otimes \mu(x))$. Évidemment $Q_2(x, G)$ dépend seulement sur la restriction de G au $T_x M$. L'énergie bidimensionnelle est donnée par

$$E(u) = \begin{cases} \frac{1}{24} \int_M Q_2(x, S_{M,u}(x)) d\mathcal{H}^2 & \text{si } u \in \mathcal{A}, \\ +\infty & \text{si non.} \end{cases} \quad (3)$$

Le résultat principal est la Γ -convergence des fonctionnelles $h^{-3} E^{(h)}$ vers E au sens suivant.

Théorème 1.

- (i) Soit $u^{(h)} : M_h \rightarrow \mathbb{R}^3$ une suite avec $\limsup_{h \rightarrow 0} h^{-3} E^{(h)}(u^{(h)}) < \infty$. Alors il existe $u \in \mathcal{A}$, une application $R : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ et des constantes $c^{(h)}$ tels que pour une sous-suite

$$u^{(h)} \circ \psi^{(h)} - c^{(h)} \rightarrow u \circ \psi^{(0)} \quad \text{dans } W^{1,2}(\Omega; \mathbb{R}^3), \quad (4)$$

$$\nabla u^{(h)} \circ \psi^{(h)} \rightarrow R \quad \text{dans } L^2(\Omega; \mathbb{R}^{3 \times 3}), \quad (5)$$

$$R_{,3} = 0, \quad R(z) \in SO(3) \text{ p.p.,} \quad R \in W^{1,2}(\Omega; \mathbb{R}^{3 \times 3}), \quad (6)$$

$$\liminf_{h \rightarrow 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) \geq E(u). \quad (7)$$

- (ii) Si $u \in \mathcal{A}$ il existe une suite $u^{(h)} : M_h \rightarrow \mathbb{R}^3$ telle que (4) - (6) soient vérifiées et

$$\lim_{h \rightarrow 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) = E(u).$$

On pourrait ajouter à les fonctionnelles des nouvelles termes linéaires en u les et des conditions au bord (voir (25), (26)) ci-dessous). Par les arguments classiques de la Γ -convergence on obtient la convergence des applications (presque) minimisantes, voir Corollary 2 ci-dessous.

Un outil essentiel de la preuve est un résultat de rigidité pour des applications proches d'un mouvement rigide (voir Theorem 3 ci-dessous ou [3], Théorème 2).

1 Introduction

The derivation of plate and shell theories is a problem having a long history with major contributions from Euler, D. Bernoulli, Kirchhoff, Love, E. and F. Cosserat, von Karman and a great many modern authors, see Love [8] for a review of the classical lines of research and Ciarlet [2] and Antman [1] for a more recent account. Most classical approaches are based on certain ansatzes leading to a variety of theories in the literature. The question which theory, if any, is predicted by nonlinear elasticity of thin objects has been open for a long time. The first result for large deformations was the derivation of nonlinear membrane theory by Le Dret and Raoult [6, 7] through the use of Γ -convergence. The more delicate case of bending theory for plates was settled recently in [3, 4], see also [9]. Here we extend this result to the nonlinear bending theory of shells. A key ingredient is an optimal oscillation estimate for deformations whose gradient is close to $SO(n)$ (see Theorem 3 below), first derived in [3] and generalizing earlier work of John [5].

2 The setting

The three dimensional problem. Let M be an oriented surface in \mathbb{R}^3 of class C^2 . For $x \in M$ let $\mu(x)$ denote the normal at x and consider, for sufficiently small h , the thickened set M_h of thickness h

$$M_h := \left\{ x + s\mu(x) : x \in M, s \in \left(-\frac{h}{2}, \frac{h}{2}\right) \right\}.$$

In the following we will assume for convenience that M is given by a single C^2 chart

$$\psi : \Omega' \subset \mathbb{R}^2 \rightarrow M,$$

where Ω' is a bounded Lipschitz domain. Then M_h is parametrized by the map

$$\begin{aligned}\psi^{(h)} : \Omega = \Omega' \times (-\frac{1}{2}, \frac{1}{2}) &\rightarrow M_h, \\ \psi^{(h)}(z', z_3) &= \psi(z') + h z_3 \eta(z'),\end{aligned}\tag{8}$$

where

$$\eta(z') = (\mu \circ \psi)(z').\tag{9}$$

The case of general surfaces can be handled by standard localization arguments.

For a map $u : M_h \rightarrow \mathbb{R}^3$ its elastic energy is

$$E^{(h)}(u) = \int_{M_h} W(\nabla u) dx,$$

where the stored-energy density $W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty]$ is Borel measurable and satisfies

$$W \in C^2 \text{ in a neighbourhood of } SO(3),\tag{10}$$

$$W \text{ is frame indifferent : } W(F) = W(RF) \text{ for all } R \in SO(3),\tag{11}$$

$$\begin{aligned}W(F) &\geq C \operatorname{dist}^2(F, SO(3)), \quad C > 0, \\ W(F) &= 0 \text{ if } F \in SO(3).\end{aligned}\tag{12}$$

The two dimensional problem. Let the surface M be as above and consider the following set of (infinitesimal) isometries of M

$$\mathcal{A} = \{u \in W^{2,2}(M; \mathbb{R}^3) : (\nabla_{\tan} u)^T (\nabla_{\tan} u) = I \text{ a.e. on } M\}.\tag{13}$$

Here the tangential derivative $\nabla_{\tan} u(x)$ is viewed as linear map from $T_x M$ to \mathbb{R}^3 . Sometimes it will be convenient to extend this map to a proper rotation in \mathbb{R}^3 . The two-dimensional energy will depend on a map $S_{M,u}(x) : T_x M \rightarrow T_x M$ which measures the difference between the second fundamental form of $N = u(M)$ and of M . Let $\psi : \Omega' \rightarrow M$ be the chart considered above, let $Y = u \circ \psi : \Omega' \rightarrow N$ and define $S_{M,u}(x)$ by

$$S_{M,u}(x) \nabla' \psi = R^T(x) \nabla'(\nu \circ Y) - \nabla'(\mu \circ \psi),\tag{14}$$

where ν is the normal of N , $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $R(x) = \nabla_{\tan} u(x)$. The closely related quadratic form

$$(\nabla' \psi)^T S_{M,u} \nabla' \psi = (\nabla' Y)^T \nabla'(\nu \circ Y) - (\nabla' \psi)^T \nabla'(\mu \circ \psi)\tag{15}$$

is exactly the difference of the fundamental forms of N and M . In terms of the Weingarten maps S_M and S_N given by $S_M \nabla' \psi = \nabla'(\mu \circ \psi)$ etc. we have

$$S_{M,u}(x) = R^T(x) S_N(u(x)) R(x) - S_M(x).\tag{16}$$

We thus call $S_{M,u}$ the *relative Weingarten map*.

Consider the quadratic form

$$Q_3(G) = \frac{\partial^2 W}{\partial F^2}(I)(G, G) \quad (17)$$

related to the linearization of the energy at the identity. It follows from (11), (12) that Q_3 is positive semidefinite and its kernel consists of the skew symmetric matrices. For each $x \in M$, with normal $\mu(x)$ we define the form

$$Q_2(x, G) = \min_{a \in \mathbb{R}^3} Q_3(G + a \otimes \mu(x)). \quad (18)$$

Since Q_3 vanishes on skew-symmetric matrices we have

$$Q_2(x, G) = Q_2(x, G') = Q_2(x, G'') \quad (19)$$

where

$$G' = G(I - \mu \otimes \mu), \quad G'' = (I - \mu \otimes \mu)G'.$$

In particular Q_2 depends only on the restriction of G to $T_x M$.

The two-dimensional energy is now defined by

$$E(u) = \begin{cases} \frac{1}{24} \int_M Q_2(x, S_{M,u}(x)) d\mathcal{H}^2 & \text{if } u \in \mathcal{A} \\ +\infty & \text{else.} \end{cases} \quad (20)$$

3 Convergence results

We essentially show that the functionals $\frac{1}{h^3} E^{(h)}$ Γ -converge to E . This implies that (almost) minimizers of $\frac{1}{h^3} E^{(h)}$ converge to minimizers of E , also when force terms and boundary conditions are added.

Theorem 1.

(i) (*compactness and ansatz-free lower bound*)

Suppose that $u^{(h)} : M_h \rightarrow \mathbb{R}^3$ satisfy $\limsup_{h \rightarrow 0} h^{-3} E^{(h)}(u^{(h)}) < \infty$. Then there exists a $u \in \mathcal{A}$, a map $R : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ and constants $c^{(h)}$ such that for a subsequence

$$u^{(h)} \circ \psi^{(h)} - c^{(h)} \rightarrow u \circ \psi^{(0)} \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^3), \quad (21)$$

$$\nabla u^{(h)} \circ \psi^{(h)} \rightarrow R \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}), \quad (22)$$

$$R_{,3} = 0, \quad R(z) \in SO(3) \text{ a.e.,} \quad R \in W^{1,2}(\Omega; \mathbb{R}^{3 \times 3}), \quad (23)$$

and for the subsequence under consideration,

$$\liminf_{h \rightarrow 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) \geq E(u). \quad (24)$$

(ii) (*attainment of lower bound*)

If $u \in \mathcal{A}$ then there exists a sequence $u^{(h)} : M_h \rightarrow \mathbb{R}^3$ such that (21) - (23) hold and

$$\lim_{h \rightarrow 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) = E(u).$$

Remarks

1. (body forces). Let $f \in L^2(M)$, and let $f^{(h)}(\psi^{(h)}(x)) = f(\psi(x'))$ denote the trivial extension to M_h . Then Theorem 1 remains valid if we add the term $-h^2 \int_{M_h} f^{(h)} \cdot u \, dx$ to $E^{(h)}$ and the term $-\int_M f \cdot u \, d\mathcal{H}^2$ to E .
2. (boundary conditions). Let Γ be a finite union of closed intervals in the curve $\partial\Omega'$. Let

$$\hat{Y} \in C^1(\bar{\Omega}'; \mathbb{R}^3) \cap W^{2,2}(\Omega'; \mathbb{R}^3), \quad \hat{b} \in W^{1,\infty}(\Omega'; \mathbb{R}^3).$$

To describe clamped boundary conditions on $\psi(\Gamma)$ we consider the admissible sets

$$\begin{aligned} \mathcal{A}_{BC}^{(h)} &= \{u \in W^{1,2}(M_h, \mathbb{R}^3) : \\ &\quad u^{(h)} \circ \psi^{(h)}(z', z_3) = \hat{Y}(z') + h z_3 \hat{b}(z'), \forall z' \in \Gamma\}, \end{aligned} \tag{25}$$

$$\begin{aligned} \mathcal{A}_{BC} &= \{u \in \mathcal{A} : (u \circ \psi)(z') = \hat{Y}(z'), \\ &\quad R(z') \mu(\psi(z')) = \hat{b}(z'), \forall z' \in \Gamma\}, \end{aligned} \tag{26}$$

where $R(z') \in SO(3)$ is the unique extension of the linear map $(\nabla_{\tan} u)(\psi(z'))$ from $T_{\psi(z')} M$ to \mathbb{R}^3 . Then the assertions of the Theorem also hold if $E^{(h)}$ and E are extended by $+\infty$ outside $\mathcal{A}_{BC}^{(h)}$ and \mathcal{A}_{BC} , respectively.

By standard reasoning in Γ -convergence assertions (i) and (ii) of the theorem imply the convergence of (almost) minimizers.

Corollary 2. Suppose that $\mathcal{A}_{BC} \neq \emptyset$. Let $u^{(h)} \in \mathcal{A}_{BC}^{(h)}$ be a sequence of almost minimizers of $E^{(h)}$ in $\mathcal{A}_{BC}^{(h)}$, i.e.

$$\lim_{h \rightarrow 0} \frac{1}{h^3} \left(E^{(h)}(u^{(h)}) - \inf_{\mathcal{A}_{BC}^{(h)}} E^{(h)} \right) = 0.$$

Then a subsequence converges to $u \in \mathcal{A}_{BC}$ in the sense of (21), (22) and u minimizes E in \mathcal{A}_{BC} .

Remark. The proof of Theorem 1 shows that $\mathcal{A}_{BC} \neq \emptyset$ if and only if $\limsup_{h \rightarrow 0} h^{-3} \inf_{\mathcal{A}_{BC}^{(h)}} E^{(h)} < \infty$.

4 Proofs

Compactness. The key ingredient is the following quantitative rigidity estimate (see [3], [4] for a proof and a discussion of related results).

Theorem 3. *Let U be a bounded Lipschitz domain in $\mathbb{R}^n, n \geq 2$. Then there exists a constant $C(U)$ with the following property. For each $v \in W^{1,2}(U; \mathbb{R}^n)$ there exists an associated rotation $R \in SO(n)$ such that*

$$\|\nabla v - R\|_{L^2(U)} \leq C(U) \|\text{dist}(\nabla v, SO(n))\|_{L^2(U)}. \quad (27)$$

The proof of Theorem 3 shows that the constant $C(U)$ can be chosen independent of U for a family of sets that are Bilipschitz images of a cube (with uniform Lipschitz constants).

Consider a lattice of squares

$$S_{a,h} = a + \left(-\frac{h}{2}, \frac{h}{2}\right)^2, \quad a \in h\mathbb{Z}^2,$$

and let Ω'_h be the union of those squares with $S_{a,3h} \subset \Omega'$. For such a square we can apply Theorem 3 to the deformed cube $C_{a,h} = \psi^{(h)}(S_{a,h} \times (-\frac{1}{2}, \frac{1}{2})) \subset M_h$ since $C_{a,h}$ is a Bilipschitz image of $(-\frac{h}{2}, \frac{h}{2})^3$ under the map $\Psi^{(h)}(z) = \psi(z') + z_3\eta(z')$. We thus obtain a map $R^{(h)} : \Omega'_h \rightarrow SO(3)$ which is constant on each $S_{a,h}$ and satisfies

$$\begin{aligned} & \int_{-\frac{1}{2}S_{a,h}}^{\frac{1}{2}} \int |(\nabla u^{(h)}) \circ \psi^{(h)} - R^{(h)}|^2 \det \nabla \psi^{(h)} dz' dz_3 \\ & \leq C \int_{C_{a,h}} W(\nabla u^{(h)}) dx. \end{aligned} \quad (28)$$

Applying the same estimate to a neighbouring cell $S_{b,h}$ and to $S_{a,3h} \supset S_{b,h}$ and taking into account that $\det \nabla \psi^{(h)} \sim h$ we easily deduce (see [4]) for $|\zeta|_\infty := \max(|\zeta_1|, |\zeta_2|) \leq h$

$$\int_{\Omega'_h} |R^{(h)}(z' + \zeta) - R^{(h)}(z')|^2 \leq \frac{C}{h} \int_{M_h} W(\nabla u^{(h)}) dx.$$

If $\zeta \in \mathbb{R}^2$ is a general translation vector and $\omega' \subset \Omega'$ with $\text{dist}(\omega', \partial\Omega') \geq C|\zeta|$ then iterative application of this estimate yields

$$\begin{aligned} \int_{\omega'} |R^{(h)}(z' + \zeta) - R^{(h)}(z')|^2 & \leq \frac{C}{h^3} |\zeta|^2 \int_{M_h} W(\nabla u^{(h)}) \\ & \leq C|\zeta|^2. \end{aligned} \quad (29)$$

From this one deduces (21) - (23) by standard means (see [4] for the details, including convergence and regularity up to the boundary). In particular one deduces from (28) after summation over the relevant lattice points a

$$\int_{\Omega'_h \times (-\frac{1}{2}, \frac{1}{2})} |(\nabla u^{(h)}) \circ \psi^{(h)} - R^{(h)}|^2 dz \leq Ch^2. \quad (30)$$

Lower bound. Let $F^{(h)} := (\nabla u^{(h)}) \circ \psi^{(h)}$. In view of (30) it is natural to introduce the scaled deviation from $SO(3)$

$$G^{(h)} = \frac{(R^{(h)})^T F^{(h)} - I}{h} \chi_{\Omega'_h}.$$

Then, for a subsequence, $G^{(h)} \rightharpoonup G$ in $L^2(\Omega)$. Let $E_h = \{|G^{(h)}| > h^{-\frac{1}{2}}\}$. Then $\text{meas } E_h \rightarrow 0$ and using positivity of W , Taylor expansion for $z \in \Omega'_h \setminus E_h$ and the fact that $h^{-1} \det \nabla \psi^{(h)}(z) \rightarrow \det((\nabla \psi)^T (\nabla \psi))^{\frac{1}{2}}(z') =: J_\psi(z')$ uniformly one deduces (see [4])

$$\begin{aligned} \liminf_{h \rightarrow 0} \frac{1}{h^3} \int_{M_h} W(\nabla u^{(h)}) dx &\geq \frac{1}{2} \int_{\Omega' \times (-\frac{1}{2}, \frac{1}{2})} Q_3(G) J_\psi dx \\ &\geq \frac{1}{2} \int_{\Omega' \times (-\frac{1}{2}, \frac{1}{2})} Q_2(\psi(z'), G') J_\psi dz, \end{aligned} \quad (31)$$

where $G'(z) = G(z)(I - \eta(z) \otimes \eta(z))$ and $\eta = \mu(\psi(z))$.

The main point is now to identify G' . To show that G' is affine in z_3 we consider the difference quotient in vertical direction

$$H^{(h)} = \Delta_3^s G^{(h)} := \frac{1}{s}(G^{(h)}(\cdot + se_3) - G^{(h)}).$$

Let $Y^{(h)} = u^{(h)} \circ \psi^{(h)}$. Then $\nabla Y^{(h)} = F^{(h)} \nabla \psi^{(h)}$ and a short calculation shows that

$$R^{(h)} H^{(h)} \nabla' \psi = \nabla' \Delta_3^s \left(\frac{1}{h} Y^{(h)} \right) - \Delta_3^s \tilde{F}^{(h)} \nabla' \eta,$$

where $\tilde{F}^{(h)}(z) = z_3 F^{(h)}(z)$. Since $H^{(h)} \rightharpoonup H$, $F^{(h)} \rightarrow R$, $R^{(h)} \rightarrow R$ and

$$\begin{aligned} \Delta_3^s \frac{1}{h} Y^{(h)} &= \frac{1}{s} \int_0^s \frac{1}{h} Y_3^{(h)}(\cdot + \sigma e_3) d\sigma \\ &= \frac{1}{s} \int_0^s (F^{(h)} \eta)(\cdot + \sigma e_3) d\sigma \rightarrow R\eta \end{aligned}$$

we get

$$RH \nabla' \psi = \nabla' (R\eta) - R \nabla' \eta = RS_{M,u} \nabla' \psi$$

where $S_{M,u}$ is the relative Weingarten map introduced in (14), (16). Thus $H' = S_{M,u}$ and $G''(z', z_3) = G'_0(z') + z_3 S_{M,u}$. Expanding the quadratic form Q_2 we easily deduce (24) from (31).

Attainment of lower bound. Let $u \in \mathcal{A}_{BC}, Y = u \circ \psi : \Omega' \rightarrow \mathbb{R}^3, b = R\eta : \Omega' \rightarrow \mathbb{R}^3$ where $R(z) \in SO(3)$ is the unique extension of $(\nabla_{\tan} u)(\psi(z))$ from $T_{\psi(z)} M$ to \mathbb{R}^3 . Then b is the unit normal, $b = Y_{,1} \wedge Y_{,2} / |Y_{,1} \wedge Y_{,2}|$. As in the case of plates [4], one can define suitable approximations $Y_\lambda \in W^{2,\infty}$ and $q_\lambda \in W^{1,\infty}$ which agree with Y and b , respectively, on a large set and satisfy the boundary conditions. Then the ansatz

$$Y^{(h)}(z', z_3) = Y_{\lambda_h}(z') + h z_3 q_{\lambda_h}(z') + \frac{1}{2} h^2 z_3^2 d_h(z'), \\ u^{(h)} = Y^{(h)} \circ (\psi^{(h)})^{-1},$$

where $\lambda_h = c/h, d_h \in W_0^{1,\infty}(\Omega'; \mathbb{R}^3), h ||d_h||_{W^{1,\infty}} \rightarrow 0, d_h \rightarrow d$ in $L^2(\Omega'; \mathbb{R}^3)$ yields the desired assertion similarly to [4]. The map d is chosen such that $Q_3(S_{M,u} + R^T d \otimes \mu) = Q_2(x, S_{M,u})$.

Boundary conditions for the two dimensional problem. The passage from $\mathcal{A}_{BC}^{(h)}$ to \mathcal{A}_{BC} is achieved as in [4]. The main point is a version of the difference quotient estimate (29) which holds up to the boundary. That allows one to define a trace for (a mollification of) $R^{(h)}$ and obtain its convergence in L^2 .

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