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The Riemann Function, Singular Entropies, and the Structure of Oscillations in Systems of two Conservation Laws

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THE RIEMANN FUNCTION, SINGULAR ENTROPIES, AND THE STRUCTURE OF OSCILLATIONS IN SYSTEMS OF TWO CONSERVATION LAWS

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Abstract. We use singular entropies, and the connection with the fundamental solution of the entropy equation and its adjoint operator, in order to derive a new formula describing the coupling of oscillations between the two characteristic fields in systems of two conservation laws.

1. Introduction

The theory of compensated compactness [Ta] provides a framework for the analysis of oscillations in scalar and systems of two conservation laws. It has been effective in establishing existence theorems for the equations of elasticity [Dp_1, Lin, Sh] and the equations of isentropic gas dynamics [Dp_2, DCL, LPT_2, LPS, CL], and for obtaining information on propagation and cacellation of oscillations to solutions of systems of two conservation laws [Se_1, Se_2, Ch_1]. The kinetic formulation [LPT_1, LPT_2] has provided a novel perspective for this problem leading to existence results for the equations of isentropic gas dynamics [LPS, CL] and the equations of chromatography [JPP].

The objective of the present work is to apply the machinery of singular entropies developed in [PTz] for strictly hyperbolic 2×2 systems to the study of propagation and cancellation of oscillations. The kinetic formulation is based on an efficient representation of the entropy structure of the problem, and this representation provides a concrete object for the study of Tartar's commutation relation. This leads to (i) simplified proofs for the results on cancellations of oscillations developed in [Se_1], and (ii) to a new formula for the coupling of oscillations between the two characteristic fields.

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We consider a strictly hyperboilic 2×2 system with characteristic speeds $\lambda_1 < \lambda_2$. It is shown in [PTz] that the equations generating entropy-entropy flux pairs

$$\mathcal{L}_{w,z}[\eta] := \eta_{wz} - \frac{g_z}{g}\eta_w - \frac{f_w}{f}\eta_z = 0\,, \qquad \begin{array}{c} q_w = \lambda_1\eta_w\,, \\ q_z = \lambda_2\eta_z\,, \end{array} \tag{1.1}$$

where $\frac{g_z}{g} = -\frac{\lambda_{1z}}{\lambda_1-\lambda_2}$, $\frac{f_w}{f} = \frac{\lambda_{2w}}{\lambda_1-\lambda_2}$, admit singular (distributional) solutions of the form

$$\begin{aligned} &H\,\mathbb{1}^k(w,\xi)\,, \quad (\lambda_1(\xi,\zeta)+Q)\,\mathbb{1}^k(w,\xi)\,, \\ &H\,\mathbb{1}^k(z,\zeta)\,, \quad (\lambda_2(\xi,\zeta)+Q)\,\mathbb{1}^k(z,\zeta)\,, \\ &\qquad \text{where} \qquad \mathbb{1}^k(w,\xi) = \begin{cases} \mathbb{1}_{k<\xi<w} & \text{if } k<w \\ 0 & \text{if } k=w \\ -\mathbb{1}_{w<\xi<k} & \text{if } w<k \end{cases}\,, \end{aligned} \tag{1.2}$$

$H = H(w,z;\xi,\zeta)$, $Q = Q(w,z;\xi,\zeta)$ solves the Goursat problem (2.26) and $k\in\mathbb{R}$ is a parameter. (Typically, we are interested in the values of k equal to 0, ∞ or $-\infty$ and this dependence is omitted.) We show that the universal entropy pair $H-Q$ generating (1.2) is precisely the generator of the fundamental solution of $\mathcal{L}$, that is

$$\mathcal{L}_{w,z}\big[H\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)\big] = \delta(w-\xi)\,\delta(z-\zeta)\,. \tag{1.3}$$

This observation leads to introduce the fundamental solution of $\mathcal{L}^T$, the adjoint operator to $\mathcal{L}$,

$$\mathcal{L}^T_{w,z}\big[\Theta\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)\big] = \delta(w-\xi)\,\delta(z-\zeta)\,. \tag{1.4}$$

The fundamental solution of $\mathcal{L}^T$ is generated by the so called Riemann function $\Theta = \Theta(w,z;\xi,\zeta)$, defined by the Goursat problem (2.30) and satisfying $H(w,z;\xi,\zeta) = \Theta(\xi,\zeta;w,z)$. The theory of fundamental solutions for linear hyperbolic operators is classical (see [So] and the Appendix), but to our knowledge the presentation in terms of simple distributions is novel. In particular, the relations of singular entropies and fundamental solutions leads to efficient representation formulas for entropy pairs $\eta - q$ (see section 2).

Next, we consider an oscillating family of solutions that satisfies the usual compensated compactness framework and introduce the Young measure ν describing their oscillations

$$wk\star-\lim h(w^\varepsilon, z^\varepsilon) = \int h(w,z)\,d\nu_{x,t}(w,z) = \overline{h}\,. \tag{1.5}$$

The singular entropies is an efficient tool for localizing the support of the Young measure ν and extracting information from Tartar's commutation relation

$$\overline{\eta_1 q_2 - \eta_2 q_1} = \overline{\eta_1}\,\overline{q_2} - \overline{\eta_2}\,\overline{q_1} \tag{1.6}$$

In section 3, we study cancellations of oscillations of the same characteristic field, by coupling singular entropies (1.2) belonging to the same characteristic family. This culminates to an alternative proof of a result by Serre [Se$_1$], stating that for a distributed Young measure we have the formulas

$$\overline{\frac{\partial\lambda_1}{\partial w}(w,z)g^2(w,z)\varphi(w)} = 0\,, \qquad \overline{\frac{\partial\lambda_2}{\partial z}(w,z)f^2(w,z)\psi(z)} = 0\,, \tag{1.7}$$

for any $\varphi(w)$, $\psi(z)$ continuous. In particular, formulas (1.7) imply strong convergence when both characteristic fields are genuinely nonlinear, or for the equations of elasticity with exactly one inflection point in the stress-strain constitutive relation [Se$_3$].

In Section 4, we derive a novel formula describing the coupling of oscillations among the two characteristic fields. We explain briefly the approach. First, from the compensated compactness bracket and the singular entropies, we derive the formula

$$\frac{1}{\delta-\gamma}\overline{\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}\mathbb{1}_{w<\xi}\mathbb{1}_{z<\zeta}} = \overline{\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}\mathbb{1}_{w<\xi}}\;\overline{\frac{\Theta(\xi,\zeta;\bar w,\bar z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}\mathbb{1}_{\bar z<\zeta}} \tag{1.8}$$

where $\Theta = \Theta(w,z;\xi,\zeta)$ is the Riemann function, while $\delta-\gamma$ is a constant denoting the normalization factor

$$\frac{1}{\delta-\gamma} = \overline{\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}} \tag{1.9}$$

Equation (1.8) may be viewed as a constraint on the joint distribution function of the random variables (w,z) describing the oscillations of the Riemann invariants. To extract information, we use certain remarkable properties of the kernel $\frac{\Theta}{\lambda_2-\lambda_1}$. Namely, there is a differential operator

$$\mathcal{N} := \partial^2_{\xi\zeta} + \frac{\lambda_{2\zeta}}{\lambda_2-\lambda_1}\partial_\xi - \frac{\lambda_{1\xi}}{\lambda_2-\lambda_1}\partial_\zeta \tag{1.10}$$

such that

$$\begin{aligned}
&\mathcal{N}\Big(\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{w<\xi}\mathbb{1}_{z<\zeta}\Big) = \frac{1}{(\lambda_2-\lambda_1)(\xi,\zeta)}\delta(w-\xi)\delta(z-\zeta)\\
&\mathcal{N}\Big(\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{w<\xi}\Big) = \mathcal{N}\Big(\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{z<\zeta}\Big) = \mathcal{N}\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big) = 0
\end{aligned} \tag{1.11}$$

The reader will surely notice the explicit effect of genuine nonlinearity on the form of the operator $\mathcal{N}$. By applying the operator $\mathcal{N}$ to (1.8) and using (1.11) we obtain

$$
\begin{aligned}
&\frac{1}{\delta-\gamma}\,\overline{\frac{1}{(\lambda_2-\lambda_1)(\xi,\zeta)}\delta(w-\xi)\delta(z-\zeta)} \\
&\quad = \overline{\partial_\xi\Big(\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{w<\xi}\Big)}\;\overline{\partial_\zeta\Big(\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{z<\zeta}\Big)} + \overline{\partial_\zeta\Big(\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{w<\xi}\Big)}\;\overline{\partial_\xi\Big(\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{z<\zeta}\Big)}
\end{aligned}
\tag{1.12}
$$

Formula (1.12) is an abstract formula describing the coupling of oscillations among the two characteristic fields, *without recourse to any structural hypotheses* on the fields beyond strict hyperbolicity. In addition, it contains as special cases the specialized formulas developed by Serre [Se_1, Se_2], concerning the coupling of oscillations between two linearly degenerate fileds, or between a linearly degenerate and a general field (see section 4).

2. Singular entropy pairs and the fundamental solution

We consider a strictly hyperbolic system of two conservation laws

$$
\begin{aligned}
u_t + a(u,v)_x &= 0 \\
v_t + b(u,v)_x &= 0
\end{aligned}
\tag{2.1}
$$

with characteristic speeds $\lambda_1 < \lambda_2$, right eigenvectors r_1, r_2 and left eigenvectors l_1, l_2. The eigenvectors are normalized so that $r_i \cdot l_j = \delta_{ij}$.

Let $w = w(u,v)$ and $z = z(u,v)$ be the 1- and 2-Riemann invariants defined by $\nabla w = l_1$ and $\nabla z = l_2$, or equivalently by

$$
\begin{aligned}
\nabla w \cdot r_2 = 0\,, &\quad \nabla w \cdot r_1 = 1\,, \\
\nabla z \cdot r_2 = 1\,, &\quad \nabla z \cdot r_1 = 0\,.
\end{aligned}
$$

The Riemann invariants induce a map $T : (u,v) \to (w,z)$ which is locally invertible. For certain special systems, like the equations of elastodynamics, T can be a globally defined and invertible map.

A given field ψ may be expressed in terms of the conserved variables (u,v) or in terms of the Riemann coordinates (w,z). The two representations are connected through the formulas

$$
\begin{gathered}
\psi(u,v) = \hat{\psi}(w(u,v), z(u,v))\,, \\
\frac{\partial \hat{\psi}}{\partial w} \circ (w,z) = \big(r_1 \cdot \nabla\big)\psi\,, \quad \frac{\partial \hat{\psi}}{\partial z} \circ (w,z) = \big(r_2 \cdot \nabla\big)\psi\,.
\end{gathered}
$$

(In the sequel we drop the hats and use the same notation in both domains.) Genuine nonlinearity of the characteristic fields is expressed by

$$r_1 \cdot \nabla\lambda_1 > 0 \quad \text{or} \quad \frac{\partial\lambda_1}{\partial w} > 0 \quad \text{and respectively} \quad r_2 \cdot \nabla\lambda_2 > 0 \quad \text{or} \quad \frac{\partial\lambda_2}{\partial z} > 0\,.$$

2.a Entropy pairs.

Let $U = (u, v)$ be the conserved variable and $F(U) = (a(u, v), b(u, v))$ be the flux. A function $\eta(U)$ is called an entropy with corresponding entropy flux $q(U)$ if every smooth solution of (2.1) satisfies the additional conservation law

$$\partial_t\eta(U) + \partial_x q(U) = 0\,. \tag{2.2}$$

Entropy pairs $\eta(U) - q(U)$ satisfy the equation

$$\nabla q(U) = \nabla\eta(U) \cdot \nabla F(U)\,, \tag{2.3}$$

and describe the nonlinear structure of (2.1). For strictly hyperbolic systems of two conservation laws, (2.3) is a determined linear hyperbolic system whose solutions generate the entropy pairs $\eta - q$.

Proposition 2.1. *Let $F \in C^2$ give rise to a stritly hyperbolic system. Then $\eta \in C^2$ is an entropy if and only if*

$$r_\alpha \cdot \nabla^2\eta\, r_\beta = 0 \quad \textit{for all } \alpha \neq \beta\ \ . \tag{2.4}$$

If η is an entropy then η is strictly convex if and only if

$$r_\alpha \cdot \nabla^2\eta\, r_\alpha > 0 \quad \textit{for any } \alpha\,. \tag{2.5}$$

Proof. The compatibility relation for the linear system (2.3) gives that $\eta \in C^2$ is an entropy if and only if $\nabla^2\eta\nabla F = \big(\nabla^2\eta\nabla F\big)^T = (\nabla F)^T\nabla^2\eta$. This relation, when expressed in the coordinate system $\{r_\alpha\}$, gives

$$(\lambda_\alpha - \lambda_\beta) r_\alpha \cdot \nabla^2\eta\, r_\beta = 0\,,$$

from where (2.4) follows. Let a be a vector with coordinates a_α in the coordinate system $\{r_\alpha\}$. Then

$$a \cdot \nabla^2\eta\, a = \sum_\alpha |a_\alpha|^2 r_\alpha \cdot \nabla^2\eta\, r_\alpha$$

and (2.5) follows. $\square$

In Riemann coordinates, (2.3) takes the particularly simple form

$$
\begin{aligned}
q_w &= \lambda_1 \eta_w \,, \\
q_z &= \lambda_2 \eta_z \,.
\end{aligned}
\tag{2.6}
$$

The compatibility equation for (2.6) is

$$
\eta_{wz} = \frac{\lambda_{2w}}{\lambda_1 - \lambda_2}\,\eta_z - \frac{\lambda_{1z}}{\lambda_1 - \lambda_2}\,\eta_w \,. \tag{2.7}
$$

Entropy pairs $\eta - q$ can then be probuced by solving the hyperbolic equation (2.7) and then integrating the exact system (2.6). It is convenient to introduce the functions f, g, defined (within a multiplicative factor) by

$$
\begin{aligned}
f(w,z) &= e^{\int^w \frac{\lambda_{2w}}{\lambda_1-\lambda_2} ds}\,, \quad & g(w,z) &= e^{-\int^z \frac{\lambda_{1z}}{\lambda_1-\lambda_2} ds}\,, \\
\text{and solving} \quad f_w &= \frac{\lambda_{2w}}{\lambda_1 - \lambda_2}\, f \,, & g_z &= -\frac{\lambda_{1z}}{\lambda_1 - \lambda_2}\, g \,.
\end{aligned}
\tag{2.8}
$$

Then (2.7) is expressed in the form

$$
\mathcal{L}[\eta] := \eta_{wz} - \frac{g_z}{g}\eta_w - \frac{f_w}{f}\eta_z = 0 \,. \tag{2.9}
$$

The convexity of entropies can be checked via the relations (2.5), which in Riemann coordinates take the form:

$$
\begin{aligned}
r_1 \cdot \nabla^2 \eta\, r_1 &= \partial_w^2 \eta + \big(r_1 \cdot \nabla^2 w\, r_1\big)\partial_w \eta + \big(r_1 \cdot \nabla^2 z\, r_1\big)\partial_z \eta \,, \\
r_2 \cdot \nabla^2 \eta\, r_2 &= \partial_z^2 \eta + \big(r_2 \cdot \nabla^2 w\, r_2\big)\partial_w \eta + \big(r_2 \cdot \nabla^2 z\, r_2\big)\partial_z \eta \,.
\end{aligned}
\tag{2.10}
$$

(see [Da$_2$] for the derivation). In general, relations (2.5) or (2.10) only provide criteria for local convexity of the entropies. Nevertheless, in special cases criteria for global convexity of entropies can also be derived (see [Da$_1$], [PTz]).

2.b Singular entropy pairs and associated representation formulas.

We outline here certain results from [PTz] concerning a class of singular entropies that satisfy (2.9) in the sense of distributions and generate representation formulas for solutions of the Goursat problem :

$$
\begin{aligned}
q_w &= \lambda_1 \eta_w \\
q_z &= \lambda_2 \eta_z
\end{aligned}
\quad \text{with data} \quad
\begin{cases}
\eta(w,0) = F(w) \\
\eta(0,z) \;= G(z)
\end{cases}
\,, \tag{2.11}
$$

where F, G smooth with $F(0) = G(0)$.

Define the indicator function

$$\mathbb{1}^k(w,\xi) = \begin{cases} \mathbb{1}_{k<\xi<w} & \text{if } k < w \\ 0 & \text{if } k = w \\ -\mathbb{1}_{w<\xi<k} & \text{if } w < k \end{cases}$$

where k is a parameter. $\mathbb{1}^k(w,\xi)$ has the following properties:

$$\begin{aligned} \mathbb{1}^k(w,\xi) &= \text{sign } (w-k)\ \mathbb{1}_{\{\xi\in(w,k)\cup(k,w)\}}. \\ \partial_w \mathbb{1}^k(w,\xi) &= \delta(w-\xi)\,, \\ \partial_\xi \mathbb{1}^k(w,\xi) &= -\delta(w-\xi) + \delta(\xi-k)\,. \end{aligned} \tag{2.12}$$

We are mainly interested in the limiting cases $k = \pm\infty$, when $\mathbb{1}^k(w,\xi)$ take the form

$$\mathbb{1}^{-\infty}(w,\xi) = \mathbb{1}_{\xi<w}\,, \quad \mathbb{1}^{\infty}(w,\xi) = -\mathbb{1}_{w<\xi}. \tag{2.13}$$

and in the case $k = 0$ denoted by $\mathbb{1}_w(\xi) = \mathbb{1}(w,\xi)$.

For $H = H(w,z)$ a C^2 function, the computation

$$\mathcal{L}[H\mathbb{1}(w,\xi)] = \mathcal{L}[H]\mathbb{1}(w,\xi) + \big(H_z - \frac{g_z}{g}H\big)\delta(w-\xi) \tag{2.14}$$

shows that if H is an entropy that satisfies $H = \tau g$ at $w = \xi$ then $H\mathbb{1}(w,\xi)$ is a singular entropy. In a similar fashion, the companion computation

$$\mathcal{L}[H\mathbb{1}(z,\zeta)] = \mathcal{L}[H]\mathbb{1}(z,\zeta) + \big(H_w - \frac{f_w}{f}H\big)\delta(z-\zeta) \tag{2.15}$$

shows that if an entropy H satisfies $H = \tau f$ at $z = \zeta$ then $H\mathbb{1}(z,\zeta)$ is a singular entropy.

This observation is the basis for the following conclusions (see [PTz] for the details):

(i) Let $H_r = H_r(w,z;\xi,\zeta)$ be the solution of (2.9) with data

$$\begin{cases} H_r = \frac{g(\xi,z)}{g(\xi,\zeta)} & \text{at } w = \xi\ , \\ H_r = 1 & \text{at } z = \zeta\ , \end{cases} \tag{2.16}$$

and let $Q_r = Q_r(w,z;\xi,\zeta)$ be the associated flux

$$\lambda_1(\xi,\zeta) + Q_r(w,z;\xi,\zeta) = \lambda_1(\xi,z)\frac{g(\xi,z)}{g(\xi,\zeta)} + \int_\xi^w \lambda_1(x,z)\,\frac{\partial H_r}{\partial w}(x,z;\xi,\zeta)\,dx\,, \tag{2.17}$$

satisfying

$$\begin{cases} Q_r = \lambda_1(\xi,z)\frac{g(\xi,z)}{g(\xi,\zeta)} - \lambda_1(\xi,\zeta) & \text{at } w=\xi\ , \\ Q_r = 0 & \text{at } z=\zeta\ . \end{cases} \tag{2.18}$$

Then $H_r - Q_r$ generates a singular entropy pair $(H_r \mathbb{1}_w(\xi), (\lambda_1(\xi,\zeta) + Q_r)\mathbb{1}_w(\xi))$.

(ii) Let $H_d = H_d(w,z;\xi,\zeta)$ solve (2.9) with data

$$\begin{cases} H_d = 1 & \text{at } w=\xi\ , \\ H_d = \frac{f(w,\zeta)}{f(\xi,\zeta)} & \text{at } z=\zeta\ , \end{cases} \tag{2.19}$$

and let $Q_d = Q_d(w,z;\xi,\zeta)$ the associated flux

$$\lambda_2(\xi,\zeta) + Q_d(w,z;\xi,\zeta) = \lambda_2(w,\zeta)\frac{f(w,\zeta)}{f(\xi,\zeta)} + \int_\zeta^z \lambda_2(w,y)\,\frac{\partial H_d}{\partial z}(w,y;\zeta)\,dy\ , \tag{2.20}$$

satisfying

$$\begin{cases} Q_d = 0 & \text{at } w=\xi\ , \\ Q_d = \lambda_2(w,\zeta)\frac{f(w,\zeta)}{f(\xi,\zeta)} - \lambda_2(\xi,\zeta) & \text{at } z=\zeta\ . \end{cases} \tag{2.21}$$

Then $H_d - Q_d$ generates a singular entropy pair $(H_d \mathbb{1}_z(\zeta), (\lambda_2(\xi,\zeta) + Q_d)\mathbb{1}_z(\zeta))$.

Note that the entropies H_r can ge cut in the direction right-left while the entropies H_d can be cut in the direction up-down; hence, the notation.

(iii) The solution of the Goursat problem (2.11) is represented in terms of the above singular pairs via the formulas

$$\begin{aligned} \eta(w,z) &= \eta(0,0) + \int_{\mathbb{R}} H_r(w,z;\xi,0)\mathbb{1}_w(\xi)\,F'(\xi)d\xi + \int_{\mathbb{R}} H_d(w,z;0,\zeta)\mathbb{1}_z(\zeta)\,G'(\zeta)d\zeta \\ q(w,z) &= q(0,0) + \int_{\mathbb{R}} \big(\lambda_1(\xi,0) + Q_r(w,z;\xi,0)\big)\mathbb{1}_w(\xi)\,F'(\xi)d\xi \\ &\quad + \int_{\mathbb{R}} \big(\lambda_2(0,\zeta) + Q_d(w,z;0,\zeta)\big)\mathbb{1}_z(\zeta)\,G'(\zeta)d\zeta \end{aligned} \tag{2.23}$$

2.c Fundamental solution and the universal entropy pair.

The indicator functions provide a simple perspective to solve the problem

$$\mathcal{L}[\mathcal{G}] = \delta(w-\xi)\delta(z-\zeta) \tag{2.24}$$

and thus to express the fundamental solution of the hyperbolic operator $\mathcal{L}$. Let $H = H(w,z) \in C^2$ be a smooth function and consider the computation

$$\begin{aligned} \mathcal{L}[H\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)] &= \mathcal{L}[H]\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta) + \big(H_z - \frac{g_z}{g}H\big)\delta(w-\xi)\mathbb{1}(z,\zeta) \\ &\quad + \big(H_w - \frac{f_w}{f}H\big)\mathbb{1}(w,\xi)\delta(z-\zeta) + H\delta(w-\xi)\delta(z-\zeta) \end{aligned} \tag{2.25}$$

This computation suggests the following construction: Fix a point (ξ,ζ) in the plane, and let $H = H(w,z;\xi,\zeta)$ be the solution of the Goursat problem

$$\mathcal{L}[H] = 0\,, \qquad \begin{cases} H = \frac{g(\xi,z)}{g(\xi,\zeta)} & \text{at } w=\xi \\ H = \frac{f(w,\zeta)}{f(\xi,\zeta)} & \text{at } z=\zeta \end{cases}, \qquad H(\xi,\zeta;\xi,\zeta) = 1\,. \tag{2.26}$$

H satisfies automatically the normalization condition $H(\xi,\zeta;\xi,\zeta)=1$ and the relations

$$\begin{cases} H_z - \frac{g_z}{g}H = 0 & \text{at } w=\xi \\ H_w - \frac{f_w}{f}H = 0 & \text{at } z=\zeta \end{cases}$$

Define next $Q = Q(w,z;\xi,\zeta)$ to be the corresponding flux normalized by $Q(\xi,\zeta;\xi,\zeta) = 0$. Integrating (2.6) and using the formulas

$$\lambda_1 f_w = \partial_w(\lambda_2 f) \quad \text{and} \quad \lambda_2 g_z = \partial_z(\lambda_1 g)$$

we see that

$$\begin{aligned} Q(w,z;\xi,\zeta) &= \lambda_1(\xi,z)H(\xi,z;\xi,\zeta) - \lambda_1(\xi,\zeta) + \int_\xi^w \lambda_1(x,z)H_w(x,z;\xi,\zeta)\,dx \\ &= \lambda_2(w,\zeta)H(w,\zeta;\xi,\zeta) - \lambda_2(\xi,\zeta) + \int_\zeta^z \lambda_2(w,y)H_z(w,y;\xi,\zeta)\,dy\,. \end{aligned} \tag{2.27}$$

The pair $H-Q$ has the following properties:

(iv$_1$) $H = H(w,z;\xi,\zeta)$ generates the fundamental solution of (2.9), that is

$$\mathcal{L}\big[H\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)\big] = \delta(w-\xi)\,\delta(z-\zeta)\,.$$

(iv$_2$) By the results of Section 2.b the pair $H-Q$ forms a "universal" entropy pair, in the sense that it can be cut in both directions to produce singular entropy pairs

$$\begin{aligned} &H\,\mathbb{1}(w,\xi)\,, \quad (\lambda_1(\xi,\zeta)+Q)\,\mathbb{1}(w,\xi)\,, \\ &H\,\mathbb{1}(z,\zeta)\,, \quad (\lambda_2(\xi,\zeta)+Q)\;\mathbb{1}(z,\zeta)\,. \end{aligned} \tag{2.28}$$

Consider next the adjoint of the operator $\mathcal{L}$, defined by

$$\mathcal{L}^T[\theta] := \theta_{wz} + \partial_w\big(\frac{g_z}{g}\theta\big) + \partial_z\big(\frac{f_w}{f}\theta\big)$$

The fundamental solution of $\mathcal{L}^T$ is computed by a similar argument. For $\Theta(w,z) \in C^2$, a computation gives

$$\begin{aligned} \mathcal{L}^T[\Theta\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)] &= \mathcal{L}^T[\Theta]\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta) + \big(\Theta_z + \frac{g_z}{g}\Theta\big)\delta(w-\xi)\mathbb{1}(z,\zeta) \\ &\quad + \big(\Theta_w + \frac{f_w}{f}\Theta\big)\mathbb{1}(w,\xi)\delta(z-\zeta) + \Theta\delta(w-\xi)\delta(z-\zeta) \end{aligned} \tag{2.29}$$

If $\Theta = \Theta(w, z; \xi, \zeta)$ is defined as the solution of the Goursat problem,

$$\mathcal{L}^T[\Theta] = 0\,, \qquad \begin{cases} \Theta = \frac{g(\xi,\zeta)}{g(\xi,z)} & \text{at } w = \xi \\ \Theta = \frac{f(\xi,\zeta)}{f(w,\zeta)} & \text{at } z = \zeta \end{cases}, \qquad \Theta(\xi,\zeta;\xi,\zeta) = 1\,, \tag{2.30a}$$

then Θ satisfies

$$\begin{cases} \Theta_z + \frac{g_z}{g}\Theta = 0 & \text{at } w = \xi \\ \Theta_w + \frac{f_w}{f}\Theta = 0 & \text{at } z = \zeta \end{cases} \tag{2.30b}$$

and $\Theta \mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)$ is the fundamental solution of the adjoint operator

$$\mathcal{L}^T\big[\Theta \mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)\big] = \delta(w-\xi)\,\delta(z-\zeta)\,. \tag{2.31}$$

The generating function Θ is called Riemann function.

From the Green's formulas in the Appendix we obtain :

(v_1) The solution η of the Goursat problem (2.11) can be represented in terms of the Riemann function Θ via the formula

$$\begin{aligned} \eta(w,z) = \Theta(0,0;w,z)\eta(0,0) &+ \int_0^w \Theta(x,0;w,z)\Big(F'(x) - \frac{f_x(x,0)}{f(x,0)}F(x)\Big)dx \\ &+ \int_0^z \Theta(0,y;w,z)\Big(G'(y) - \frac{g_y(0,y)}{g(0,y)}G(y)\Big)dy \end{aligned} \tag{2.32}$$

(v_2) The generators H and Θ of the fundamental solutions obey the symmetry

$$H(w,z;\xi,\zeta) = \Theta(\xi,\zeta;w,z) \tag{2.33}$$

Note that (v_1) is an immediate consequence of (A.9). (v_2) is again obtained from (A.9) if we select $u = H(w,z;\xi,\zeta)$ defined in (2.26) and $v = \Theta(\xi,\zeta;w,z)$ defined in (2.30).

(vi) This analysis culminates to a representation formula for the solution (η, q) of (2.11) in terms of the universal entropy pair $H - Q$ in (2.26)-(2.27) or, equivalently, the singular pair (2.28):

$$\begin{aligned} \eta(w,z) = H(w,z;0,0)\eta(0,0) &+ \int_{\mathbb{R}} H(w,z;x,0)\mathbb{1}_w(x)f(x,0)\frac{d}{dx}\frac{F(x)}{f(x,0)}\,dx \\ &+ \int_{\mathbb{R}} H(w,z;0,y)\mathbb{1}_z(y)g(0,y)\frac{d}{dy}\frac{G(y)}{g(0,y)}\,dy \end{aligned} \tag{2.34a}$$

$$\begin{aligned} q(w,z) = q(0,0) + Q(w,z;0,0)\eta(0,0) &+ \int_{\mathbb{R}} \big(\lambda_1(x,0) + Q(w,z;x,0)\big)\mathbb{1}_w(x)f(x,0)\frac{d}{dx}\frac{F(x)}{f(x,0)}\,dx \\ &+ \int_{\mathbb{R}} \big(\lambda_2(0,y) + Q(w,z;0,y)\big)\mathbb{1}_z(y)g(0,y)\frac{d}{dy}\frac{G(y)}{g(0,y)}\,dy \end{aligned} \tag{2.34b}$$

The assertion that the solution (η, q) of (2.11) is given by the formulas (2.34) can be checked directly using (2.26), (2.27) and (2.28). We do not present here the lengthy yet straightforward computations. Note that (2.34a) is just a rewriting of (2.32) using (2.33) while (2.34b) is the companion formula representing the fluxes. We remark that (2.23) and (2.34) offer two alternative representations of the solutions to (2.11) using different entropy kernels.

3. Cancellation of oscillations for 2 × 2 systems

In the next two sections we discuss the properties of oscillating families of solutions to 2×2 systems, using the fundamental solution of the entropy equations and the universal entropy-entropy flux pair. Consider the following framework: $(u^\varepsilon, v^\varepsilon)$ is a family of solutions to (2.1) taking values within the domain of invertibility of the Riemann coordinates. We assume that the Riemann invariants $(w^\varepsilon(x,t), z^\varepsilon(x,t))$ are uniformly stable in $L^\infty_{x,t}$, that is

$$(\mathrm{A}_1) \qquad |w^\varepsilon| + |z^\varepsilon| \leq K$$

for some constant K, and satisfy the compactness framework that the dissipation measure,

$$(\mathrm{A}_2) \qquad \partial_t \eta(w^\varepsilon, z^\varepsilon) + \partial_x q(w^\varepsilon, z^\varepsilon) \in_c H^{-1}_{loc,x,t}\,,$$

is precompact in $H^{-1}_{loc,x,t}$ for all smooth entropy pairs $\eta - q$.

By (A_1) a subsequence converges to some functions $w(x,t), z(x,t) \in L^\infty$,

$$w^\varepsilon \rightharpoonup w \quad z^\varepsilon \rightharpoonup z\,, \qquad \text{weak-}\star \text{ in } L^\infty.$$

Let us introduce the Young measure $\nu_{x,t}$ associated with (a subsequence of) $(w^\varepsilon, z^\varepsilon)$ and representing the weak-$\star$ limits of $(w^\varepsilon, z^\varepsilon)$:

$$wk\star - \lim h(w^\varepsilon, z^\varepsilon) = \int h(w,z)\, d\nu_{x,t}(w,z) \tag{3.1}$$

For a.e (x,t) the measure $\nu = \nu_{x,t}$ has compact support contained in the ball of radius K. We will at times use the notation

$$\overline{h} = \int h(w,z)\, d\nu(w,z)\,.$$

The div-curl lemma and (A_2) imply that for any two entropy pairs $\eta_i - q_i$, $i = 1, 2$, we have Tartar's commutation bracket [Ta]

$$\overline{\eta_1 q_2 - \eta_2 q_1} = \overline{\eta_1}\,\overline{q_2} - \overline{\eta_2}\,\overline{q_1}$$

Our objective is to analyze the commutation bracket using the universal entropy pair $H - Q$, which represents all possible entropy pairs (see (2.34)).

The universal pair $H(w,z;\xi,\zeta)$-$Q(w,z;\xi,\zeta)$ is defined in Section 2.c. H is the entropy satisfying the Goursat data

$$H(\xi,z;\xi,\zeta) = \frac{g(\xi,z)}{g(\xi,\zeta)}\,, \quad H(w,\zeta;\xi,\zeta) = \frac{f(w,\zeta)}{f(\xi,\zeta)} \tag{3.2}$$

while Q is the associated flux

$$\begin{aligned} Q(w,z;\xi,\zeta) &= \lambda_1(w,z)H(w,z;\xi,\zeta) - \lambda_1(\xi,\zeta) - \int_\xi^w \frac{\partial\lambda_1}{\partial w}(x,z)H(x,z;\xi,\zeta)\,dx \\ &= \lambda_2(w,z)H(w,z;\xi,\zeta) - \lambda_2(\xi,\zeta) - \int_\zeta^z \frac{\partial\lambda_2}{\partial z}(w,y)H(w,y;\xi,\zeta)\,dy\,. \end{aligned} \tag{3.3}$$

Recall that for any (ξ,ζ) the pairs

$$\begin{aligned} &H(w,z;\xi,\zeta)\,\mathbb{1}(w,\xi)\,, \quad (\lambda_1(\xi,\zeta) + Q(w,z;\xi,\zeta))\,\mathbb{1}(w,\xi)\,, \\ &H(w,z;\xi,\zeta)\,\mathbb{1}(z,\zeta)\,, \quad (\lambda_2(\xi,\zeta) + Q(w,z;\xi,\zeta))\,\mathbb{1}(z,\zeta)\,, \end{aligned} \tag{3.4}$$

are distributional solutions of (2.6), and that

$$\mathcal{L}\big[H(w,z;\xi,\zeta)\mathbb{1}(w,\xi)\mathbb{1}(z,\zeta)\big] = \delta(w-\xi)\,\delta(z-\zeta)\,.$$

The functions $\mathbb{1}(w,\xi)$, $\mathbb{1}(z,\zeta)$ in the above relations may be replaced by the functions in (2.13).

First, the commutation bracket is expressed in terms of the universal pair $H - Q$. In what follows, we suppress the w,z dependence of H and Q (with respect to which weak limits are taken), and use the notation

$$Q_1(\xi,\zeta) = \lambda_1(\xi,\zeta) + Q(\xi,\zeta)\,, \quad Q_2(\xi,\zeta) = \lambda_2(\xi,\zeta) + Q(\xi,\zeta)\,. \tag{3.5}$$

Proposition 3.1. *For a.e. ξ,θ,*

$$\begin{aligned} &\overline{\big(Q_1(\theta,\zeta)H(\xi,\zeta) - Q_1(\xi,\zeta)H(\theta,\zeta)\big)\mathbb{1}_{w>\theta}\mathbb{1}_{w<\xi}} \\ &\qquad = \overline{Q_1(\theta,\zeta)\mathbb{1}_{w>\theta}}\;\overline{H(\xi,\zeta)\mathbb{1}_{w<\xi}} - \overline{Q_1(\xi,\zeta)\mathbb{1}_{w<\xi}}\;\overline{H(\theta,\zeta)\mathbb{1}_{w>\theta}} \end{aligned} \tag{3.6}$$

For a.e ζ,k,

$$\begin{aligned} &\overline{\big(Q_2(\xi,\zeta)H(\xi,k) - Q_2(\xi,k)H(\xi,\zeta)\big)\mathbb{1}_{z>\zeta}\mathbb{1}_{z<k}} \\ &\qquad = \overline{Q_2(\xi,\zeta)\mathbb{1}_{z>\zeta}}\;\overline{H(\xi,k)\mathbb{1}_{z<k}} - \overline{Q_2(\xi,k)\mathbb{1}_{z<k}}\;\overline{H(\xi,\zeta)\mathbb{1}_{z>\zeta}} \end{aligned} \tag{3.7}$$

Proof. Let ζ be fixed and suppress the ζ dependence. For φ, $\psi \in C_c^\infty(\mathbb{R})$ we have

$$\partial_t \int H(w^\varepsilon, z^\varepsilon; \xi)\mathbb{1}_{w^\varepsilon<\xi}\varphi(\xi)d\xi + \partial_x \int Q_1(w^\varepsilon, z^\varepsilon; \xi)\mathbb{1}_{w^\varepsilon<\xi}\varphi(\xi)d\xi \quad \in_c H^{-1}_{loc}$$
$$\partial_t \int H(w^\varepsilon, z^\varepsilon; \theta)\mathbb{1}_{w^\varepsilon>\theta}\psi(\theta)d\theta + \partial_x \int Q_1(w^\varepsilon, z^\varepsilon; \theta)\mathbb{1}_{w^\varepsilon>\theta}\psi(\theta)d\theta \quad \in_c H^{-1}_{loc}$$

The div-curl lemma then implies that for a.e. fixed (x, t)

$$\begin{aligned}&\overline{\int_\xi \int_\theta \big(Q_1(\theta)H(\xi) - Q_1(\xi)H(\theta)\big)\mathbb{1}_{w<\xi}\mathbb{1}_{w>\theta}\varphi(\xi)\psi(\theta)\, d\xi d\theta} \\ &\quad = \overline{\int_\xi H(\xi)\mathbb{1}_{w<\xi}\varphi(\xi)\, d\xi}\; \overline{\int_\theta Q_1(\theta)\mathbb{1}_{w>\theta}\psi(\theta)\, d\theta} - \overline{\int_\theta H(\theta)\mathbb{1}_{w>\theta}\psi(\theta)\, d\theta}\; \overline{\int_\xi Q_1(\xi)\mathbb{1}_{w<\xi}\varphi(\xi)\, d\xi}\end{aligned}$$

and by Fubini

$$\begin{aligned}\int_\xi\int_\theta \Big\{&\overline{\big(H(\xi)Q_1(\theta) - H(\theta)Q_1(\xi)\big)\mathbb{1}_{w<\xi}\mathbb{1}_{w>\theta}} \\ &- \Big(\overline{H(\xi)\mathbb{1}_{w<\xi}}\;\overline{Q_1(\theta)\mathbb{1}_{w>\theta}} - \overline{H(\theta)\mathbb{1}_{w>\theta}}\;\overline{Q_1(\xi)\mathbb{1}_{w<\xi}}\Big)\Big\}\varphi(\xi)\psi(\theta)\, d\xi d\theta = 0\end{aligned}$$

Since φ, ψ are arbitrary, (3.6) follows. The same argument gives (3.7). $\square$

Due to (A$_1$) the Young measure is for each fixed (x, t) compactly supported. Fix (x, t) and define the smallest rectangle in the $w - z$ plane containing the support of ν. Let ξ_{min}, ξ_{max} be the locations of the vertical boundaries and ζ_{min}, ζ_{max} the locations of the horizontal boundaries of the enclosing rectangle.

Lemma 3.2. *(i) If $\xi_{min} < \xi_{max}$ there is a constant γ so that*

$$\begin{aligned}\overline{Q_1(\xi,\zeta)\mathbb{1}_{w>\xi}} &= \gamma\overline{H(\xi,\zeta)\mathbb{1}_{w>\xi}} \\ \overline{Q_1(\xi,\zeta)\mathbb{1}_{w<\xi}} &= \gamma\overline{H(\xi,\zeta)\mathbb{1}_{w<\xi}} \qquad \text{for a.e. } \xi_{min} < \xi < \xi_{max} \\ \overline{Q_1(\xi,\zeta)} &= \gamma\overline{H(\xi,\zeta)}\end{aligned} \tag{3.8}$$

and

$$\overline{\big(Q_1(\theta,\zeta)H(\xi,\zeta) - Q_1(\xi,\zeta)H(\theta,\zeta)\big)\mathbb{1}_{w>\theta}\mathbb{1}_{w<\xi}} = 0 \qquad a.e.\ \xi_{min} < \theta < \xi < \xi_{max} \tag{3.9}$$

(ii) If $\zeta_{min} < \zeta_{max}$ there is a constant δ so that

$$\begin{aligned}\overline{Q_2(\xi,\zeta)\mathbb{1}_{z>\zeta}} &= \delta\overline{H(\xi,\zeta)\mathbb{1}_{z>\zeta}} \\ \overline{Q_2(\xi,\zeta)\mathbb{1}_{z<\zeta}} &= \delta\overline{H(\xi,\zeta)\mathbb{1}_{z<\zeta}} \qquad \text{for a.e. } \zeta_{min} < \zeta < \zeta_{max} \\ \overline{Q_2(\xi,\zeta)} &= \delta\overline{H(\xi,\zeta)}\end{aligned} \tag{3.10}$$

and

$$\overline{\big(Q_2(\xi,\zeta)H(\xi,k) - Q_2(\xi,k)H(\xi,\zeta)\big)\mathbb{1}_{z>\zeta}\mathbb{1}_{z<k}} = 0 \qquad a.e.\ \zeta_{min} < \zeta < k < \zeta_{max} \tag{3.11}$$

(iii) If $\xi_{min} < \xi_{max}$ and $\zeta_{min} < \zeta_{max}$ then

$$(\delta - \gamma)\overline{H(\xi,\zeta)} = \overline{Q_2(\xi,\zeta) - Q_1(\xi,\zeta)} = \lambda_2(\xi,\zeta) - \lambda_1(\xi,\zeta)\,. \tag{3.12}$$

Proof. Fix ζ and again suppress the ζ dependence. Let a, b be two points such that $\xi_{min} < a, b < \xi_{max}$. The entropy $H(w,z;\xi)$ is continuous in all arguments and $H(\xi_{min}, z; \xi_{min}) > 0$ for $z \in \mathbb{R}$ by (3.2). If a_0 is selected sufficiently near ξ_{min} then $H(w,z;a) > 0$ on the strip $(w,z) \in (\xi_{min}, a_0) \times (\zeta_{min}, \zeta_{max})$ for all $\xi_{min} < a < a_0$. For $\alpha \in (\xi_{min}, a_0)$ it is

$$\overline{H(a)\mathbb{1}_{w<a}} = \int_{w<a} H(w,z;a)\,d\nu(w,z) > 0$$

The bracket (3.6) gives for $\theta > a$

$$\begin{aligned}&\overline{Q_1(\theta)\mathbb{1}_{w>\theta}}\;\overline{H(a)\mathbb{1}_{w<a}} - \overline{Q_1(a)\mathbb{1}_{w<a}}\;\overline{H(\theta)\mathbb{1}_{w>\theta}} \\ &\quad = \overline{\big(Q_1(\theta)H(a) - Q_1(a)H(\theta)\big)\mathbb{1}_{w>\theta}\mathbb{1}_{w<a}} = 0\end{aligned}$$

and hence

$$\overline{Q_1(\theta)\mathbb{1}_{w>\theta}} = \frac{\overline{Q_1(a)\mathbb{1}_{w<a}}}{\overline{H(a)\mathbb{1}_{w<a}}}\;\overline{H(\theta)\mathbb{1}_{w>\theta}} =: \gamma_{min}\;\overline{H(\theta)\mathbb{1}_{w>\theta}} \quad \text{for } a < \theta. \tag{3.13}$$

Similarly, if b is selected suficiently near ξ_{max}, we have $\overline{H(b)\mathbb{1}_{w>b}} > 0$ and

$$\overline{Q_1(\xi)\mathbb{1}_{w<\xi}} = \frac{\overline{Q_1(b)\mathbb{1}_{w>b}}}{\overline{H(b)\mathbb{1}_{w>b}}}\;\overline{H(\xi)\mathbb{1}_{w<\xi}} =: \gamma_{max}\;\overline{H(\xi)\mathbb{1}_{w<\xi}} \quad \text{for } b < \xi. \tag{3.14}$$

We conclude that $\gamma_{min} = \gamma_{max} =: \gamma$ and that

$$\overline{Q_1(\xi)\mathbb{1}_{w<\xi}} = \gamma\overline{H(\xi)\mathbb{1}_{w<\xi}}\;, \quad \overline{Q_1(\xi)\mathbb{1}_{w>\xi}} = \gamma\overline{H(\xi)\mathbb{1}_{w>\xi}}\;, \quad \text{for a.e. } \xi \in (\xi_{min}, \xi_{max})\;.$$

If we now take $\xi_n \to \xi+$ from the right, then $\mathbb{1}_{w<\xi_n} \to \mathbb{1}_{w\le\xi}\ d\nu - a.e$ and we have by the dominated convergence theorem that

$$\overline{Q_1(\xi)\mathbb{1}_{w\le\xi}} = \gamma\overline{H(\xi)\mathbb{1}_{w\le\xi}}$$

Thus the last identity in (3.8) follows; (3.9) is then an immediate consequence of (3.8) and (3.6). The second part of the lemma, (3.10) and(3.11), is proved by a similar argument, while (3.12) follows upon subtracting (3.8) and (3.10) and using (3.5). □

Next, we show that the brackets (3.9) and (3.11) entail information on cancellation of oscillations within each characteristic field.

Proposition 3.3. *(i) If $\xi_{min} < \xi_{max}$ then for any continuous φ*

$$\int \frac{\partial \lambda_1}{\partial w}(w,z) g^2(w,z)\varphi(w)\, d\nu(w,z) = 0\,. \tag{3.15}$$

(ii) If $\zeta_{min} < \zeta_{max}$ then for any continuous ψ

$$\int \frac{\partial \lambda_2}{\partial z}(w,z) f^2(w,z)\psi(z)\, d\nu(w,z) = 0\,. \tag{3.16}$$

Proof. Fix ζ and suppress the ζ dependence. Let $\xi \in (\xi_{min}, \xi_{max})$ and $h > 0$ and note that (3.9) gives

$$\overline{\big(Q_1(\xi+h)H(\xi) - Q_1(\xi)H(\xi+h)\big)\mathbb{1}_{\xi<w<\xi+h}} \tag{3.17}$$

A computation using (3.3) gives

$$\begin{aligned} I(\xi+h;\xi) &= Q_1(w,z;\xi+h)H(w,z;\xi) - Q_1(w,z;\xi)H(w,z;\xi+h) \\ &= \int_\xi^{\xi+h} \frac{\partial \lambda_1}{\partial w}(x,z) H(x,z;\xi+h)H(w,z;\xi)\, dx \\ &\quad + \int_\xi^w \frac{\partial \lambda_1}{\partial w}(x,z)\big[H(x,z;\xi)H(w,z;\xi+h) - H(x,z;\xi+h)H(w,z;\xi)\big]\, dx \\ &=: \int_\xi^{\xi+h} F(x,w,\xi,h)dx + \int_\xi^w G(x,w,\xi,h)dx \end{aligned}$$

From (3.17) we form the quotient

$$\frac{1}{h}\overline{I(\xi+h;\xi)\frac{1}{h}\mathbb{1}_{\xi<w<\xi+h}} = 0 \tag{3.18}$$

and obtain

$$\frac{1}{h^2}\int_\xi \Big(\int d\nu(w,z) I(\xi+h;\xi)\mathbb{1}_{\xi<w<\xi+h}\Big)\varphi(\xi)d\xi = 0$$

for a test function φ. Using Fubini's theorem this is expressed as

$$\int d\nu(w,z)\frac{1}{h^2}\Big(\int_{w-h}^w \int_\xi^{\xi+h} \varphi(\xi)F(x,w,\xi,h)\, dxd\xi + \int_{w-h}^w \int_\xi^w \varphi(\xi)G(x,w,\xi,h)\, dxd\xi\Big) = 0$$

The domain of integration of the first integral is a parallelepiped of area h^2 and has the Taylor expansion

$$\int_{w-h}^w \int_\xi^{\xi+h} \varphi(\xi)F(x,w,\xi,h)\, dxd\xi = F(w,w,w,0)\varphi(w)h^2 + o(h^2)$$

The second integral is over a triangle of area $\frac{1}{2}h^2$ and has the Taylor expansion

$$\int_{w-h}^{w}\int_{\xi}^{w}\varphi(\xi)G(x,w,\xi,h)\,dxd\xi = G(w,w,w,0)\frac{1}{2}h^2 + o(h^2)$$

Also note that

$$\begin{aligned}F(w,w,w,0) &= \frac{\partial\lambda_1}{\partial w}(w,z)H^2(w,z;w,\zeta) = \frac{\partial\lambda_1}{\partial w}(w,z)\Big(\frac{g(w,z)}{g(w,\zeta)}\Big)^2\\ G(w,w,w,0) &= 0\end{aligned}$$

Passing to the limit $h\to 0$ in (3.18), we deduce

$$\int d\nu(w,z)\frac{\partial\lambda_1}{\partial w}(w,z)g^2(w,z)\frac{\varphi(w)}{g^2(w,\zeta)} = 0$$

The final result (3.15) follows by redefining the test function φ, while (3.16) is proved via a similar argument. □

Next, we write a commutation bracket involving singular entropy pairs associated with different characteristic fields. Let (ξ,ζ) and (ξ',ζ') be two distinct points. An argument as in the proof of Proposition 3.1, in conjunction with (3.8) and (3.10), yields

$$\begin{aligned}&\overline{\Big(\big(\lambda_2(\xi',\zeta')+Q(\xi',\zeta')\big)H(\xi,\zeta)-\big(\lambda_1(\xi,\zeta)+Q(\xi,\zeta)\big)H(\xi',\zeta')\Big)\mathbb{1}_{z<\zeta'}\mathbb{1}_{w<\xi}}\\ &\quad=\overline{H(\xi,\zeta)\mathbb{1}_{w<\xi}}\;\overline{(\lambda_2(\xi',\zeta')+Q(\xi',\zeta'))\mathbb{1}_{z<\zeta'}}-\overline{H(\xi',\zeta')\mathbb{1}_{z<\zeta'}}\;\overline{(\lambda_1(\xi,\zeta)+Q(\xi,\zeta))\mathbb{1}_{w<\xi}}\\ &\quad=(\delta-\gamma)\overline{H(\xi,\zeta)\mathbb{1}_{w<\xi}}\;\overline{H(\xi',\zeta')\mathbb{1}_{z<\zeta'}}\end{aligned} \tag{3.19}$$

Equation (3.19) contains the most general information regarding the coupling of the two characteristic fields. In the sequel we will use a simplified bracket, obtained from (3.19) by taking the limits $\xi'\to\xi$ and $\zeta\to\zeta'$. The simplified bracket reads

$$(\lambda_2-\lambda_1)(\xi,\zeta)\,\overline{H(\xi,\zeta)\mathbb{1}_{w<\xi}\mathbb{1}_{z<\zeta}} = (\delta-\gamma)\overline{H(\xi,\zeta)\mathbb{1}_{w<\xi}}\;\overline{H(\xi,\zeta)\mathbb{1}_{z<\zeta}} \tag{3.20}$$

Obviously, identical relations hold if $\mathbb{1}_{w<\xi}$ and/or $\mathbb{1}_{z<\zeta}$ are replaced by $\mathbb{1}_{w>\xi}$ and/or $\mathbb{1}_{z>\zeta}$.

Lemma 3.4. *If $\xi_{min}<\xi_{max}$ and $\zeta_{min}<\zeta_{max}$ then $\delta>\gamma$ and the corners of the rectangle $[\xi_{min},\xi_{max}]\times[\zeta_{min}<\zeta_{max}]$ belong to the support of ν.*

Proof. The function $H(\xi, \zeta)$ satisfies the boundary conditions (3.2) and is thus strictly positive on the lines $w = \xi$ and $z = \zeta$. Consider now any point (ξ, ζ) so that the sector $\{w < \xi, z < \zeta\}$ meets the support on a small set. In this case $\overline{H(\xi, \zeta) \mathbb{1}_{w<\xi} \mathbb{1}_{z<\zeta}} > 0$, and (3.19) implies that no term on the right side can vanish. Hence $\delta \neq \gamma$.

Consider now the point (ξ, ζ) to be inside the rectangle $[\xi_{min}, \xi_{max}] \times [\zeta_{min} < \zeta_{max}]$ and near the corner (ξ_{min}, ζ_{min}). Then $\overline{H(\xi, \zeta) \mathbb{1}_{w<\xi}} > 0$ and $\overline{H(\xi, \zeta) \mathbb{1}_{z<\zeta}} > 0$. Again (3.19) and the fact that $\delta \neq \gamma$ imply $\overline{H(\xi, \zeta) \mathbb{1}_{w<\xi} \mathbb{1}_{z<\zeta}} \neq 0$. Thus the corner $(\xi_{min}, \zeta_{min}) \in \operatorname{supp} \nu$ and $\delta > \gamma$. Similar arguments show that all corners belong to the support of ν. $\square$

4. Coupling of oscillations between the two characteristic fields

Next, we study the coupling of oscillations between the two characteristic fields. The starting point is the relation (3.20), where $H = (w, z; \xi, \zeta)$ is the generator of the fundamental solution of the operator

$$\mathcal{L} = \partial^2_{wz} + \frac{\lambda_{2w}}{\lambda_2 - \lambda_1} \partial_z - \frac{\lambda_{1z}}{\lambda_2 - \lambda_1} \partial_w$$

(see section 2.c). In section 4.a we derive a formula describing the coupling of oscillations of the two characteristic fields, in terms of the Riemann function and the characteristic speeds. Then in sections 4.b,c we show that this formula produces as special cases certain formulas obtained by Serre [Se_1, Se_2] valid for the cases of two linearly degenerate fields, or one linearly degenerate field coupled with a general one.

4.a The coupling formula for general fields.

The analysis will use the Riemann function $\Theta = \Theta(\xi, \zeta; w, z)$ defined in (2.30). Recall that Θ generates the fundamental solution of the adjoint operator $\mathcal{L}^T$ in (2.31), and is connected to H through the symmetry relation

$$\Theta(\xi, \zeta; w, z) = H(w, z; \xi, \zeta) \tag{4.1}$$

Equation (3.20) is expressed in the form

$$\frac{1}{\delta - \gamma} \overline{\frac{\Theta(\xi, \zeta; w, z)}{(\lambda_2 - \lambda_1)(\xi, \zeta)} \mathbb{1}_{w<\xi} \mathbb{1}_{z<\zeta}} = \overline{\frac{\Theta(\xi, \zeta; w, z)}{(\lambda_2 - \lambda_1)(\xi, \zeta)} \mathbb{1}_{w<\xi}} \; \overline{\frac{\Theta(\xi, \zeta; \bar{w}, \bar{z})}{(\lambda_2 - \lambda_1)(\xi, \zeta)} \mathbb{1}_{\bar{z}<\zeta}} \tag{4.2}$$

The averages in (4.2) denote integration with respect to the Young measure ν in the variables (w, z) or $(\bar{w}, \bar{z})$. The factor $\delta - \gamma$ is constant, and, as seen from (3.12), satisfies for (ξ, ζ) in the support of ν

$$\frac{1}{\delta - \gamma} = \overline{\frac{\Theta(\xi, \zeta; w, z)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}} \tag{4.3}$$

We view it as a normalization factor and just carry it along in the analysis. The dependence of Θ in (w, z) is henceforth suppressed.

The forthcoming analysis is based on some remarkable properties of the kernel $\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}$. Let $\mathcal{N}$ be the differential operator

$$\mathcal{N} := \partial^2_{\xi\zeta} + \frac{\lambda_{2\zeta}}{\lambda_2 - \lambda_1}\partial_\xi - \frac{\lambda_{1\xi}}{\lambda_2 - \lambda_1}\partial_\zeta \tag{4.4}$$

Then, we have

Lemma 4.1. *The function* $\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}$ *satisfies*

$$\mathcal{N}\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\mathbb{1}_{w<\xi}\mathbb{1}_{z<\zeta}\Big) = \frac{1}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\delta(w - \xi)\delta(z - \zeta) \tag{4.5}$$

$$\mathcal{N}\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\mathbb{1}_{w<\xi}\Big) = \mathcal{N}\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\mathbb{1}_{z<\zeta}\Big) = \mathcal{N}\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\Big) = 0 \tag{4.6}$$

Proof. Let $\Theta = \Theta(\xi, \zeta; w, z)$. From (2.8), (2.30a) and (2.30b) we see that $\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}$ satisfies

$$\mathcal{N}\Big(\frac{\Theta}{(\lambda_2 - \lambda_1)}\Big) = 0\,, \qquad \begin{cases} \Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)_\zeta + \frac{\lambda_{2\zeta}}{\lambda_2-\lambda_1}\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big) = 0 & \text{at } \xi = w \\ \Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)_\xi - \frac{\lambda_{1\xi}}{\lambda_2-\lambda_1}\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big) = 0 & \text{at } \zeta = z \end{cases} \tag{4.7}$$

Then (4.5) and (4.6) follow by a direct computation. $\square$

The identities in Lemma 4.1 allow to extract from (4.2) information on the way oscillations in the two characteristic fields couple. We first give a formal derivation that will be justified in Proposition 4.2. Apply the operator $\mathcal{N}$ on both sides of (4.2) and interchange derivatives and averages. Using (4.5) and (4.6) we obtain

$$\begin{aligned} \frac{1}{\delta - \gamma}\overline{\frac{1}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\delta(w - \xi)\delta(z - \zeta)} = {} & \overline{\partial_\xi\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\mathbb{1}_{w<\xi}\Big)}\ \overline{\partial_\zeta\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\mathbb{1}_{z<\zeta}\Big)} \\ & + \overline{\partial_\zeta\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\mathbb{1}_{w<\xi}\Big)}\ \overline{\partial_\xi\Big(\frac{\Theta(\xi, \zeta)}{(\lambda_2 - \lambda_1)(\xi, \zeta)}\mathbb{1}_{z<\zeta}\Big)} \end{aligned} \tag{4.8}$$

Furthermore, performing the differentiations in (4.8), we obtain (4.9). Next, we give a rigorous derivation of (4.9).

Proposition 4.2. *If $\xi_{min} < \xi_{max}$ and $\zeta_{min} < \zeta_{max}$, then the coupling of oscillations is described by the formula*

$$
\begin{aligned}
\frac{1}{\delta-\gamma}\,\overline{\frac{1}{\lambda_2-\lambda_1}\delta(w-\xi)\delta(z-\zeta)} &= \overline{\frac{\Theta}{\lambda_2-\lambda_1}\delta(\xi-w)}\;\;\overline{\frac{\Theta}{\lambda_2-\lambda_1}\delta(\zeta-z)} \\
&+ \overline{\frac{\Theta}{\lambda_2-\lambda_1}\delta(\xi-w)}\;\;\overline{\partial_\zeta\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)\mathbb{1}_{z<\zeta}} + \overline{\partial_\xi\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)\mathbb{1}_{w<\xi}}\;\;\overline{\frac{\Theta}{\lambda_2-\lambda_1}\delta(\zeta-z)} \\
&+ \overline{\partial_\xi\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)\mathbb{1}_{w<\xi}}\;\;\overline{\partial_\zeta\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)\mathbb{1}_{z<\zeta}} + \overline{\partial_\zeta\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)\mathbb{1}_{w<\xi}}\;\;\overline{\partial_\xi\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)\mathbb{1}_{z<\zeta}}
\end{aligned}
\tag{4.9}
$$

where $\Theta = \Theta(\xi, \zeta; w, z)$ is the Riemann function, defined as the solution of the Goursat problem (2.30), and $\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_1)(\xi, \zeta)$.

Remark. The precise meaning of the terms in (4.9) is provided by the formula (4.18) and will be made clear in the course of the proof. Formula (4.18) indicates that the action of the Young measure is described by a sum of four product measures each describing specific couplings of the two oscillating fields. The study of special cases in sections 4.b and 4.c further illustrates the nature of various terms.

Proof. We will use the notation

$$
\begin{aligned}
a &= -\frac{\lambda_{2\zeta}}{\lambda_2-\lambda_1}\,, \qquad & b &= \frac{\lambda_{1\xi}}{\lambda_2-\lambda_1} \\
\Theta &= \Theta(\xi,\zeta;w,z) \qquad & \bar{\Theta} &= \Theta(\xi,\zeta;\bar{w},\bar{z})
\end{aligned}
\tag{4.10}
$$

Note that $\frac{\Theta}{\lambda_2-\lambda_1}$ satisfies

$$
\begin{aligned}
&\mathcal{N}\frac{\Theta}{\lambda_2-\lambda_1} = (\partial^2_{\xi\zeta} - a\partial_\xi - b\partial_\zeta)\frac{\Theta}{\lambda_2-\lambda_1} = 0 \\
&\begin{cases} (\partial_\zeta - a)\frac{\Theta}{\lambda_2-\lambda_1} = 0 & \text{at } \xi = w \\ (\partial_\xi - b)\frac{\Theta}{\lambda_2-\lambda_1} = 0 & \text{at } \zeta = z \end{cases}
\end{aligned}
\tag{4.11}
$$

while $\frac{\bar{\Theta}}{\lambda_2-\lambda_1}$ satisfies the same Goursat problem with the boundary conditions applied at $\xi = \bar{w}$ and $\zeta = \bar{z}$.

We write (4.2) in the expanded form

$$
\begin{aligned}
&\frac{1}{\delta-\gamma}\int_{(w,z)} d\nu(w,z)\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{\xi>w}\mathbb{1}_{\zeta>z} \\
&\quad = \int_{(w,z)} d\nu(w,z)\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{\xi>w} \int_{(\bar{w},\bar{z})} d\nu(\bar{w},\bar{z})\frac{\bar{\Theta}}{\lambda_2-\lambda_1}\mathbb{1}_{\zeta>\bar{z}}
\end{aligned}
\tag{4.12}
$$

Let $\mathcal{N}^T$ be the adjoint operator of $\mathcal{N}$,

$$\mathcal{N}^T := \partial^2_{\xi\zeta} + \partial_\xi(a\cdot) + \partial_\zeta(b\cdot)\,, \tag{4.13}$$

and $\varphi = \varphi(\xi,\zeta)$ be a test function with compact support. We multiply (4.12) with $\mathcal{N}^T\varphi$ and integrate the resulting identity over (ξ,ζ). The outcome of the operation is calculated below for the left and right parts of (4.12) separately.

Step 1. The left part of (4.12) gives

$$\begin{aligned} I_l &= \frac{1}{\delta-\gamma}\int_\xi\int_\zeta\Big(\int_{(w,z)} d\nu(w,z)\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{\xi>w}\mathbb{1}_{\zeta>z}\mathcal{N}^T\varphi\Big)d\xi d\zeta \\ &= \frac{1}{\delta-\gamma}\int_{(w,z)} d\nu(w,z)\int_{\xi>w}\int_{\zeta>z}\frac{\Theta}{\lambda_2-\lambda_1}\mathcal{N}^T\varphi d\xi d\zeta \end{aligned}$$

The Green's identity (A.4) in the Appendix, applied to the functions $u=\frac{\Theta}{\lambda_2-\lambda_1}$, $v=\varphi$ on the rectangle SQMP with corners $S=(w,z)$ and $M=(W,Z)$, gives

$$\begin{aligned} \iint_{SQMP}(\varphi\mathcal{N}\frac{\Theta}{\lambda_2-\lambda_1} - \frac{\Theta}{\lambda_2-\lambda_1}\mathcal{N}^T\varphi)d\xi d\zeta &= \varphi\frac{\Theta}{\lambda_2-\lambda_1}\Big|_{(W,Z)} - \varphi\frac{\Theta}{\lambda_2-\lambda_1}\Big|_{(w,z)} \\ &- \int_{z<\zeta<Z}\frac{\Theta}{\lambda_2-\lambda_1}(\partial_\zeta+a)\varphi\Big|_{\xi=W}d\zeta - \int_{w<\xi<W}\frac{\Theta}{\lambda_2-\lambda_1}(\partial_\xi+b)\varphi\Big|_{\zeta=Z}d\xi \\ &- \int_{z<\zeta<Z}\varphi(\partial_\zeta-a)\frac{\Theta}{\lambda_2-\lambda_1}\Big|_{\xi=w}d\zeta - \int_{w<\xi<W}\varphi(\partial_\xi-b)\frac{\Theta}{\lambda_2-\lambda_1}\Big|_{\zeta=z}d\xi \end{aligned}$$

We now send $W,Z\to\infty$. Since φ has compact support and $\frac{\Theta}{\lambda_2-\lambda_1}$ satisfies (4.11), we arrive at the formula

$$\int_{\xi>w}\int_{\zeta>z}\frac{\Theta}{\lambda_2-\lambda_1}\mathcal{N}^T\varphi d\xi d\zeta = \varphi(w,z)\frac{\Theta(w,z;w,z)}{(\lambda_2-\lambda_1)(w,z)} = \frac{\varphi(w,z)}{(\lambda_2-\lambda_1)(w,z)} \tag{4.14}$$

Hence,

$$\begin{aligned} I_l &= \frac{1}{\delta-\gamma}\int_\xi\int_\zeta\Big(\int_{(w,z)} d\nu(w,z)\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{\xi>w}\mathbb{1}_{\zeta>z}N^T\varphi\Big)d\xi d\zeta \\ &= \frac{1}{\delta-\gamma}\int_{(w,z)}\frac{\varphi(w,z)}{(\lambda_2-\lambda_1)(w,z)}d\nu(w,z) \end{aligned} \tag{4.15}$$

Step 2. The right part of (4.12) gives

$$\begin{aligned} I_r &= \int_\xi\int_\zeta\Big(\int_{(w,z)} d\nu(w,z)\frac{\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{\xi>w}\int_{(\bar w,\bar z)} d\nu(\bar w,\bar z)\frac{\bar\Theta}{\lambda_2-\lambda_1}\mathbb{1}_{\zeta>\bar z}\Big)(\mathcal{N}^T\varphi)d\xi d\zeta \\ &= \int_{(w,z)} d\nu(w,z)\int_{(\bar w,\bar z)} d\nu(\bar w,\bar z)\Big(\int_{\xi>w}\int_{\zeta>\bar z}\frac{\Theta}{\lambda_2-\lambda_1}\frac{\bar\Theta}{\lambda_2-\lambda_1}\mathcal{N}^T\varphi d\xi d\zeta\Big) \end{aligned} \tag{4.16}$$

Again we apply Green's identity (A.4) in the Appendix, now to the functions $u = \frac{\Theta\bar{\Theta}}{(\lambda_2-\lambda_1)^2}$, $v = \varphi$, on the rectangle SQMP with corners $S = (w, \bar{z})$ and $M = (W, \bar{Z})$. We now obtain

$$\iint_{SQMP} \left(\varphi(\mathcal{N} \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2}) - \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2} \mathcal{N}^T \varphi \right) d\xi d\zeta = \varphi \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2}\Big|_{(W,\bar{Z})} - \varphi \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2}\Big|_{(w,\bar{z})}$$
$$- \int_{\bar{z}<\zeta<\bar{Z}} \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2} (\partial_\zeta + a)\varphi\Big|_{\xi=W} d\zeta - \int_{w<\xi<W} \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2} (\partial_\xi + b)\varphi\Big|_{\zeta=\bar{Z}} d\xi$$
$$- \int_{\bar{z}<\zeta<\bar{Z}} \varphi(\partial_\zeta - a) \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2}\Big|_{\xi=w} d\zeta - \int_{w<\xi<W} \varphi(\partial_\xi - b) \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2}\Big|_{\zeta=\bar{z}} d\xi$$

Note that (4.11) implies

$$\mathcal{N} \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2} = \Big(\partial_\xi \frac{\Theta}{\lambda_2 - \lambda_1}\Big)\Big(\partial_\zeta \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big) + \Big(\partial_\zeta \frac{\Theta}{\lambda_2 - \lambda_1}\Big)\Big(\partial_\xi \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big)$$
$$\begin{cases} (\partial_\zeta - a)\frac{\Theta\bar{\Theta}}{(\lambda_2-\lambda_1)^2} = \frac{\Theta}{\lambda_2-\lambda_1}\Big(\partial_\zeta \frac{\bar{\Theta}}{\lambda_2-\lambda_1}\Big) & \text{at } \xi = w \\ (\partial_\xi - b)\frac{\Theta\bar{\Theta}}{(\lambda_2-\lambda_1)^2} = \frac{\bar{\Theta}}{\lambda_2-\lambda_1}\Big(\partial_\xi \frac{\Theta}{\lambda_2-\lambda_1}\Big) & \text{at } \zeta = \bar{z} \end{cases}$$

Sending $W, \bar{Z} \to \infty$ and using that φ has compact support, we deduce

$$\begin{aligned} &\int_{\xi>w} \int_{\zeta>z} \frac{\Theta\bar{\Theta}}{(\lambda_2 - \lambda_1)^2} \mathcal{N}^T \varphi d\xi d\zeta \\ &= \int_{\xi>w} \int_{\zeta>z} \varphi \Big(\partial_\xi \frac{\Theta}{\lambda_2 - \lambda_1}\Big)\Big(\partial_\zeta \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big) + \Big(\partial_\zeta \frac{\Theta}{\lambda_2 - \lambda_1}\Big)\Big(\partial_\xi \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big) d\xi d\zeta \\ &+ \int_{\{\zeta>\bar{z}, \xi=w\}} \varphi \frac{\Theta}{\lambda_2 - \lambda_1}\Big(\partial_\zeta \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big) d\zeta + \int_{\{\xi>w, \zeta=\bar{z}\}} \varphi \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big(\partial_\xi \frac{\Theta}{\lambda_2 - \lambda_1}\Big) d\xi \\ &+ \varphi \frac{\Theta}{\lambda_2 - \lambda_1} \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big|_{\xi=w, \zeta=\bar{z}} \end{aligned} \tag{4.17}$$

Combining (4.15), (4.16) and (4.17) we arrive at the formula

$$\begin{aligned} &\frac{1}{\delta - \gamma} \int_{(w,z)} d\nu(w,z) \frac{\varphi(w,z)}{(\lambda_2 - \lambda_1)(w,z)} \\ &= \int_{(w,z)} d\nu(w,z) \int_{(\bar{w},\bar{z})} d\nu(\bar{w},\bar{z}) \Big\{ \varphi(w,\bar{z}) \frac{\Theta}{\lambda_2 - \lambda_1}(w,\bar{z}) \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}(w,\bar{z}) \\ &+ \int_{\{\zeta>\bar{z}, \xi=w\}} \frac{\Theta}{\lambda_2 - \lambda_1}\Big(\partial_\zeta \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big) \varphi d\zeta + \int_{\{\xi>w, \zeta=\bar{z}\}} \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big(\partial_\xi \frac{\Theta}{\lambda_2 - \lambda_1}\Big) \varphi d\xi \\ &+ \int_{\xi>w} \int_{\zeta>\bar{z}} \Big[\Big(\partial_\xi \frac{\Theta}{\lambda_2 - \lambda_1}\Big)\Big(\partial_\zeta \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big) + \Big(\partial_\zeta \frac{\Theta}{\lambda_2 - \lambda_1}\Big)\Big(\partial_\xi \frac{\bar{\Theta}}{\lambda_2 - \lambda_1}\Big)\Big] \varphi d\xi d\zeta \Big\} \end{aligned} \tag{4.18}$$

which is precisely (4.9). $\square$

It is expedient to rewrite (4.18) in other formats that illustrate aspects of the formula. To this end, note that (2.8) gives

$$
\begin{aligned}
\frac{f(w,z)}{f(\xi,z)} &= e^{-\int_\xi^w \frac{\lambda_{2w}}{\lambda_2-\lambda_1}(x,z)dx} &&= \frac{(\lambda_2-\lambda_1)(\xi,z)}{(\lambda_2-\lambda_1)(w,z)} e^{-\int_\xi^w \frac{\lambda_{1w}}{\lambda_2-\lambda_1}(x,z)dx} \\
\frac{g(w,z)}{g(w,\zeta)} &= e^{\int_\zeta^z \frac{\lambda_{1z}}{\lambda_2-\lambda_1}(w,y)dy} &&= \frac{(\lambda_2-\lambda_1)(w,\zeta)}{(\lambda_2-\lambda_1)(w,z)} e^{\int_\zeta^z \frac{\lambda_{2z}}{\lambda_2-\lambda_1}(w,y)dy}
\end{aligned}
\tag{4.19}
$$

where the last expressions involve the coefficients of genuine nonlinearity λ_{1w} and λ_{2z}. Furthermore, since

$$
H(w,z;w,\zeta) = \frac{g(w,z)}{g(w,\zeta)}, \qquad H(w,z;\xi,z) = \frac{f(w,z)}{f(\xi,z)}, \tag{4.20}
$$

and $\Theta(\xi,\zeta;w,z) = H(w,z;\xi,\zeta)$, we have

$$
\begin{aligned}
\frac{\Theta}{\lambda_2-\lambda_1}\Big|_{\xi=w} &= \frac{\Theta(w,\zeta;w,z)}{(\lambda_2-\lambda_1)(w,\zeta)} = \frac{g(w,z)}{g(w,\zeta)} \frac{1}{(\lambda_2-\lambda_1)(w,\zeta)} \\
\frac{\bar\Theta}{\lambda_2-\lambda_1}\Big|_{\zeta=\bar z} &= \frac{\Theta(\xi,\bar z;\bar w,\bar z)}{(\lambda_2-\lambda_1)(\xi,\bar z)} = \frac{f(\bar w,\bar z)}{f(\xi,\bar z)} \frac{1}{(\lambda_2-\lambda_1)(\xi,\bar z)}
\end{aligned}
\tag{4.21}
$$

Using (4.21), we rewrite (4.18) as

$$
\begin{aligned}
&\frac{1}{\delta-\gamma}\int_{(w,z)} d\nu(w,z) \frac{\varphi(w,z)}{(\lambda_2-\lambda_1)(w,z)} \\
&= \int_{(w,z)} d\nu(w,z) \int_{(\bar w,\bar z)} d\nu(\bar w,\bar z) \Bigg\{ \varphi(w,\bar z) \frac{g(w,z)}{g(w,\bar z)} \frac{f(\bar w,\bar z)}{f(w,\bar z)} \frac{1}{(\lambda_2-\lambda_1)^2(w,\bar z)} \\
&\quad + \int_{\zeta>\bar z} \varphi(w,\zeta) \frac{g(w,z)}{g(w,\zeta)} \frac{1}{(\lambda_2-\lambda_1)(w,\zeta)} \Big(\partial_\zeta \frac{\bar\Theta}{\lambda_2-\lambda_1}\Big)(w,\zeta) d\zeta \\
&\quad + \int_{\xi>w} \varphi(\xi,\bar z) \frac{f(\bar w,\bar z)}{f(\xi,\bar z)} \frac{1}{(\lambda_2-\lambda_1)(\xi,\bar z)} \Big(\partial_\xi \frac{\Theta}{\lambda_2-\lambda_1}\Big)(\xi,\bar z) d\xi \\
&\quad + \int_{\xi>w}\int_{\zeta>z} \Big[\Big(\partial_\xi \frac{\Theta}{\lambda_2-\lambda_1}\Big)\Big(\partial_\zeta \frac{\bar\Theta}{\lambda_2-\lambda_1}\Big) + \Big(\partial_\zeta \frac{\Theta}{\lambda_2-\lambda_1}\Big)\Big(\partial_\xi \frac{\bar\Theta}{\lambda_2-\lambda_1}\Big)\Big] \varphi\, d\xi d\zeta \Bigg\}
\end{aligned}
\tag{4.22}
$$

For the choice of test function $\varphi = (\lambda_2-\lambda_1)^2 g f \psi$, with $\psi = \psi(\xi,\zeta)$ a new test function, (4.22) takes the form

(4.23)

$$\frac{1}{\delta-\gamma}\int_{(w,z)} d\nu(w,z)(\lambda_2-\lambda_1)(w,z)g(w,z)f(w,z)\psi(w,z)$$

$$= \int_{(w,z)} d\nu(w,z)\int_{(\bar w,\bar z)} d\nu(\bar w,\bar z)\Bigg\{\psi(w,\bar z)g(w,z)f(\bar w,\bar z)$$

$$+\Bigg(\int_{\zeta>\bar z}\psi(w,\zeta)(\lambda_2-\lambda_1)(w,\zeta)f(w,\zeta)\Big(\partial_\zeta\frac{\bar\Theta}{\lambda_2-\lambda_1}\Big)(w,\zeta)d\zeta\Bigg)g(w,z)$$

$$+\Bigg(\int_{\xi>w}\psi(\xi,\bar z)(\lambda_2-\lambda_1)(\xi,\bar z)g(\xi,\bar z)\Big(\partial_\xi\frac{\Theta}{\lambda_2-\lambda_1}\Big)(\xi,\bar z)d\xi\Bigg)f(\bar w,\bar z)$$

$$+\int_{\xi>w}\int_{\zeta>z}\left[\Big(\partial_\xi\frac{\Theta}{\lambda_2-\lambda_1}\Big)\Big(\partial_\zeta\frac{\bar\Theta}{\lambda_2-\lambda_1}\Big)+\Big(\partial_\zeta\frac{\Theta}{\lambda_2-\lambda_1}\Big)\Big(\partial_\xi\frac{\bar\Theta}{\lambda_2-\lambda_1}\Big)\right](\lambda_2-\lambda_1)^2 gf\psi d\xi d\zeta\Bigg\}$$

For product test functions $\psi(\xi,\zeta)=\Phi(\xi)\Psi(\zeta)$, the first term in the right factorizes into a product of Young measures. By contrast, the remaining three terms will not in general factorize, since they contain couplings of the variables (w,z) and $(\bar w,\bar z)$ (recall that $\bar\Theta=\Theta(\xi,\zeta,\bar w,\bar z)$ while $\Theta=\Theta(\xi,\zeta;w,z)$). The implications of (4.22) or (4.23) on various special cases are studied below.

4.b Two linearly degenerate characteristic fields.

Consider the case that both characteristic fields are linearly degenerate, that is $\frac{\partial\lambda_1}{\partial w}=\frac{\partial\lambda_2}{\partial z}=0$. In this case, (4.19) reads

$$\frac{f(w,z)}{f(\xi,z)}=\frac{(\lambda_2-\lambda_1)(\xi,z)}{(\lambda_2-\lambda_1)(w,z)}\,,\qquad \frac{g(w,z)}{g(w,\zeta)}=\frac{(\lambda_2-\lambda_1)(w,\zeta)}{(\lambda_2-\lambda_1)(w,z)}$$

while the Goursat problem (4.7) becomes

$$\partial^2_{\xi\zeta}\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)=0\,,\qquad \begin{cases}\partial_\zeta\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)=0 & \text{at } \xi=w\\ \partial_\xi\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big)=0 & \text{at } \zeta=z\end{cases}\qquad \Theta(w,z;w,z)=1\,. \tag{4.24}$$

The solution of (4.25) is

$$\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)}=\frac{1}{(\lambda_2-\lambda_1)(w,z)} \tag{4.25}$$

and it satisfies $\partial_\xi\big(\frac{\Theta}{\lambda_2-\lambda_1}\big)=\partial_\zeta\big(\frac{\Theta}{\lambda_2-\lambda_1}\big)=0$. Most of the coupling terms in (4.18) drop out and it simplifies to

$$\frac{1}{\delta-\gamma}\int_{(w,z)} d\nu(w,z)\frac{\varphi(w,z)}{(\lambda_2-\lambda_1)(w,z)}$$

$$=\int_{(w,z)} d\nu(w,z)\int_{(\bar w,\bar z)} d\nu(\bar w,\bar z)\varphi(w,\bar z)\frac{1}{(\lambda_2-\lambda_1)(w,z)}\frac{1}{(\lambda_2-\lambda_1)(\bar w,\bar z)}$$

For $\varphi = \Phi(\xi)\Psi(\zeta)$ a product this, in turn, gives

$$\frac{1}{\delta-\gamma}\int d\nu \frac{\Phi(w)\Psi(z)}{(\lambda_2-\lambda_1)(w,z)} = \Big(\int d\nu \frac{\Phi(w)}{(\lambda_2-\lambda_1)(w,z)}\Big)\Big(\int d\nu \frac{\Psi(z)}{(\lambda_2-\lambda_1)(w,z)}\Big) \tag{4.26}$$

Furthermore, by (4.3) and (4.25),

$$\frac{1}{\delta-\gamma} = \int \frac{d\nu}{\lambda_2-\lambda_1} \tag{4.27}$$

Equation (4.26) may be viewed as stating that the probability measure $\mu := \Big(\overline{\frac{1}{\lambda_2-\lambda_1}}\Big)^{-1}\frac{1}{\lambda_2-\lambda_1}\nu$ can be factorized into $\mu = \mu_1 \otimes \mu_2$, a product of its marginals.

4.c Coupling of a linearly degenerate with a general field.

Consider now the case that the first characteristic field is general while the second characteristic field is linearly degenerate, $\lambda_1 = \lambda_1(w,z)$ while $\lambda_2 = \lambda_2(w)$. In this case, (4.19) reads

$$\begin{aligned} \frac{f(w,z)}{f(\xi,z)} &= \frac{(\lambda_2-\lambda_1)(\xi,z)}{(\lambda_2-\lambda_1)(w,z)} e^{-\int_\xi^w \frac{\lambda_{1w}}{\lambda_2-\lambda_1}(x,z)dx} \\ \frac{g(w,z)}{g(w,\zeta)} &= \frac{(\lambda_2-\lambda_1)(w,\zeta)}{(\lambda_2-\lambda_1)(w,z)} \end{aligned} \tag{4.28}$$

The Goursat problem (4.4), (4.7) takes the form

$$\begin{aligned} &\Big(\partial^2_{\xi\zeta} - \frac{\lambda_{1\xi}}{\lambda_2-\lambda_1}\partial_\zeta\Big)\frac{\Theta}{\lambda_2-\lambda_1} = 0 \\ &\begin{cases} \partial_\zeta \frac{\Theta}{\lambda_2-\lambda_1} = 0 & \text{at } \xi = w \\ \Big(\partial_\xi - \frac{\lambda_{1\xi}}{\lambda_2-\lambda_1}\Big)\frac{\Theta}{\lambda_2-\lambda_1} = 0 & \text{at } \zeta = z \end{cases} \\ &\Theta(w,z;w,z) = 1 \end{aligned} \tag{4.29}$$

Integrating the problem and using (4.28), we obtain

$$\frac{\Theta(\xi,\zeta;w,z)}{(\lambda_2-\lambda_1)(\xi,\zeta)} = \frac{1}{(\lambda_2-\lambda_1)(w,z)} e^{-\int_\xi^w \frac{\lambda_{1w}}{\lambda_2-\lambda_1}(x,z)dx} = \frac{1}{(\lambda_2-\lambda_1)(\xi,z)}\,\frac{f(w,z)}{f(\xi,z)} \tag{4.30}$$

which, in particular, implies $\partial_\zeta\Big(\frac{\Theta}{\lambda_2-\lambda_1}\Big) = 0$.

Again many terms in (4.22) drop out, and the resulting identity reads

$$\begin{aligned} &\frac{1}{\delta-\gamma}\int_{(w,z)} d\nu(w,z)\frac{\varphi(w,z)}{(\lambda_2-\lambda_1)(w,z)} \\ &= \int_{(w,z)} d\nu(w,z)\int_{(\bar w,\bar z)} d\nu(\bar w,\bar z)\Big\{\varphi(w,\bar z)\frac{g(w,z)}{g(w,\bar z)}\frac{f(\bar w,\bar z)}{f(w,\bar z)}\frac{1}{(\lambda_2-\lambda_1)^2(w,\bar z)} \\ &\quad + \Big(\int_{\xi>w}\varphi(\xi,\bar z)\frac{1}{(\lambda_2-\lambda_1)(\xi,\bar z)f(\xi,\bar z)}\partial_\xi\Big(\frac{1}{(\lambda_2-\lambda_1)(\xi,z)f(\xi,z)}\Big)d\xi\Big) f(w,z)f(\bar w,\bar z)\Big\} \end{aligned} \tag{4.29}$$

For a test function $\varphi = (\lambda_2 - \lambda_1) f \Phi\Psi$, with $\Phi = \Phi(\xi)$, $\Psi = \Psi(\zeta)$, we conclude

(4.30)
$$\frac{1}{\delta-\gamma}\int_{(w,z)} d\nu(w,z) f(w,z)\Phi(w)\Psi(z) = \Bigg(\int_{(\bar w,\bar z)} d\nu(\bar w,\bar z) f(\bar w,\bar z)\Psi(\bar z)\Bigg)$$
$$\times\Bigg(\int_{(w,z)} d\nu(w,z)\Big[\frac{\Phi(w)}{(\lambda_2-\lambda_1)(w,z)} + f(w,z)\int_{\xi>w}\Phi(\xi)\partial_\xi\Big(\frac{1}{(\lambda_2-\lambda_1)(\xi,z)f(\xi,z)}\Big)d\xi\Big]\Bigg)$$

It can be further expressed in the form

$$\begin{aligned}\frac{1}{\delta-\gamma}\int \Phi(w)\Psi(z) f(w,z) d\nu &= \Big(\int \Psi(z) f(w,z) d\nu\Big)\\ &\times\Big(\int\Big[-\int_\xi \frac{1}{(\lambda_2-\lambda_1)(\xi,z)f(\xi,z)}\mathbb{1}_{w<\xi}\partial_\xi\Phi(\xi)\Big] f(w,z) d\nu\Big)\end{aligned} \tag{4.31}$$

which states that the measure $\mu := f\nu$ is a product measure.

Note that (4.3) and (4.30) imply for the constant $\delta-\gamma$ the formula

$$\frac{1}{\delta-\gamma} = \int \frac{\Theta}{\lambda_2-\lambda_1} d\nu = \int \frac{1}{(\lambda_2-\lambda_1)(w,z)} e^{-\int_\xi^w \frac{\lambda_{1w}}{\lambda_2-\lambda_1}(x,z)dx} d\nu \tag{4.32}$$

and in turn, by differentiating with respect to ξ, that

$$\int \frac{\lambda_{1w}}{(\lambda_2-\lambda_1)^2}(w,z) e^{-\int_\xi^w \frac{\lambda_{1w}}{\lambda_2-\lambda_1}(x,z)dx} d\nu = 0 \tag{4.33}$$

The information in (4.33) is already covered by (3.15), but interestingly the derivation is independent of (3.15).

A. Appendix. Green's formulas for hyperbolic operators

We list here some material concerning Green's formulas for hyperbolic operators (see Sobolev [So]). Consider the hyperbolic operator in two variables

$$\mathcal{L}u = u_{wz} - a u_w - b u_z - c u\,, \tag{A.1}$$

where a, b, c are functions of (w,z), and its adjoint operator

$$\mathcal{L}^T v = v_{wz} + \partial_w(av) + \partial_z(bv) - cv\,. \tag{A.2}$$

One can easily derive the identities

$$\begin{aligned} v\mathcal{L}u - u\mathcal{L}^T v &= \partial_{wz}(uv) - \partial_w\big[u(v_z + av)\big] - \partial_z\big[u(v_w + bv)\big]\\ &= -\partial_{wz}(uv) + \partial_w\big[v(u_z - au)\big] + \partial_z\big[v(u_w - bu)\big]\\ &= \partial_w\big(\frac{1}{2}vu_z - \frac{1}{2}uv_z - auv\big) + \partial_z\big(\frac{1}{2}vu_w - \frac{1}{2}uv_w - buv\big)\,,\end{aligned} \tag{A.3}$$

(A.3) gives rise to various Green type formulas.

Let $SQMP$ be a rectangle in the $w-z$ plane with coordinates $S=(\xi_0,\zeta_0)$, $Q=(\xi_1,\zeta_0)$, $M=(\xi_1,\zeta_1)$ and $P=(\xi_0,\zeta_1)$. Integrating the third identity in (A.3) over the rectangle $SQMP$ we obtain after some integrations by parts the Green formula

$$\begin{aligned}\iint_{SQMP}(v\mathcal{L}u-u\mathcal{L}^Tv) &= vu\Big|_M - vu\Big|_S - \int_Q^M u(v_z+av)dz - \int_P^M u(v_w+bv)dw \\ &\quad - \int_S^P v(u_z-au)dz - \int_S^Q v(u_w-bu)dw\end{aligned} \tag{A.4}$$

Suppose next that SP is a vertical segment on the $w-z$ plane and assume that the function v is of compact support. Integrating the second identity in (A.3) over the strip $\mathbb{R}\times SP$ we obtain

$$\iint_{(-\infty,\infty)\times SP}(v\mathcal{L}u-u\mathcal{L}^Tv) = \int_{-\infty}^{\infty} v(u_w-bu)\Big|_{z=\zeta_1}dw - \int_{-\infty}^{\infty} v(u_w-bu)\Big|_{z=\zeta_0}dw\,. \tag{A.5}$$

Similarly, if SQ is a horizontal segment on the $w-z$ plane and v is of compact support then, upon integrating the second identity in (A.3) over the strip $SQ\times\mathbb{R}$, we obtain

$$\iint_{SQ\times(-\infty,\infty)}(v\mathcal{L}u-u\mathcal{L}^Tv) = \int_{-\infty}^{\infty} v(u_z-au)\Big|_{w=\xi_1}dz - \int_{-\infty}^{\infty} v(u_z-au)\Big|_{w=\xi_0}dz\,. \tag{A.6}$$

Formula (A.4) yields a representation formula for the solution u of the Goursat problem

$$\mathcal{L}u=h(w,z)\ ,\quad \begin{cases} u=F(w) & \text{at } z=\zeta_0 \\ u=G(z) & \text{at } w=\xi_0 \end{cases}\ , \tag{A.7}$$

where $G(\zeta_0)=F(\xi_0)=:u(\xi_0,\zeta_0)$. Define the Riemann function $v=v(w,z;\xi_1,\zeta_1)$ as the solution of the problem

$$\mathcal{L}^Tv=0\ ,\quad \begin{cases} v_z+av=0 & \text{at } w=\xi_1 \\ v_w+bv=0 & \text{at } z=\zeta_1 \end{cases}\ ,\quad v(\xi_1,\zeta_1;\xi_1,\zeta_1)=1\,. \tag{A.8}$$

Then (A.4) gives

(A.9)

$$\begin{aligned}u(\xi_1,\zeta_1) &= v(\xi_0,\zeta_0;\xi_1,\zeta_1)u(\xi_0,\zeta_0) + \int_S^P v(w,z;\xi_1,\zeta_1)(u_z-au)(w,z)\Big|_{w=\xi_0}dz \\ &\quad + \int_S^Q v(w,z;\xi_1,\zeta_1)(u_w-bu)(w,z)\Big|_{z=\zeta_0}dw + \iint_{SQMP} h(w,z)v(w,z;\xi_1,\zeta_1)\,dwdz\,.\end{aligned}$$

Since (ξ_1,ζ_1) is arbitrary, (A.9) provides a representation of the solution to (A.7) in terms of the Goursat data F, G and the Riemann function.

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