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The Principle of the Fermionic
Projector II, Derivation of the Effective
Gauge Group

by

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Abstract

We study the principle of the fermionic projector for the two-point action corresponding to the Lagrangian

$$\mathcal{L}[A] = |A^2|^2 - \mu |A|^4, \quad \mu \in \mathbb{R}$$

and a fermionic projector which in the vacuum is the direct sum of seven identical massive sectors and one massless left-handed sector, each of which is composed of three Dirac seas. It is shown under general assumptions and for an interaction via general chiral and (pseudo)scalar potentials that the sectors spontaneously form pairs, which are referred to as blocks. The resulting so-called effective interaction can be described by chiral potentials corresponding to the effective gauge group

$$SU(2) \otimes SU(3) \otimes U(1)^3.$$

This model has striking similarity to the standard model if the block containing the left-handed sector is identified with the leptons and the three other blocks with the quarks. Namely, the effective gauge fields have the following properties.

- The $SU(3)$ corresponds to an unbroken gauge symmetry. The $SU(3)$ gauge fields couple to the quarks exactly as the strong gauge fields in the standard model.
- The $SU(2)$ potentials are left-handed and couple to the leptons and quarks exactly as the weak gauge potentials in the standard model. Similar to the CKM mixing in the standard model, the off-diagonal components of these potentials must involve a non-trivial mixing of the generations. The $SU(2)$ gauge symmetry is spontaneously broken.
- The $U(1)$ of electrodynamics can be identified with an Abelian subgroup of the effective gauge group.

The effective gauge group is larger than the gauge group of the standard model, but this is not inconsistent because a more detailed analysis of our variational principle should give further constraints for the Abelian gauge potentials. Moreover, there are the following differences to the standard model, which we derive mathematically without working out their physical implications in detail.

- The $SU(2)$ gauge field tensor F must be simple in the sense that $F = \Lambda s$ for a real 2-form Λ and an $su(2)$ -valued function s .
- In the lepton block, the off-diagonal $SU(2)$ gauge potentials are associated with a new type of potential, called nil potential, which couples to the right-handed component.

These results give a strong indication that the principle of the fermionic projector is of physical significance.

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The principle of the fermionic projector is an attempt to formulate physics using a new type of variational principle in space-time. In the first paper dedicated to this principle [1], the main concern was the so-called continuum limit, which gives a procedure for analyzing our variational principles in the setting of relativistic quantum mechanics and classical field theory. In the present second part, we shall work out the continuum limit more specifically under the assumption that the configuration of the fermions is as in the standard model. Our aim is to verify if the interaction between the fermions described by our variational principles is in agreement with the fundamental forces of nature.

We will develop the theory mainly in the example of the model variational principle introduced in [1, Section 2.5]. But our methods apply similarly to other variational principles, and we will try to give an outlook on the general situation as well.

1 The Fermion Configuration of the Standard Model

Guided by the configuration of the leptons and quarks in the standard model, in this chapter we will introduce a continuum fermionic projector which seems appropriate for the formulation of a realistic physical model. In preparation, we begin with the simplified situation where we take into account only the first generation of elementary particles, i.e. the quarks d , u and the leptons e , ν_e . In the vacuum, each of these particles is described by the corresponding Dirac sea (see [1, Section 3.1])

$$P(x, y) = X t_m(x, y) , \quad (1.1)$$

where t_m is the integral over the lower mass shell

$$t_m(x, y) = \int \frac{d^4 k}{(2\pi)^4} (\not{k} + m) \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)} . \quad (1.2)$$

(Θ is the Heaviside function. As in [3, 4], we use small letters for solutions of the Dirac equation and capital letters for the corresponding solutions of the Klein-Gordon equation, i.e. $t_m = (i\not{\partial} + m) T_{m^2}$ with T_{m^2} according to [1, eqn (3.4)].) Here m is the mass of the Dirac particles, and the 4×4 matrix X describes their chirality. More precisely, for the massive particles d , u , and e , X is the identity matrix, whereas for the massless left-handed neutrinos, we choose $X = \chi_L$, where $\chi_{L/R}$ are the chiral projectors $\chi_{L/R} = \frac{1}{2}(1 \mp \gamma^5)$ (we assume as in the standard model that the neutrinos have zero rest mass; massive neutrinos are briefly discussed in Remark 4.17). The simplest fermionic projector involving the Dirac seas of the particles d , u , e , and ν_e is obtained by taking their direct sum, i.e.

$$P(x, y) = \bigoplus_{l=1}^4 X_l t_{m_l}(x, y) \quad (1.3)$$

with $m_1 = m_d$, $m_2 = m_u$, $m_3 = m_e$, $m_4 = 0$ and $X_1 = X_2 = X_3 = \mathbf{1}$, $X_4 = \chi_L$. The spin dimension in (1.3) is $(8, 8)$. Interpreting isometries of the spin scalar product as local gauge transformations (see [1, Section 2.1]), the gauge group is $U(8, 8)$. Clearly, the ordering of the Dirac seas in the direct sum in (1.3) is a pure convention. Nevertheless, our choice is no loss of generality, because any other ordering can be obtained from (1.3) by a suitable global gauge transformation.

In the standard model, the quarks come in three “colors,” with an underlying $SU(3)$ symmetry. We can build in this symmetry here by taking three identical copies of each

quark Dirac sea. This leads us to consider instead of (1.3) the fermionic projector

$$P(x, y) = \bigoplus_{l=1}^f X_l t_{m_l}(x, y) \quad (1.4)$$

with $f = 8$ and $m_1 = m_2 = m_3 = m_d$, $m_4 = m_5 = m_6 = m_u$, $m_7 = m_e$, $m_8 = 0$, and $X_1 = \dots = X_7 = \mathbb{1}$, $X_8 = \chi_L$. Now the spin dimension is $(16, 16)$, and the gauge group is $U(16, 16)$. It is convenient to introduce as in [2] the matrix notation

$$t(x, y) = \bigoplus_{l=1}^f t_{m_l}(x, y), \quad X = \bigoplus_{l=1}^f X_l, \quad Y = \frac{1}{m} \bigoplus_{l=1}^f m_l, \quad (1.5)$$

where $m > 0$ is an arbitrary mass parameter (e.g. one may choose $m = \max_l m_l$). The matrices X and Y are referred to as the *chiral asymmetry matrix* and the *mass matrix*, respectively. With this notation, the fermionic projector of the vacuum (1.4) takes the form

$$P(x, y) = X t(x, y), \quad (1.6)$$

and it satisfies the Dirac equation

$$(i\partial_x - mY) P(x, y) = 0. \quad (1.7)$$

The interaction is introduced as follows. First, in an interacting system the Dirac particles are moving in a bosonic field (e.g. a Yang-Mills field or a gravitational field). To describe this mathematically, we insert into the Dirac equation (1.7) an operator \mathcal{B} which is composed of the bosonic potentials,

$$(i\partial_x + \mathcal{B} - mY) P(x, y) = 0. \quad (1.8)$$

Second, an interacting system in general contains particles and anti-particles. In our formulation, this becomes apparent in the unique decomposition of the fermionic projector of the form

$$P(x, y) = P^{\text{sea}}(x, y) + c \sum_{a=1}^{n_f} |\Psi_a(x) \rangle \langle \Psi_a(y)| - c \sum_{a=1}^{n_a} |\Phi_a(x) \rangle \langle \Phi_a(y)|, \quad (1.9)$$

where $(\Psi_a)_{a=1, \dots, n_f}$ and $(\Phi_a)_{a=1, \dots, n_a}$ are an orthonormal basis for the particle and anti-particle states, respectively, and c is a normalization constant (see [1, eqn (2.24)]). Here the projector P^{sea} is built up of the Dirac seas in the presence of the bosonic field \mathcal{B} . It can be defined via the causal perturbation expansion [2], provided that the Dirac operator is *causality compatible* with the chiral asymmetry, i.e. if

$$X^* (i\partial + \mathcal{B} - mY) = (i\partial + \mathcal{B} - mY) X, \quad (1.10)$$

where “*” denotes the adjoint with respect to the spin scalar product (see [2] for details; we also remark that this condition will be weakened later on, see Def. 4.1). The condition (1.10) gives a constraint for the interaction of the neutrinos with the other fermions. Finally, the light-cone expansion [3] is a powerful tool for analyzing P^{sea} in position space (cf. the short discussion in [1, Section 1.5]).

Let us now consider the realistic situation of three generations. Grouping the elementary particles according to their lepton number and isospin, we get the four families

(d, s, b) , (u, c, t) , $(\nu_e, \nu_\mu, \nu_\tau)$, and (e, μ, τ) . In the standard model, the particles within each family couple in the same way to the gauge fields. If one thinks of a Dirac equation of type (1.8), this means that the bosonic potentials contained in the operator \mathcal{B} should act similarly on the Dirac seas of each family. This can be arranged by taking the (ordinary) sum of these Dirac seas. Thus we define the *fermionic projector of the vacuum* by

$$P(x, y) = \bigoplus_{n=1}^8 \sum_{\alpha=1}^3 X_n t_{m_{n\alpha}}(x, y) \quad (1.11)$$

with $X_1 = \dots = X_7 = \mathbf{1}$ and $X_8 = \chi_L$; furthermore $m_{11} = m_{21} = m_{31} = m_d$, $m_{12} = m_{22} = m_{32} = m_s$, $m_{13} = m_{23} = m_{33} = m_b$, $m_{41} = m_{51} = m_{61} = m_u$, \dots , $m_{71} = m_e$, $m_{72} = m_\mu$, $m_{73} = m_\tau$, and $m_{81} = m_{82} = m_{83} = 0$. We refer to the direct summands in (1.11) as *sectors*. The spin dimension in (1.11) is again $(16, 16)$.

Since each sector is composed of states of the three (in general different) masses $m_{n\alpha}$, $\alpha = 1, 2, 3$, the fermionic projector (1.11) cannot be written as a solution of a Dirac equation of the form (1.7), and thus the methods of [2, 3] for describing the interaction cannot be applied. To get around this problem, we proceed as follows. We first introduce as an auxiliary object a new fermionic projector where we replace the ordinary sum in (1.11) by a direct sum. Thus combining the indices n and α to one index $l = (n\alpha)$, which takes the values $l = 1, \dots, 8 \times 3 = 24 =: f$, this *auxiliary fermionic projector* is again given by (1.4). In matrix notation (1.5), it satisfies the auxiliary Dirac equation (1.7). In order to describe the case with interaction, we can according to (1.8) insert a perturbation operator \mathcal{B} into the auxiliary Dirac equation. Notice that \mathcal{B} acts as a matrix on the 24 components of the direct sum; we write for clarity $\mathcal{B} = \mathcal{B}_{(b\beta)}^{(a\alpha)}$ with $a, b = 1, \dots, 8$ and $\alpha, \beta = 1, 2, 3$. Under the assumption that the auxiliary Dirac operator is causality compatible (1.10), the results of [2, 3] apply and yield a decomposition of the auxiliary fermionic projector of the form (1.9) as well as formulas for the light-cone expansion of P^{sea} . In order to relate these results to the original fermionic projector (1.11), we must reduce the number of components of the direct sum to get back to eight sectors. To this end, we sum over the generations, i.e. we define the fermionic projector with interaction $(P_b^a)_{a,b=1,\dots,8}$ by

$$P_b^a(x, y) = \sum_{\alpha, \beta=1}^3 P_{(b\beta)}^{(a\alpha)}(x, y). \quad (1.12)$$

We refer to the summations in (1.12) as the *partial trace* over the generations.

With (1.4) and (1.12), we have introduced the fermionic projector according to the fermion configuration in the standard model. The only free parameters are the 9 masses of the elementary leptons and quarks. The operator \mathcal{B} which describes the interaction between the fermions is constrained by the condition that the Dirac operator should be causality compatible (1.10). But we point out that, apart from this mathematical consistency condition, the operator \mathcal{B} can be arbitrary. Thus in contrast to the standard model, we do not put in the structure of the fundamental interactions here, i.e. we do not specify the gauge groups, the coupling of the gauge fields to the fermions, the coupling constants, the CKM matrix, the Higgs mechanism, the masses of the W - and Z -bosons, etc. The reason is that in our description, the form of the physical interactions is to be described and determined by our variational principle in space-time.

2 The Euler-Lagrange Equations in the Vacuum

Following [1, Section 2.5], we want to describe the physical interactions using a variational principle in discrete space-time. In this chapter, we shall consider the general two-point action [1, eqn (2.35)]

$$S = \sum_{x,y \in M} \mathcal{L}[P(x,y) P(y,x)] \quad (2.1)$$

and study for which Lagrangians \mathcal{L} the corresponding Euler-Lagrange (EL) equations are satisfied in the vacuum (a problem arising for actions other than two-point actions will be discussed in Remark 3.9 below). We first derive the EL equations corresponding to (2.1). We set

$$A_{xy} = P(x,y) P(y,x) \quad (2.2)$$

and for simplicity often omit the subscript “ xy ” in what follows. In a gauge, A is represented by a $4N \times 4N$ matrix, with $N = 8$ for the fermion configuration of the standard model (1.11). We write the matrix components with Greek indices, $A = (A_{\beta}^{\alpha})_{\alpha,\beta=1,\dots,4N}$. The Lagrangian in (2.1) is a functional on $4N \times 4N$ matrices. Denoting its gradient by \mathcal{M} ,

$$\mathcal{M}[A] = (\mathcal{M}[A]_{\beta}^{\alpha})_{\alpha,\beta=1,\dots,4N} \quad \text{with} \quad \mathcal{M}[A]_{\beta}^{\alpha} = \frac{\partial \mathcal{L}[A]}{\partial A_{\alpha}^{\beta}},$$

the variation of \mathcal{L} is given by

$$\delta \mathcal{L}[A] = \sum_{\alpha,\beta=1}^{4N} \mathcal{M}[A]_{\beta}^{\alpha} \delta A_{\alpha}^{\beta} = \text{Tr}(\mathcal{M}[A] \delta A), \quad (2.3)$$

where “Tr” denotes the trace of $4N \times 4N$ matrices. Summing over x and y yields the variation of the action,

$$\delta S = \sum_{x,y \in M} \delta \mathcal{L}[A_{xy}] = \sum_{x,y \in M} \text{Tr}(\mathcal{M}[A_{xy}] \delta A_{xy}).$$

We substitute in the identity

$$\delta A_{xy} = \delta P(x,y) P(y,x) + P(x,y) \delta P(y,x) \quad (2.4)$$

and use the symmetry $x \leftrightarrow y$ as well as the fact that the trace is cyclic to obtain

$$\delta S = 4 \sum_{x,y \in M} \text{Tr}(Q(x,y) \delta P(y,x)) \quad (2.5)$$

with

$$Q(x,y) = \frac{1}{4} (\mathcal{M}[A_{xy}] P(x,y) + P(x,y) \mathcal{M}[A_{yx}]). \quad (2.6)$$

We also write (2.5) in the compact form

$$\delta S = 4 \text{tr}(Q \delta P), \quad (2.7)$$

where “tr” denotes the trace in the scalar product space H , and Q is the operator on H with kernel (2.6). Exactly as in [1, Section 2.5], we consider unitary variations of P with finite support, i.e. [1, eqn (2.43)]

$$\delta P = i [B, P],$$

where B is a Hermitian operator of finite rank. Substituting into (2.7) and cyclically commuting the operators in the trace yields that

$$\delta S = 4i \operatorname{tr}(Q [B, P]) = 4i \operatorname{tr}([P, Q] B) .$$

Since B is arbitrary, we conclude that

$$[P, Q] = 0 \tag{2.8}$$

with Q according to (2.6). These are the EL equations.

2.1 The Spectral Decomposition of $P(x, y) P(y, x)$

As outlined in [1, Section 2.5] in a model example, the EL equations (2.8) can be analyzed using the spectral decomposition of A_{xy} . On the other hand, it was explained in [1, Section 3.3] that A should be looked at in an expansion about the light cone. We shall now combine these methods and compute the eigenvalues and spectral projectors of A using the formalism of [1, Section 3.5].

Since the fermionic projector of the vacuum (1.11) is a direct sum, we can study the eight sectors separately. We first consider the *neutrino sector* $n = 8$, i.e.

$$P(x, y) = \sum_{\alpha=1}^3 \chi_L t_{m_\alpha} \quad \text{with} \quad m_\alpha = 0.$$

If we assume that the regularized Dirac seas have a vector-scalar structure [1, eqn (3.4)] and regularize as explained after [1, eqn (3.76)], the regularized fermionic projector, which with a slight abuse of notation we denote again by $P(x, y)$, is of the form

$$P(x, y) = \chi_L g_j(x, y) \gamma^j \tag{2.9}$$

with suitable functions g_j . Since P is Hermitian, $P(y, x)$ is given by

$$P(y, x) = P(x, y)^* = \chi_L \overline{g_j(x, y)} \gamma^j .$$

Omitting the arguments (x, y) of the functions g_j , we obtain for the 4×4 matrix A

$$A = \chi_L \not{g} \chi_L \bar{\not{g}} = \chi_L \chi_R \not{g} \bar{\not{g}} = 0 . \tag{2.10}$$

Hence in the neutrino sector, A_{xy} is identically equal to zero. We refer to cancellations like in (2.10), which come about because the neutrino sector contains only left-handed particles, as *chiral cancellations*.

Next we consider the *massive sectors* $n = 1, \dots, 7$ in (1.11), i.e.

$$P(x, y) = \sum_{\alpha=1}^3 t_{m_\alpha} . \tag{2.11}$$

Again assuming that the regularized Dirac seas have a vector-scalar structure, the regularized fermionic projector is

$$P(x, y) = g_j(x, y) \gamma^j + h(x, y) \tag{2.12}$$

with suitable functions g_j and h . Using that P is Hermitian,

$$P(y, x) = \overline{g_j(x, y)} \gamma^j + \overline{h(x, y)}. \quad (2.13)$$

Again omitting the arguments (x, y) , we obtain for the 4×4 matrix A_{xy}

$$A = \not{g} \bar{\not{g}} + h \bar{\not{g}} + \not{g} \bar{h} + h \bar{h}. \quad (2.14)$$

It is useful to decompose A in the form

$$A = A_1 + A_2 + \mu$$

with

$$A_1 = \frac{1}{2} [\not{g}, \bar{\not{g}}], \quad A_2 = h \bar{\not{g}} + \not{g} \bar{h}, \quad \mu = g \bar{g} + h \bar{h}$$

and $g \bar{g} \equiv g_j \bar{g}^j$. Namely, A_1 and A_2 anti-commute, and thus

$$(A - \mu)^2 = A_1^2 + A_2^2 = (g \bar{g})^2 - g^2 \bar{g}^2 + (g \bar{h} + h \bar{g})^2. \quad (2.15)$$

The right of (2.15) is a multiple of the identity matrix, and so (2.15) is a quadratic equation for A . The roots λ_{\pm} of this equation,

$$\lambda_{\pm} = g \bar{g} + h \bar{h} \pm \sqrt{(g \bar{g})^2 - g^2 \bar{g}^2 + (g \bar{h} + h \bar{g})^2}, \quad (2.16)$$

are the zeros of the characteristic polynomial of A . However, we must be careful about associating eigenspaces to λ_{\pm} because A need not be diagonalizable. Let us first consider the case that the two eigenvalues in (2.16) are distinct. If we assume that A is diagonalizable, then λ_{\pm} are the two eigenvalues of A , and the corresponding spectral projectors F_{\pm} are computed to be

$$F_{\pm} = \frac{\mathbb{1}}{2} \pm \frac{1}{\lambda_1 - \lambda_2} \left(A - \frac{1}{2} (\lambda_1 + \lambda_2) \mathbb{1} \right) \quad (2.17)$$

$$= \frac{\mathbb{1}}{2} \pm \frac{\frac{1}{2} [\not{g}, \bar{\not{g}}] + h \bar{\not{g}} + \not{g} \bar{h}}{2 \sqrt{(g \bar{g})^2 - g^2 \bar{g}^2 + (g \bar{h} + h \bar{g})^2}}. \quad (2.18)$$

The explicit formula (2.18) even implies that A is diagonalizable. Namely, a short calculation yields that

$$A F_{\pm} = \lambda_{\pm} F_{\pm} \quad \text{and} \quad F_+ + F_- = \mathbb{1},$$

proving that the image of F_+ and F_- are indeed eigenspaces of A which span \mathbb{C}^4 . Moreover, a short computation using (2.12) and (2.18) yields that

$$F_{\pm} P(x, y) = \frac{\not{g} + h}{2} \pm \frac{\not{g} (g \bar{g} + h \bar{h}) - \bar{\not{g}} (g^2 - h^2) + (g \bar{g}) h + g^2 \bar{h}}{2 \sqrt{(g \bar{g})^2 - g^2 \bar{g}^2 + (g \bar{h} + h \bar{g})^2}}. \quad (2.19)$$

Writing out for clarity the dependence on x and y , the spectral decomposition of A is

$$A_{xy} = \sum_{s=\pm} \lambda_s^{xy} F_s^{xy}. \quad (2.20)$$

The following lemma relates the spectral representation of A_{xy} to that of A_{yx} .

Lemma 2.1

$$\lambda_{\pm}^{xy} = \lambda_{\mp}^{yx} \quad (2.21)$$

$$F_{\pm}^{xy} P(x, y) = P(x, y) F_{\mp}^{yx}, \quad (2.22)$$

Proof. According to (2.12) and (2.13), A_{yx} is obtained from A_{xy} by the transformations $g_j \leftrightarrow \overline{g_j}$, $h \leftrightarrow \overline{h}$. The eigenvalues (2.16) are invariant under these transformations. Our convention for labelling the eigenvalues is (2.21). Using this convention, we obtain from (2.17) that

$$\begin{aligned} F_{\pm}^{xy} P(x, y) &= \frac{1}{2} P(x, y) \pm \frac{1}{\lambda_1^{xy} - \lambda_2^{xy}} \left(A_{xy} P(x, y) - \frac{1}{2} (\lambda_1^{xy} + \lambda_2^{xy}) P(x, y) \right) \\ P(x, y) F_{\mp}^{yx} &= \frac{1}{2} P(x, y) \pm \frac{1}{\lambda_1^{xy} - \lambda_2^{xy}} \left(P(x, y) A_{yx} - \frac{1}{2} (\lambda_1^{xy} + \lambda_2^{xy}) P(x, y) \right). \end{aligned}$$

The identity $P(x, y) A_{yx} = P(x, y) P(y, x) P(x, y) = A_{xy} P(x, y)$ yields (2.22). \blacksquare

If the eigenvalues in (2.16) are equal, the matrix A need not be diagonalizable (namely, the right side of (2.15) may be zero without (2.14) being a multiple of the identity matrix). We shall treat this degenerate case by taking the limits $\lambda_+ - \lambda_- \rightarrow 0$ in the spectral representation (2.20).

Before going on, we point out that according to (2.16), the 4×4 matrix A has at most two distinct eigenvalues. In order to understand better how this degeneracy comes about, it is useful to consider the space V of real vectors which are orthogonal to g_j and $\overline{g_j}$,

$$V = \{v \mid v_j g^j = 0 = v_j \overline{g^j}\}.$$

Since we must satisfy two conditions in four dimensions, $\dim V \geq 2$. Furthermore, a short calculation using (2.14) shows that for every $v \in V$,

$$[A, v_j \gamma^j \gamma^5] = 0. \quad (2.23)$$

Thus the eigenspaces of A are invariant subspaces of the operators $v_j \gamma^j \gamma^5$. In the case when the two eigenvalues (2.16) are distinct, the family of operators $(v_j \gamma^j \gamma^5)_{v \in V}$ acts transitively on the two-dimensional eigenspaces of A . Notice that the operators $v_j \gamma^j \gamma^5$ map left-handed spinors into right-handed spinors and vice versa. Thus one may regard (2.23) as describing a symmetry between the left- and right-handed component of A . We refer to the fact that A has at most two distinct eigenvalues as the *chiral degeneracy* of the massive sectors in the vacuum.

Our next step is to rewrite the spectral representation using the formalism of [1, Section 3.5]. Expanding in powers of m and regularizing gives for a Dirac sea t_m the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i \not{g}}{2} T_{[2n]}^{(n-1)}(x, y) + T_{[2n+1]}^{(n)}(x, y) \right). \quad (2.24)$$

In composite expressions, one must carefully keep track that every factor ξ is associated to a corresponding factor $T_{[p]}^{(n)}$. In [1, Section 3.6] this was accomplished by putting a bracket around both factors. In order to have a more flexible notation, we here allow the two factors to be written separately, but in this case the pairing is made manifest by adding

an index $^{(n)}$, and if necessary also a subscript $_{[r]}$, to the factors ξ . With this notation, the contraction rules [1, eqns (3.93)–(3.95)] can be written as

$$(\xi_{[r_1]}^{(n_1)})_j (\xi_{[r_2]}^{(n_2)})^j = \frac{1}{2} (z_{[r_1]}^{(n_1)} + z_{[r_2]}^{(n_2)}) , \quad (\xi_{[r_1]}^{(n_1)})_j (\overline{\xi_{[r_2]}^{(n_2)}})^j = \frac{1}{2} (z_{[r_1]}^{(n_1)} + \overline{z_{[r_2]}^{(n_2)}}) \quad (2.25)$$

(and similar for the complex conjugates), where we introduced factors $z_{[r]}^{(n)}$ which by definition combine with the corresponding factor $T_{[r]}^{(n)}$ according to

$$z_{[r]}^{(n)} T_{[r]}^{(n)} = -4 (n T_{[r]}^{(n+1)} + T_{\{r\}}^{(n+2)}) + (\text{smooth functions}) . \quad (2.26)$$

In our calculations, most separate factors ξ and z will be associated to $T_{[0]}^{(-1)}$. Therefore, we shall in this case often omit the indices, i.e.

$$\xi \equiv \xi_{[0]}^{(-1)} , \quad z \equiv z_{[0]}^{(-1)} .$$

We point out that the calculation rule (2.26) is valid only modulo smooth functions. This is because in [1] we analyzed the effects of the ultraviolet regularization, but disregarded the “regularization” for small momenta related to the logarithmic mass problem. However, this is not a problem because the smooth contribution in (2.26) is easily determined from the behavior away from the light cone, where the factors $T_{\bullet}^{(n)}$ are known smooth functions and $z_{[r]}^{(n)} = \xi^2$.

Each summand in (2.11) is regularized according to (2.24). However, the regularization functions involved may be different for each Dirac sea. Introducing new regularization functions, we can write the sum of the three Dirac seas again in the form (2.24). More precisely, in the integrands of [1, eqns (3.77)–(3.79)] we make the following replacements,

$$\left. \begin{aligned} h a^{\frac{p-1}{2}} &\rightarrow \sum_{\alpha=1}^3 h_{\alpha} a_{\alpha}^{\frac{p-1}{2}} , & h a^{\frac{p-1}{2}} b &\rightarrow \sum_{\alpha=1}^3 h_{\alpha} a_{\alpha}^{\frac{p-1}{2}} b_{\alpha} \\ g a^{\frac{p}{2}} &\rightarrow \sum_{\alpha=1}^3 g_{\alpha} a_{\alpha}^{\frac{p}{2}} , & g a^{\frac{p}{2}} b &\rightarrow \sum_{\alpha=1}^3 g_{\alpha} a_{\alpha}^{\frac{p}{2}} b_{\alpha} \end{aligned} \right\} \quad (2.27)$$

As mentioned in [1, Remark 3.1], the calculation rules for monomials are valid also when different regularization functions are involved. This implies that the contraction rules (2.25),(2.26) are valid for the sums of Dirac seas as well. Using (2.24) and the contraction rules, we can expand the spectral decomposition around the singularities on the light cone. Our expansion parameter is the *degree on the light cone*, also denoted by “deg”. It is defined by

$$\deg(T_{\bullet}^{(n)}) = 1 - n , \quad \deg(z^{(n)}) = -1 ,$$

and the degree of a function which is smooth and non-zero on the light cone is set to zero. The degree of a product is obtained by adding the degrees of all factors, and of a quotient by taking the difference of the degrees of the numerator and denominator. The leading contribution to the eigenvalues is computed as follows,

$$\begin{aligned} g\bar{g} + h\bar{h} &= \frac{1}{4} \left((\xi_j T_{[0]}^{(-1)}) (\overline{\xi^j T_{[0]}^{(-1)}}) + (\xi_j T_{[0]}^{(-1)}) (\overline{\xi^j T_{[2]}^{(0)}}) + (\xi_j T_{[2]}^{(0)}) (\overline{\xi^j T_{[0]}^{(-1)}}) \right) \\ &\quad + (\deg < 3) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} (z + \bar{z}) T_{[0]}^{(-1)} \overline{T_{[0]}^{(-1)}} + (\deg < 3) \\
&= \frac{1}{2} \left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} + T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right) + (\deg < 3) \\
(g\bar{g})^2 - g^2 \bar{g}^2 &= \frac{1}{16} \left((\xi_j T_{[0]}^{(-1)}) (\overline{\xi^j T_{[0]}^{(-1)}}) \right)^2 - \frac{1}{16} (\xi T_{[0]}^{(-1)})^2 (\overline{\xi T_{[0]}^{(-1)}})^2 + (\deg < 6) \\
&= \frac{1}{4} \left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} + T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right)^2 - T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} + (\deg < 6) \\
&= \frac{1}{4} \left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right)^2 + (\deg < 6) \\
(g\bar{h} + h\bar{g})^2 &= \frac{1}{4} \left((i\xi T_{[0]}^{(-1)}) \overline{T_{[1]}^{(0)}} + T_{[1]}^{(0)} (\overline{i\xi T_{[0]}^{(-1)}}) \right)^2 + (\deg < 6) = (\deg < 6),
\end{aligned}$$

and thus

$$\begin{aligned}
\lambda_{\pm} &= \frac{1}{2} \left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} + T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right) + (\deg < 3) \\
&\quad \pm \frac{1}{2} \sqrt{\left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right)^2 + (\deg < 6)} \\
&= \frac{1}{2} \left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} + T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right) + (\deg < 3) \\
&\quad \pm \frac{1}{2} \left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right) \pm \frac{(\deg < 6)}{4 \left(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \right)} + \dots \\
&= \begin{cases} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} & \text{for “+”} \\ T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} & \text{for “-”} \end{cases} + (\deg < 3). \tag{2.28}
\end{aligned}$$

The spectral projectors are calculated similarly,

$$F_{\pm} = \frac{1}{2} \pm \frac{\frac{1}{8} [\not{\xi} T_{[0]}^{(-1)}, \overline{\not{\xi} T_{[0]}^{(-1)}}] - \frac{i}{2} \left(T_{[1]}^{(0)} (\overline{\not{\xi} T_{[0]}^{(-1)}}) - (\not{\xi} T_{[0]}^{(-1)}) \overline{T_{[1]}^{(0)}} \right)}{T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}}} + (\deg < 0) \tag{2.29}$$

$$\begin{aligned}
F_{\pm} P(x, y) &= \frac{i}{4} (\not{\xi} T_{[0]}^{(-1)}) + (\deg < 2) \\
&\quad \pm \frac{i}{4} \frac{(\not{\xi} T_{[0]}^{(-1)}) (T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} + T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}}) - 2 (\not{\xi} T_{[0]}^{(-1)}) T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}}}{T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}}}. \tag{2.30}
\end{aligned}$$

The last expression contains inner factors ξ . In situations when these factors are not contracted to other inner factors in a composite expression, we can treat them as outer factors. Then (2.30) simplifies to

$$F_{\pm} P(x, y) = \begin{cases} 0 & \text{for “+”} \\ \frac{i}{2} \not{\xi} T_{[0]}^{(-1)} & \text{for “-”} \end{cases} + (\deg < 2). \tag{2.31}$$

By expanding, one can compute the eigenvalues and spectral projectors also to lower degree on the light cone. We do not want to enter the details of this calculation here because in this chapter we only need that the lower degrees involve the masses of the Dirac seas. This is illustrated by the following expansion of the eigenvalues,

$$\begin{aligned} \lambda_{\pm} = \frac{1}{4} \times & \begin{cases} (z T_{[0]}^{(-1)}) \overline{T_{[0]}^{(-1)}} + (z T_{[2]}^{(0)}) \overline{T_{[0]}^{(-1)}} + (z T_{[0]}^{(-1)}) \overline{T_{[2]}^{(0)}} & \text{for “+”} \\ T_{[0]}^{(-1)} \overline{(z T_{[0]}^{(-1)})} + T_{[0]}^{(-1)} \overline{(z T_{[2]}^{(0)})} + T_{[2]}^{(0)} \overline{(z T_{[0]}^{(-1)})} & \text{for “-”} \end{cases} \\ & + T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \mp \frac{T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}}}{T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}}} (T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}}) + (\deg < 2) . \end{aligned} \quad (2.32)$$

Similarly, the contributions to degree < 2 involve even higher powers of the masses.

Let us specify how we can give the above spectral decomposition of A_{xy} mathematical meaning. A priori, our formulas for λ_{\pm} and F_{\pm} are only formal expressions because the formalism of [1, Section 3.5] applies to monomials in $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$, but dividing by $(T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}})$ is not a well-defined operation. In order to make mathematical sense of the spectral decomposition and in order to ensure at the same time that the EL equations have a well-defined continuum limit, we shall only consider Lagrangians for which all relevant expressions obtained by substituting the spectral representation of A into the EL equations are polynomials in $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$. Under this assumption, working with the spectral representation of A_{xy} can be regarded merely as a convenient formalism for handling the the EL equations, the latter being well-defined according to [1, Chapter 3]. Having Lagrangians of this type in mind, we can treat A in the massive sectors as a diagonalizable matrix with two distinct eigenvalues λ_{\pm} and corresponding spectral projectors F_{\pm} .

The explicit formulas (2.28) and (2.29) show that the eigenvalues of A are to leading degree not real, but appear in *complex conjugate pairs*, i.e.

$$\overline{\lambda_+} = \lambda_- \quad \text{and} \quad F_+^* = F_- , \quad (2.33)$$

where “*” denotes the adjoint with respect to the spin scalar product. If one considers perturbations of these eigenvalues by taking into account the contributions of lower degree, λ_+ and λ_- will clearly still be complex. As explained after [1, eqn (2.41)], this implies that the relations (2.33) remain valid (as is e.g. the case in (2.32)). We conclude that in our expansion about the singularities on the light cone, the eigenvalues appear to every degree in complex conjugate pairs (2.33).

We finally summarize the results obtained in the neutrino and massive sectors and introduce a convenient notation for the eigenvalues and spectral projectors of A_{xy} . We found that within the formalism of [1, Section 3.6], A can be treated as a diagonalizable matrix. We denote the distinct eigenvalues of A by $(\lambda_k)_{k=1,\dots,K}$ and the corresponding spectral projectors by F_k . Since A vanishes in the neutrino sector, zero is an eigenvalue of A of multiplicity four; we choose the numbering such that $\lambda_1 = 0$. Due to the chiral degeneracy, all eigenspaces are at least two-dimensional. Furthermore, all non-zero eigenvalues of A are complex and appear in complex conjugate pairs (2.33). It is useful to also consider the eigenvalues counting their multiplicities. We denote them by $(\lambda_{\alpha})_{\alpha=1,\dots,4N}$ or also by $(\lambda_{ncs})_{n=1,\dots,8, c=L/R, s=\pm}$, where n refers to the sectors and c, s count the eigenvalues

within each sector. More precisely,

$$\lambda_{8a} = 0 \quad \text{and} \quad \lambda_{nc\pm} = \lambda_{\pm}^{(n)}, \quad n = 1, \dots, 7 \quad (2.34)$$

with $\lambda_{\pm}^{(n)}$ as given by (2.16) or (2.32), whereby the index “ (n) ” emphasizes that the eigenvalues λ_{\pm} depend on the regularization functions in the corresponding sector.

2.2 Pointwise Spectral Analysis of the Euler-Lagrange Equations

In this Section, we shall derive conditions which ensure that the EL equations (2.8),(2.6) are satisfied in the vacuum and argue why we want to choose our Lagrangian in such a way that these sufficient conditions are fulfilled. Since the Lagrangian $\mathcal{L}[A]$ must be independent of the matrix representation of A , it depends only on the eigenvalues λ_{α} ,

$$\mathcal{L}[A_{xy}] = \mathcal{L}(\lambda_1^{xy}, \dots, \lambda_{4N}^{xy}),$$

and furthermore $\mathcal{L}(\lambda_1^{xy}, \dots, \lambda_{4N}^{xy})$ is symmetric in its arguments. In preparation, we first consider the case when the eigenvalues of A are non-degenerate. Then the variation of the eigenvalues is given in first order perturbation theory by $\delta\lambda_{\alpha} = \text{Tr}(F_{\alpha} \delta A)$. Let us assume that \mathcal{L} depends smoothly on the λ_{α} , but is not necessarily holomorphic (in particular, \mathcal{L} is allowed to be a polynomial in $|\lambda_{\alpha}|^2$). Then

$$\delta L = \sum_{\alpha=1}^{4N} \left(\frac{\partial \mathcal{L}(\lambda)}{\partial \text{Re} \lambda_{\alpha}} \text{Re Tr}(F_{\alpha} \delta A) + \frac{\partial \mathcal{L}(\lambda)}{\partial \text{Im} \lambda_{\alpha}} \text{Im Tr}(F_{\alpha} \delta A) \right) \quad (2.35)$$

$$= \text{Re} \sum_{\alpha=1}^{4N} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_{\alpha}} \text{Tr}(F_{\alpha} \delta A), \quad (2.36)$$

where we set

$$\begin{aligned} \frac{\partial \mathcal{L}(\lambda)}{\partial \text{Re} \lambda_{\alpha}} &= \lim_{\mathbb{R} \ni \varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mathcal{L}(\lambda_1, \dots, \lambda_{\alpha-1}, \lambda_{\alpha} + \varepsilon, \lambda_{\alpha+1}, \dots, \lambda_{4N}) - \mathcal{L}(\lambda_1, \dots, \lambda_{4N})) \\ \frac{\partial \mathcal{L}(\lambda)}{\partial \text{Im} \lambda_{\alpha}} &= \lim_{\mathbb{R} \ni \varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mathcal{L}(\lambda_1, \dots, \lambda_{\alpha-1}, \lambda_{\alpha} + i\varepsilon, \lambda_{\alpha+1}, \dots, \lambda_{4N}) - \mathcal{L}(\lambda_1, \dots, \lambda_{4N})) \end{aligned}$$

and

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_{\alpha}} \equiv \frac{\partial \mathcal{L}(\lambda)}{\partial \text{Re} \lambda_{\alpha}} - i \frac{\partial \mathcal{L}(\lambda)}{\partial \text{Im} \lambda_{\alpha}}. \quad (2.37)$$

If some of the λ_{α} s coincide, we must apply perturbation theory with degeneracies. One obtains in generalization of (2.36) that

$$\delta L = \text{Re} \sum_{k=1}^K \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_{\alpha}} \text{Tr}(F_k \delta A) \Big|_{\lambda_{\alpha}=\lambda_k}. \quad (2.38)$$

Here our notation means that we choose the index α such that $\lambda_{\alpha} = \lambda_k$. Clearly, α is not unique iff λ_k is a degenerate eigenvalue; in this case α can be chosen arbitrarily due to the symmetry of \mathcal{L} . We also write (2.38) in the shorter form

$$\delta L = \text{Re} \sum_{k=1}^K \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_k} \text{Tr}(F_k \delta A). \quad (2.39)$$

In this formula it is not necessary to take the real part. Namely, as the Lagrangian is real and symmetric in its arguments, we know that $\overline{\mathcal{L}(\lambda_1, \dots, \lambda_{4N})} = \mathcal{L}(\overline{\lambda_1}, \dots, \overline{\lambda_{4N}})$, and thus according to (2.37),

$$\overline{\frac{\partial \mathcal{L}(\lambda_1, \dots, \lambda_{4N})}{\partial \lambda_k}} = \frac{\partial \mathcal{L}(\overline{\lambda_1}, \dots, \overline{\lambda_{4N}})}{\partial \overline{\lambda_k}}.$$

Using furthermore that the eigenvalues of A appear in complex conjugate pairs (2.33), one sees that the operator

$$\sum_{k=1}^K \frac{\partial \mathcal{L}}{\partial \lambda_k} F_k$$

is Hermitian. Hence we can write the sum in (2.39) as the trace of products of two Hermitian operators on H , being automatically real. We conclude that

$$\delta L = \sum_{k=1}^K \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_k} \text{Tr}(F_k \delta A). \quad (2.40)$$

Comparing (2.40) with (2.3) gives

$$\mathcal{M}[A_{xy}] = \sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_k^{xy}} F_k^{xy}. \quad (2.41)$$

We substitute this identity into (2.6) and apply Lemma 2.1 in each sector to obtain

$$Q(x, y) = \frac{1}{2} \sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}} F_k^{xy} P(x, y) = \frac{1}{2} \sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}} P(x, y) F_k^{yx}. \quad (2.42)$$

Using these relations, we can write the EL equations (2.8) as

$$\int d^4 z \left(P(x, z) P(z, y) \sum_{k=1}^{K_{zy}} \frac{\partial \mathcal{L}(\lambda^{zy})}{\partial \lambda_k^{zy}} F_k^{yz} - \sum_{k=1}^{K_{xz}} \frac{\partial \mathcal{L}(\lambda^{xz})}{\partial \lambda_k^{xz}} F_k^{xz} P(x, z) P(z, y) \right) = 0. \quad (2.43)$$

This equation splits into separate equations on the eight sectors. In the neutrino sector, we have according to (2.9),

$$P(x, z) P(z, y) = \chi_L \not{g}(x, z) \chi_L \not{g}(z, y) = \chi_L \chi_R \not{g}(x, z) \not{g}(z, y) = 0. \quad (2.44)$$

Since both summands in (2.43) contain a factor $P(x, z) P(z, y)$, the EL equations are trivially satisfied in the neutrino sector due to chiral cancellations. In the massive sectors, there are no chiral cancellations. As shown in [1, Appendix D], there are no further cancellations in the commutator if generic perturbations of the physical system are taken into account (see also [1, Section 3.6]). This means that (2.43) will be satisfied if and only if Q vanishes in the massive sectors, i.e.

$$\sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}} F_k^{xy} P(x, y) X X^* = 0. \quad (2.45)$$

As explained on page 12, we shall only consider Lagrangians for which (2.45) is a linear combination of monomials in $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$. Thus we can evaluate (2.45) weakly near the

light cone and apply the integration-by-parts rules [1, eqn (3.107)]. However, it is reasonable to impose that (2.45) should be satisfied even *pointwise* (i.e. without weak evaluation), as the following arguments show. First, one should keep in mind that the integration-by-parts rules are valid only to leading order in $(lE_P)^{-1}$. As is worked out in [1, Appendix C], the restriction to the leading order in $(lE_P)^{-1}$ is crucial when perturbations by the bosonic fields are considered (basically because the microscopic form of the bosonic potentials is unknown). But in the vacuum, one can consider the higher orders in $(lE_P)^{-1}$ as well (see the so-called regularization expansion in [1, Section 3.4]). Therefore, it is natural to impose that in the vacuum the EL equations should be satisfied to all orders in $(lE_P)^{-1}$. Then the integration-by-parts rules do not apply, and weak evaluation becomes equivalent to pointwise evaluation. A second argument in favor of a pointwise analysis is that even if we restricted attention to the leading order in $(lE_P)^{-1}$ and allowed for integrating by parts, this would hardly simplify the equations (2.45), because the integration-by-parts rules depend on the indices n of the involved factors $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$. More precisely, the relations between the monomials given by the integration-by-parts rules are different to every degree, and thus the conditions (2.45) could be satisfied only by imposing to every degree conditions on the regularization parameters. It seems difficult to satisfy all these extra conditions with our finite number of regularization functions. Clearly, this last argument does not rule out the possibility that there might be a Lagrangian together with a special regularization such that (2.45) is satisfied to leading order in $(lE_P)^{-1}$ only after applying the integration-by-parts rules. But apart from not being quite satisfactory due to our first argument, such Lagrangians are certainly difficult to handle, and we shall not consider them here.

For these reasons, we here restrict attention to Lagrangians for which (2.45) is satisfied pointwise. Then (2.45) simplifies to the conditions

$$\frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_{nc}^{xy}} = 0 \quad \text{for } n = 1, \dots, 7. \quad (2.46)$$

2.3 Motivation of the Lagrangian, the Mass Degeneracy Assumption

Let us discuss the conditions (2.46) in concrete examples. We begin with the class of Lagrangians which are *polynomial* in the eigenvalues λ_{α} of A . Since different powers of λ_{α} have a different degree on the light cone, there cannot be cancellations between them. Thus it suffices to consider polynomials which are *homogeneous of degree h* , $h \geq 1$. Furthermore, as the Lagrangian should be independent of the matrix representation of A , it can be expressed in terms of traces of powers of A . Thus we consider Lagrangians of the form

$$\mathcal{L}[A] = \mathcal{P}_h[A], \quad (2.47)$$

where \mathcal{P}_l denotes a polynomial in $\text{Tr}(A^p)$ homogeneous in A of degree l , i.e.

$$\mathcal{P}_l = \sum_n c_n R_{p_1} \cdots R_{p_{\max(n)}} \quad \text{with} \quad \sum_{j=1}^{\max(n)} p_j = l, \quad (2.48)$$

$$R_p = \text{Tr}(A^p) = \sum_{\alpha=1}^{4N} \lambda_{\alpha}^p \quad (2.49)$$

and coefficients c_n , which for simplicity we assume to be rational. In the example of degree $h = 3$,

$$\mathcal{L}[A] = c_1 R_3 + c_2 R_1 R_2 + c_3 R_1 R_1 R_1 \quad (2.50)$$

with three coefficients $c_n \in \mathbb{Q}$ (only two of which are of relevance because the normalization of \mathcal{L} has no effect on the EL equations). We assume that \mathcal{L} is *non-trivial* in the sense that at least one of the coefficients c_n in the definition (2.47),(2.48) of \mathcal{L} should be non-zero. The EL equations corresponding to (2.47) can easily be computed using that $\delta R_p = p \operatorname{Tr}(A^{p-1} \delta A)$ together with (2.4) and the fact that the trace is cyclic. The resulting operator Q in (2.8) is of the form

$$Q(x, y) = \left[\mathcal{P}_0 A^{h-1} + \mathcal{P}_1 A^{h-2} + \cdots + \mathcal{P}_{h-1} \right] P(x, y) \quad (2.51)$$

where \mathcal{P}_l are homogeneous polynomials of the form (2.48) (\mathcal{P}_0 is a rational number). In the example (2.50),

$$Q(x, y) = \left[\frac{3}{2} c_1 A^2 + c_2 R_1 A + \frac{1}{2} (c_2 R_2 + 3c_3 R_1^2) \right] P(x, y) .$$

By substituting the regularized formulas of the light-cone expansion into (2.51), one sees that $Q(x, y)$ is to every degree on the light cone a polynomial in $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$, well-defined according to [1, Section 3.5]. Thus for polynomial actions, our spectral decomposition is not needed. But it is nevertheless a convenient method for handling the otherwise rather complicated combinatorics of the Dirac matrices.

For the polynomial Lagrangian (2.47), the conditions (2.46) become

$$\mathcal{P}_0 \lambda^{h-1} + \mathcal{P}_1 \lambda^{h-2} + \cdots + \mathcal{P}_{h-1} = 0 \quad \text{for } \lambda = \lambda_{ncs}, n = 1, \dots, 7 \quad (2.52)$$

and the \mathcal{P}_l as in (2.48). It is useful to analyze these conditions algebraically as polynomial equations with rational coefficients for the eigenvalues of A . To this end, we need to introduce an abstract mathematical notion which makes precise that, according to (2.34), the eigenvalues λ_{ncs} have certain degeneracies, but that there are no further relations between them. We say that the matrix A has n *independent eigenvalues* if A has n distinct eigenvalues, one of them being zero and the others being algebraically independent. The following lemma shows that the conditions (2.52) can be fulfilled only if the degree of the Lagrangian is sufficiently large.

Lemma 2.2 *For a non-trivial Lagrangian of the form (2.47) which satisfies the conditions (2.52),*

$$h \geq n, \quad (2.53)$$

where n denotes the number of independent eigenvalues of A .

Proof. First suppose that the factors \mathcal{P}_l in (2.52) are not all zero. Then we can regard the left of (2.52) as a polynomial in λ of degree at most $h - 1$. According to (2.34), the eigenvalues λ_{8a} in the lepton sector all vanish. Thus the polynomial in (2.52) has at least $n - 1$ distinct zeros, and thus its degree must be at least $n - 1$. This proves (2.53).

It remains to consider the case when the coefficients \mathcal{P}_l in (2.52) all vanish. Since the Lagrangian is non-trivial, at least one of the \mathcal{P}_l is non-trivial, we write $\mathcal{P}_l \neq 0$. On the

other hand, $\mathcal{P}_l[A] = 0$ and furthermore $l \leq h - 1$ from (2.52). Hence to conclude the proof it suffices to show that

$$\mathcal{P}_l \neq 0 \wedge \mathcal{P}_l[A] = 0 \implies l \geq n - 1. \quad (2.54)$$

To prove (2.54), we proceed inductively in n . For $n = 1$ there is nothing to show. Assume that (2.54) holds for given n and any matrix A with n independent eigenvalues. Consider a matrix A with $n + 1$ independent eigenvalues. A given non-trivial polynomial \mathcal{P}_{l+1} can be uniquely decomposed in the form

$$\mathcal{P}_{l+1} = R_1 \mathcal{P}_l + R_2 \mathcal{P}_{l-1} + \dots + R_{l+1} \mathcal{P}_0, \quad (2.55)$$

where the polynomials \mathcal{P}_{l-k} contain only factors R_j with $j > k$. Since $\mathcal{P}_{l+1} \neq 0$, at least one of the factors \mathcal{P}_{l-k} is non-trivial. Let $k \geq 0$ be the smallest natural number such that $\mathcal{P}_{l-k} \neq 0$. The functional $\mathcal{P}_{l+1}[A]$ is a homogeneous polynomial of degree $l + 1$ in the eigenvalues $\lambda_1, \dots, \lambda_{n+1}$ of A . We pick those contributions to this polynomial which are homogeneous in $0 \neq \lambda_{n+1}$ of degree $k + 1$. These contributions all come from the summand $R_{k+1} \mathcal{P}_{l-k}$ in (2.55) because the summands to its left are trivial and the summands to its right are composed only of factors R_l with $l > k + 1$. Hence apart from the prefactor λ_{n+1}^{k+1} and up to irrelevant combinatorial factors for each of the monomials, these contributions coincide with the polynomial $\mathcal{P}_{l-k}(\lambda_1, \dots, \lambda_n)$ evaluated for a matrix with n independent eigenvalues. We conclude that if $\mathcal{P}_{l+1}(\lambda_1, \dots, \lambda_{n+1})$ vanishes, then $\mathcal{P}_{l-k}(\lambda_1, \dots, \lambda_n)$ must also be zero. The induction hypothesis yields that $l - k \geq n - 1$ and thus $l + 1 \geq n$. ■

We are looking for a Lagrangian which is as simple as possible. One strategy is to consider the general polynomial Lagrangian (2.47) and to choose the degree h minimal. According to Lemma 2.2, the degree cannot be smaller than the number of independent eigenvalues of A . Thus if we treat the eigenvalues as being algebraically independent in the sectors containing the Dirac seas (d, s, b) , (u, c, t) , and (e, μ, τ) , then h is bounded from below by $h \geq 3 \times 2 + 1 = 7$. Unfortunately, polynomials of degree ≥ 7 involve many coefficients c_n and are complicated. Therefore, it is desirable to reduce the number of independent eigenvalues. Since the eigenvalues depend on the masses and regularization functions of the particles involved (see (2.32)), we can reduce the number of distinct eigenvalues only by assuming that the massive sectors are identical. The best we can do is to assume that

$$m_u = m_d = m_e, \quad m_c = m_s = m_\mu, \quad m_t = m_b = m_\tau, \quad (2.56)$$

and that the regularization functions in the massive sectors coincide. Then the additional degeneracies in the massive sectors reduce the number of distinct eigenvalues to three. If (2.56) holds, the bound of Lemma 2.2 is even optimal. Namely, a simple calculation shows that the polynomial Lagrangian of degree three (2.50) with

$$c_1 = 14, \quad c_2 = -\frac{3}{2}, \quad c_3 = \frac{1}{28} \quad (2.57)$$

satisfies the conditions (2.46). According to Lemma 2.2, a degree $h < 2$ would make it necessary to impose relations between λ_+ and λ_- , which is impossible in our formalism (2.28). We conclude that (2.50), (2.57) is the polynomial Lagrangian of minimal degree which satisfies the conditions (2.46). We refer to (2.56) as the *mass degeneracy assumption*. In order

to understand what this condition means physically, one should keep in mind that (2.56) gives conditions for the bare masses, which due to the self-interaction are different from the effective masses (this is a bit similar to the situation in the grand unified theories, where simple algebraic relations between the bare quark and lepton masses are used with some success [7]).

Another strategy for finding promising Lagrangians is to consider homogeneous polynomials of higher degree, but which are of a particular simple form. A good example for such a Lagrangian is the determinant,

$$\mathcal{L}[A] = \det A . \quad (2.58)$$

By writing $\det A = \det(\mathbf{1} - (\mathbf{1} - A))$, expanding in powers of $(\mathbf{1} - A)$, and multiplying out, this Lagrangian can be brought into the form (2.47),(2.48) with $h = 4N$. The Lagrangian (2.58) is appealing because of its simple form. Furthermore, it has the nice property that whenever the eigenvalues of A appear in complex conjugate pairs (2.33), the product of these eigenvalues is positive, and thus $\mathcal{L} \geq 0$. Unfortunately, there is the following drawback. The matrix A vanishes identically in the neutrino sector (2.10), and so A has a zero eigenvalue of multiplicity four. As a consequence, \mathcal{L} and its variations vanish until perturbations of at least fourth order are taken into account, making the analysis rather complicated. For this reason, (2.58) does not seem the best Lagrangian for developing our methods, and we shall not consider it here.

In the polynomial Lagrangian (2.50),(2.57), we did not use that the eigenvalues of A appear in complex conjugate pairs. This fact can be exploited to construct an even simpler Lagrangian. Assume again that the masses are degenerate (2.56). Then the absolute squares $|\lambda_\alpha|^2$ of the eigenvalues of A take only the two values 0 and $|\lambda_+|^2 = |\lambda_-|^2$, with multiplicities 4 and 28, respectively. Thus if we consider homogeneous polynomials in $|\lambda_\alpha|^2$, there is already a Lagrangian of degree two which satisfies the conditions (2.46), namely

$$\mathcal{L} = \sum_{\alpha=1}^{4N} |\lambda_\alpha|^4 - \frac{1}{28} \left(\sum_{\alpha=1}^{4N} |\lambda_\alpha|^2 \right)^2 . \quad (2.59)$$

Using the notion of the spectral weight [1, eqn (2.38)],

$$|A| \equiv \left(\sum_{k=1}^K n_k |\lambda_k|^2 \right)^{\frac{1}{2}} ,$$

this Lagrangian can be written as

$$\mathcal{L}[A] = |A^2|^2 - \frac{1}{28} |A|^4 . \quad (2.60)$$

The factor $1/28$ may be replaced by a Lagrangian multiplier μ ,

$$\mathcal{L}[A] = |A^2|^2 - \mu |A|^4 , \quad (2.61)$$

because the value of $\mu = 1/28$ is uniquely determined from the condition that the EL equations should be satisfied in the vacuum. The functional (2.61) can be regarded as the effective Lagrangian corresponding to the variational principle with constraint [1, eqns (2.38),(2.39)]. We conclude that (2.60) is precisely the model variational principle introduced in [1, Section 2.5].

The operator Q corresponding to the Lagrangian (2.60) is computed using (2.42) to be

$$Q(x, y) = 2 \sum_{k=1}^{K_{xy}} \left(|\lambda_k|^2 - \frac{1}{28} |A|^2 \right) \bar{\lambda}_k F_k^{xy} P(x, y). \quad (2.62)$$

Let us verify that this expression is well-defined in the framework of [1, Section 3.6]: We introduce the so-called *spectral adjoint* \bar{A} by taking the complex conjugate of the eigenvalues in the spectral decomposition of A ,

$$\bar{A} = \sum_{k=1}^{K_{xy}} \bar{\lambda}_k F_k. \quad (2.63)$$

According to (2.16) and (2.18), the transformations $g \leftrightarrow -\bar{g}$, $h \leftrightarrow \bar{h}$ leave the eigenvalues λ_{\pm} unchanged and map the spectral projectors into their adjoints, $F_{\pm} \rightarrow F_{\pm}^*$. These transformations can be realized by exchanging x with y and anti-commuting with γ^5 , and thus

$$\bar{A}_{xy} = \sum_{k=1}^{K_{xy}} \bar{\lambda}_k^{xy} F_k^{xy} \stackrel{(2.33)}{=} \sum_{k=1}^{K_{xy}} \lambda_k^{xy} (F_k^{xy})^* = \gamma^5 A_{yx} \gamma^5. \quad (2.64)$$

Using that

$$\bar{A} = \sum_{k=1}^K \bar{\lambda}_k F_k, \quad (A\bar{A})\bar{A} = \sum_{k=1}^K |\lambda_k|^2 \bar{\lambda}_k F_k, \quad |A|^2 = \text{Tr}(A\bar{A}),$$

one can write (2.62) as a polynomial in A_{xy} , \bar{A}_{xy} , and $P(x, y)$. If now we express \bar{A}_{xy} in terms of A_{yx} , (2.64), and substitute in (2.2) as well as the expansion for $P(x, y)$, (2.24), we obtain for $Q(x, y)$ a linear combination of monomials in $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$.

The above considerations give a motivation for the Lagrangian (2.60) together with the mass degeneracy assumption (2.56). Clearly, our discussion does not rule out the possibility that there are other physically interesting actions. In particular, we only considered Lagrangians for which the corresponding kernel $Q(x, y)$ is a linear combination of monomials in $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$. This means that if the methods of [1] were extended to more general operations on the $T_{\bullet}^{(n)}$, this might give rise to other promising actions (e.g. if one succeeded in defining the absolute value $|T_{\bullet}^{(n)}|$, one could consider in analogy to (2.59) the Lagrangian $\mathcal{L} = \sum_{\alpha=1}^{4N} |\lambda_{\alpha}|^2 - \frac{1}{28} \left(\sum_{\alpha=1}^{4N} |\lambda_{\alpha}| \right)^2$). But with the methods presently available, the Lagrangian (2.60) gives the simplest promising variational principle. Also, it is nice that all the special properties of the fermionic projector of the vacuum were used. Namely, the EL equations corresponding to (2.60) are fulfilled only due to chiral cancellations in the neutrino sector and due to the fact that the eigenvalues of A appear in complex conjugate pairs.

3 The Dynamical Gauge Group

We now begin the analysis of the EL equations with interaction. The general method is to substitute the regularized formulas of the light-cone expansion into (2.8) and to take the continuum limit as described in [1, Chapter 3]. In order to work in a concrete example, we shall analyze our model variational principle (2.60). But the methods as well as many of the results carry over to other variational principles, as will be discussed in the Remarks at the end of Chapter 3 and at the end of Chapter 4.

For the bosonic potentials in the auxiliary Dirac equation (1.8) we make the ansatz

$$\mathcal{B} = \not{C} + \gamma^5 \not{E} + \Phi + i\gamma^5 \Xi \quad (3.1)$$

with a vector potential C , an axial potential E , and scalar/pseudoscalar potentials ϕ and Ξ , which again in component notation $\mathcal{B} = \mathcal{B}_{(b\beta)}^{(a\alpha)}$ we assume to be of the form

$$C = C_b^a \delta_\beta^\alpha, \quad E = E_b^a \delta_\beta^\alpha, \quad \phi = \phi_{(b\beta)}^{(a\alpha)}, \quad \Xi = \Xi_{(b\beta)}^{(a\alpha)}. \quad (3.2)$$

Exactly as in [3], it is convenient to introduce the chiral potentials

$$A_{L/R} = C \pm E, \quad (3.3)$$

and to define the *dynamical mass matrices* by

$$mY_{L/R} = mY - \phi \mp i\Xi. \quad (3.4)$$

Then the auxiliary Dirac equation takes the form

$$(i\not{\partial} + \chi_L(\not{A}_R - mY_R) + \chi_R(\not{A}_L - mY_L)) P(x, y) = 0. \quad (3.5)$$

Clearly, the potentials in (3.1) must be causality compatible. We assume in what follows that this condition is satisfied, and will specify what it means in the course of our analysis.

Let us briefly discuss the ansatz (3.1). The vector and axial potentials in (3.1) have a similar form as the gauge potentials in the standard model. Indeed, when combined to the chiral potentials (3.3), they can be regarded as the gauge potentials corresponding to the gauge group $U(8)_L \otimes U(8)_R$. This so-called *chiral gauge group* includes the gauge group of the standard model. At every space-time point, it has a natural representation as a pair of 8×8 matrices acting on the sectors; we will work in this representation throughout. Compared to the most general ansatz for the chiral potentials, the only restriction in (3.3), (3.2) is that the potentials are the same for the three generations. This can be justified from the behavior of the fermionic projector under generalized gauge transformations, as will be explained in Remark 3.7 below. The scalar potentials in (3.1) do not appear in the standard model, but as we shall see, they will play an important role in our description of the interaction (here and in what follows, we omit the word “pseudo” and by a “scalar potential” mean a scalar or a pseudoscalar potential). We point out that we do not consider a gravitational field. The reason is that in the present paper, we want to restrict attention to the interactions of the standard model. But since the principle of the fermionic projector respects the equivalence principle, one could clearly include a gravitational field; we plan to do so at a later point. Compared to a general multiplication operator, (3.1) does not contain bilinear potentials (i.e. potentials of the form $H_{jk}\sigma^{jk}$ with $\sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k]$). Clearly, bilinear potentials do not appear in the standard model, but

it is not obvious why they should be irrelevant in our description. Nevertheless, we omit bilinear potentials here in order to keep the analysis as simple as possible. To summarize, (3.1) is certainly not the most general ansatz which is worth being studied. But since the potentials in (3.1) are considerably more general than the gauge potentials in the standard model, it seems reasonable to take (3.1) as the starting point for our analysis.

3.1 The Euler-Lagrange Equations to Highest Degree on the Light Cone

We come to the detailed calculations. We again work with the spectral decomposition of A_{xy} and proceed degree by degree on the light cone. In this section, we consider the highest degree. Then the fermionic projector is influenced only by the chiral potentials (and not by the scalar potentials or the particle states), and the chiral potentials merely describe local phase transformations of the fermionic projector. More precisely, truncating all contributions of degree < 2 and denoting this “truncated fermionic projector” by $P_0(x, y)$, we have (see [3, Section 2.2])

$$P_0(x, y) = \left(\chi_L X_L \int_x^L + \chi_R X_R \int_x^R \right) \frac{i}{2} \not{x} T_{[0]}^{(-1)}(x, y), \quad (3.6)$$

where we used for the ordered exponentials the short notation

$$\int_c^y = \text{Pexp} \left(-i \int_0^1 A_c^j(\tau y + (1 - \tau)x) (y - x)_j d\tau \right) \quad (3.7)$$

with $c = L$ or R . We also truncate the matrix A_{xy} by setting

$$A_0(x, y) = P_0(x, y) P_0(y, x).$$

It follows from (3.6) that

$$A_0 = \left\{ \chi_L X_L \int_x^L \int_y^x X_R + \chi_R X_R \int_x^R \int_y^L X_L \right\} \frac{1}{4} (\not{x} T_{[0]}^{(-1)}) (\overline{\not{x} T_{[0]}^{(-1)}}). \quad (3.8)$$

We can assume in what follows that the matrix inside the curly brackets is diagonalizable; indeed, this is the generic situation, and the general case immediately follows from it by approximation. The matrix A_0 is invariant on the left- and right-handed spinors. If considered on one of these invariant subspaces, the curly brackets depend only on the sector indices $a, b = 1, \dots, 8$, whereas the factors to their right involve only Dirac matrices. This allows us to factorize the spectral decomposition of A_0 as follows. We first diagonalize the phase factor, i.e.

$$W_c \equiv X_c \int_c^y \int_{\bar{c}}^x X_{\bar{c}} = \sum_{n=1}^8 \nu_{nc} I_{nc} \quad (3.9)$$

with eigenvalues ν_{nc} (counting multiplicities) and corresponding spectral projectors I_{nc} , where \bar{c} is defined by $\bar{L} = R$ and $\bar{R} = L$. The matrices W_L and W_R are obtained from each other by taking their adjoint. Thus we can arrange that the same holds for their spectral decompositions,

$$\overline{\nu_{nc}} = \nu_{n\bar{c}}, \quad I_{nc}^* = I_{n\bar{c}}. \quad (3.10)$$

The spectral representation of the second term in (3.8) is computed exactly as described in Subsection 2.1. More precisely, it is obtained from (2.16) and (2.18) by setting h to

zero, i.e. similar to (2.28) and (2.29),

$$\frac{1}{4} (\not{x} T_{[0]}^{(-1)}) (\overline{\not{x} T_{[0]}^{(-1)}}) = \sum_{s=\pm} \lambda_s F_s \quad (3.11)$$

with

$$\lambda_s = \frac{1}{4} T_{[0]}^{(-1)} \overline{T_{[0]}^{(-1)}} \times \begin{cases} z & \text{if } s = + \\ \bar{z} & \text{if } s = - \end{cases} \quad (3.12)$$

$$F_s = \frac{1}{z - \bar{z}} \times \begin{cases} \not{x} \bar{\not{x}} - \bar{z} & \text{if } s = + \\ -\not{x} \bar{\not{x}} + z & \text{if } s = - \end{cases} \quad (3.13)$$

Combining (3.9) and (3.11) gives

$$A_0 = \sum_{n=1}^8 \sum_{c=L,R} \sum_{s=\pm} \lambda_{ncs} F_{ncs} \quad (3.14)$$

with

$$\lambda_{ncs} = \nu_{nc} \lambda_s, \quad F_{ncs} = \chi_c I_{nc} F_s. \quad (3.15)$$

It might be surprising at first sight that, although A_0 clearly is a gauge-invariant expression, the phase shifts described by the ordered exponentials in (3.6) do not drop out in (3.8). Let us explain in detail how this comes about. We first recall that under gauge transformations, the truncated fermionic projector transforms like

$$P_0(x, y) \longrightarrow U(x) P_0(x, y) U(y)^{-1}, \quad (3.16)$$

where U is unitary with respect to the spin scalar product, $U(x)^* = U(x)^{-1}$. These $U(2N, 2N)$ gauge transformations correspond to a local symmetry of the system, which is related to the freedom in choosing a local basis for the spinors (see [1, Section 2.1]). When forming the closed chain, the gauge transformations at y drop out,

$$P_0(x, y) P_0(y, x) \longrightarrow U(x) P_0(x, y) P_0(y, x) U(x)^{-1}.$$

In order to see the relation between the phase transformations in (3.6) and the above gauge transformations, it is useful to consider the situation when the chiral potentials have the form of pure gauge potentials, i.e.

$$A_c^j = iV_c (\partial^j V_c^{-1}) \quad (3.17)$$

with unitary operators $V_L, V_R \in U(8)$. Then the ordered exponential (3.7) reduces to a product of unitary transformations at the two end points,

$$\int_c^y = V_c(x) V_c(y)^{-1}.$$

Using that the potentials are causality compatible, (3.6) becomes

$$P_0(x, y) = \sum_{c=L,R} \chi_c V_c(x) X_c \left(\frac{i}{2} \not{x} T_{[0]}^{(-1)}(x, y) \right) V_c(y)^{-1}. \quad (3.18)$$

Hence the left- and right-handed components of P_0 are transformed independently by V_L and V_R , respectively. In order to write these transformations in a form similar to (3.16), we combine V_L and V_R into one operator V ,

$$V = \chi_L V_L + \chi_R V_R .$$

The effect of the chiral potentials in (3.18) is then described by the transformation

$$P_0(x, y) \longrightarrow V(x) P_0(x, y) V(y)^* ,$$

and thus the closed chain transforms according to

$$P_0(x, y) P_0(y, x) \longrightarrow V(x) P_0(x, y) V(y)^* V(y) P_0(y, x) V(x)^* . \quad (3.19)$$

The point is that the transformation V is in general not unitary, because

$$V^* = \chi_R V_L^{-1} + \chi_L V_R^{-1} \stackrel{\text{in general}}{\neq} \chi_L V_L^{-1} + \chi_R V_R^{-1} = V^{-1} .$$

More precisely, V is unitary if and only if $V_L = V_R$ at every space-time point. According to (3.17), this implies the condition $A_L \equiv A_R$. From (3.3) we conclude that V is unitary if and only if the axial potentials E in (3.1) are identically equal to zero. This means that only the subgroup $U(8) \subset U(8)_L \otimes U(8)_R$ of the chiral gauge group, which gives rise to the vector potential C in (3.1), describes local unitary transformations of the fermionic projector and thus corresponds to a local gauge symmetry in the sense of [1, Section 2.2]. We refer to this subgroup of the chiral gauge group as the *free gauge group* \mathcal{F} ; it can be identified with a subgroup of the gauge group, $\mathcal{F} \subset U(2N, 2N)$ (we remark for clarity that the other degrees of freedom of the gauge group $U(2N, 2N)$ are related to the gravitational field [5] and are thus not considered here). The axial potentials, however, describe local transformations which are not unitary and thus cannot be identified with gauge transformations in the sense of [1, Section 2.2]. These non-unitary transformations do not correspond to an underlying local symmetry of the system. The interpretation of these results is that the *chiral gauge group is spontaneously broken*, only its subgroup \mathcal{F} corresponds to an unbroken local symmetry of the system.

A simple way to understand why the chiral gauge group is spontaneously broken is that axial potentials describe relative phase shifts between the left- and right-handed components of the fermionic projector. Such relative phases do not drop out when we form composite expressions, as one sees in (3.19) or, more generally, in (3.8). By imposing that the relative phases be zero in all composite expressions, we can distinguish those systems in which the axial potentials vanish identically. In this way, one can fix the gauge up to global chiral gauge transformations (i.e. transformations of the form (3.18) with constant matrices V_c) and up to local free gauge transformations. Since this gauge fixing argument makes use of the phases which appear in $P_0(x, y)$, one may regard the chiral gauge symmetry as being spontaneously broken by the fermionic projector.

The spontaneous breaking of the chiral gauge symmetry by the fermionic projector has, at least on the qualitative level, some similarity to the Higgs mechanism in the standard model. We recall that in the Higgs mechanism one arranges by a suitable quartic potential in the classical Lagrangian that the Higgs field Φ has a non-trivial ground state, i.e. $\Phi \neq 0$ in the vacuum. The Higgs field is acted upon by a local gauge group. Since $\Phi \neq 0$, one can, by prescribing the phase of Φ , fix the gauge globally. This shows that the local gauge symmetry is spontaneously broken by the Higgs field, a fact which can then be used to give

the gauge bosons mass. In our setting, we also have in the vacuum a non-trivial object, namely the fermionic projector, which is composed of the Dirac seas corresponding to the leptons and quarks. Thus our situation is indeed quite similar to the Higgs mechanism, if one only keeps in mind that the role of the Higgs field in our description is played by the fermionic projector of the vacuum. Clearly, this analogy does not carry over to the mathematical details. But also in our description, the spontaneous symmetry breaking can give rise to massive gauge bosons (see [6]).

Since the chiral gauge symmetry is spontaneously broken, we cannot expect that the EL equations admit chiral potentials corresponding to the whole group $U(8)_L \otimes U(8)_R$. In order to quantify which restrictions for the chiral potentials we get, we must work in a more general setting and introduce a suitable mathematical notation. Contributions to the fermionic projector which involve the phases of the chiral potentials, but not the gauge fields, currents, or scalar potentials, are called *gauge terms*. Likewise, we refer to the contributions of the gauge terms to a composite expression in the fermionic projector as the gauge terms in the respective expression. The simplest examples for gauge terms are (3.6) or (3.8), but we will encounter gauge terms to lower degree on the light cone as well.

Def. 3.1 *A subgroup \mathcal{G} of the chiral gauge group is called a **dynamical gauge group** if the gauge terms of the potentials corresponding to \mathcal{G} vanish in the EL equations. Its subgroup $\mathcal{G} \cap \mathcal{F}$ is the **free dynamical gauge group**.*

Clearly, this definition does not give a unique dynamical gauge group. In particular, every subgroup of a dynamical gauge group is again a dynamical gauge group. Since we want to choose the dynamical gauge group as large as possible, we can always restrict attention to dynamical gauge groups which are *maximal* in the sense that they are not contained in a larger dynamical gauge group.

We first analyze the gauge term (3.6) in the EL equations corresponding to our variational principle (2.60) to the highest degree on the light cone. This gives the following result.

Theorem 3.2 *The eigenvalues ν_{nc} of W_c must satisfy the conditions*

$$\nu_{8c} = 0 \quad \text{and} \quad |\nu_{nc}| = |\nu_{n'c'}| \text{ for } n, n' = 1, \dots, 7 \text{ and } c, c' = L, R. \quad (3.20)$$

The dynamical gauge group \mathcal{G} is restricted by

$$\mathcal{G} \subset (U(7) \otimes U(1))_L \otimes (U(7) \otimes U(1))_R, \quad (3.21)$$

where the $U(7)$ are unitary matrices acting on the seven massive sectors, and the $U(1)$ act on the neutrino sector.

If conversely the conditions (3.20) or (3.21) are satisfied, then the EL equations are satisfied to degree 11 on the light cone.

It is easy to see that the conditions in the above theorem are sufficient for the EL equations to be satisfied, and this consideration also gives an idea of how these conditions come about. Namely, suppose that (3.21) holds. Then the dynamical gauge potentials are invariant on the massive sectors as well as on the neutrino sector. Using a block matrix notation where the first component refers to the massive sectors and the second component to the neutrino sector, we see from (3.9) that the matrices W_L and W_R have the form

$$W_L = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}, \quad W_R = \begin{pmatrix} U^* & 0 \\ 0 & 0 \end{pmatrix} \quad (3.22)$$

with U a unitary 7×7 matrix. Hence their eigenvalues ν_{nc} satisfy the conditions (3.20). Similar to the calculation leading to (2.62), the gradient of the Lagrangian is computed using (2.41) to be

$$\mathcal{M}[A] = 4 \sum_{k=1}^K \left\{ \left(|\lambda_k|^2 - \frac{1}{28} |A|^2 \right) \overline{\lambda_k} \right\} F_k . \quad (3.23)$$

We saw in Section 2.3 that the curly brackets vanish in the vacuum. If (3.20) is satisfied, the gauge terms change the eigenvalues λ_{nc} only by a phase (3.15). Since these phases drop out when absolute values are taken, the curly brackets in (3.23) are zero even with interaction (to the highest degree on the light cone). This implies according to (2.6) that Q vanishes, and so the EL equations are satisfied.

It is more difficult to show that the conditions (3.20) and (3.21) are also necessary. We give the prove in detail. In preparation, we prove that for the truncated fermionic projector, an analogue of Lemma 2.1 holds even in the presence of chiral potentials.

Lemma 3.3 *The spectral representations of $A_0(x, y)$ and $A_0(y, x)$ are related to each other by*

$$\lambda_{nc\pm}^{xy} = \lambda_{n\bar{c}\mp}^{yx} , \quad F_{nc\pm}^{xy} P_0(x, y) = P_0(x, y) F_{n\bar{c}\mp}^{yx} . \quad (3.24)$$

Proof. In the case $c = L$, we can use that $X_L = \mathbb{1}$ and obtain from (3.9) that

$$W_L^{xy} = \int_L^y W_R^{yx} \int_y^x .$$

Since the ordered exponentials are unitary, the eigenvalues of W_L^{xy} and W_R^{yx} coincide, and furthermore their spectral projectors are obtained from each other by a unitary transformation,

$$\nu_{nL}^{xy} = \nu_{nR}^{yx} , \quad I_{nL}^{xy} = \int_L^y I_{nR}^{yx} \int_y^x . \quad (3.25)$$

Hence

$$\begin{aligned} F_{nLs}^{xy} P_0(x, y) &= \chi_L I_{nL}^{xy} F_s^{xy} P_0(x, y) \\ &= \chi_L \int_L^y I_{nR}^{yx} \left\{ F_s^{xy} \int_L^x \chi_L P_0(x, y) \right\} . \end{aligned} \quad (3.26)$$

According to (3.6), the left-handed component of $P_0(x, y)$ involves the ordered exponential \int_L^y , which precisely cancels the ordered exponential \int_L^x in (3.26). Thus the curly brackets in (3.26) are independent of the chiral potentials, and so we can apply our result in the vacuum (2.22). The ordered exponential \int_L^y in (3.26) introduces the correct phase factor into $P_0(x, y)$, and we conclude that

$$F_{nL\pm}^{xy} P_0(x, y) = \chi_L P_0(x, y) I_{nR}^{yx} F_{\mp}^{yx} = P_0(x, y) F_{nR\mp}^{yx} .$$

For the right-handed component, the calculation is a bit more complicated because $X_R \neq \mathbb{1}$: Replacing $x \leftrightarrow y$ in (3.25) and solving for I_{nR}^{xy} gives

$$\nu_{nR}^{xy} = \nu_{nL}^{yx} , \quad I_{nR}^{xy} = \int_L^y I_{nL}^{yx} \int_y^x .$$

Using that X_R and \int_R commute (as we assume the potentials to be causality compatible) and that X_R is idempotent, we obtain that

$$\begin{aligned} F_{nRs}^{xy} P_0(x, y) &= \chi_R I_{nR}^{xy} F_s^{xy} P_0(x, y) = \chi_R \int_x^y I_{nL}^{yx} F_s^{xy} \int_y^x P_0(x, y) \\ &= \chi_R \int_x^y I_{nL}^{yx} \left(\int_y^x \int_x^y X_R \right) \left\{ F_s^{xy} \int_y^x \chi_R P_0(x, y) \right\}. \end{aligned}$$

Now in the factor $P_0(x, y)$ we can take out the chiral asymmetry matrix X_R (it is already contained in the round brackets to its left), and thus we can, exactly as in (3.26), in the curly brackets apply Lemma 2.1. Furthermore, the round brackets coincide with the matrix W_L^{yx} , which clearly commutes with its spectral projector I_{nL}^{yx} . This gives

$$F_{nR\pm}^{xy} = \chi_R \left(\int_x^y W_L^{yx} \int_y^x \right) P_0(x, y) I_{nL}^{yx} F_{\mp}^{yx}.$$

According to (3.9), the bracket is equal to X_R . ■

Proof of Theorem 3.2. Using the argument given after the statement of the theorem, it remains to show that the conditions (3.20) and (3.21) are necessary. Exactly as in (2.42), it follows from Lemma 3.3 that, to highest degree on the light cone,

$$Q(x, y) = \frac{1}{2} \sum_{n,c,s} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_{ncs}^{xy}} F_{ncs}^{xy} P_0(x, y) + (\deg < 11).$$

Computing the Euler-Lagrange equations similar to (2.43) and keeping track of the chiral cancellations yields that, in analogy to (2.45),

$$\sum_{n,c,s} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_{ncs}^{xy}} F_{ncs}^{xy} P_0(x, y) \int_y^z X_{\bar{c}} + (\deg < 11) = 0,$$

and by multiplying from the right by the macroscopic unitary matrix $\int_{\bar{c}}^y$, we can arrange that $z = x$. We substitute in (3.6) and the right of (3.15), and apply (2.31),

$$\sum_{nc} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_{nc-}} \chi_c I_{nc} \left\{ X_c \int_x^y \int_y^x X_{\bar{c}} \right\} \left(\frac{i}{2} \not{g} T_{[0]}^{(-1)} \right) + (\deg < 11) = 0.$$

The curly brackets coincide with the matrix W_c , and since I_{nc} is a spectral projector of this matrix, we simply get a scalar factor ν_{nc} . Furthermore, we use the particular form of our Lagrangian (2.60) as well as the left of (3.15) to obtain that

$$i \sum_{n,c} \left(|\nu_{nc}|^2 - \frac{1}{14} \sum_{n',c'} |\nu_{n'c'}|^2 \right) |\nu_{nc}|^2 \chi_c \not{g} I_{nc} |\lambda_-|^2 \overline{\lambda_-} T_{[0]}^{(-1)} = 0. \quad (3.27)$$

Using (2.28), the non-smooth factors are a monomial of degree eleven,

$$|\lambda_-|^2 \overline{\lambda_-} T_{[0]}^{(-1)} = \left(T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} \right)^2 \overline{T_{[0]}^{(0)}}. \quad (3.28)$$

We cannot assume that this monomial is equal to zero. Namely, the fermionic projector differs to highest degree on the light cone from the fermionic projector of the vacuum only by macroscopic phase factors (this is guaranteed by the gauge invariance of the regularized causal perturbation expansion [1, Appendix C]). Therefore, exactly as explained for the vacuum in Section 2.2, we can evaluate (3.27) even pointwise. In particular, we can consider the regularization expansion of (3.27) (see [1, Section 3.3]). This means that in order to set the monomial (3.28) in (3.27) equal to zero, we would have to impose an infinite number of regularization conditions. We conclude that in order to satisfy (3.27), we must assume that the macroscopic prefactor vanishes. Using that the spectral projectors I_{nc} are linearly independent, we get the conditions

$$\left(|\nu_{nc}|^2 - \frac{1}{14} \sum_{n',c'} |\nu_{n'c'}|^2 \right) |\nu_{nc}|^2 = 0 \quad \text{for all } n, c.$$

This implies that the absolute values of 14 of the eigenvalues ν_{nc} must coincide, and that the remaining two eigenvalues must be zero. The matrix W_L contains a factor X_R and is thus singular of rank one. Choosing the numbering such that $\nu_{L8} = 0$, it follows from (3.10) that also $\nu_{R8} = 0$. Hence the two zero eigenvalues are those for $n = 8$. This shows that (3.20) is a necessary condition.

Next we will show that (3.20) implies the constraint for the dynamical gauge group (3.21). We introduce for fixed x and y the abbreviations

$$U = \int_L^y \int_R^x \quad \text{and} \quad T = X_R. \quad (3.29)$$

We consider $U = (U_b^a)$ and $T = (T_b^a)$ as matrices on \mathbb{C}^8 endowed with the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$. Then U is unitary, and T is a projector of rank 7. According to (3.10), the conditions (3.20) tell us that the matrix UT must have 7 eigenvalues on the unit circle and one zero eigenvalue. For a vector u in the kernel of UT ,

$$0 = \langle UTu, UTu \rangle = \langle Tu, Tu \rangle,$$

where we used in the last step that U is unitary. Thus u is also in the kernel of T . On the other hand, if u is an eigenvector of UT corresponding to an eigenvalue on the unit circle,

$$|u|^2 = \langle UTu, UTu \rangle = \langle Tu, Tu \rangle.$$

Since for a projector, $|Tu| < |u|$ unless u is in the image of T , it follows that u is also an eigenvector of T , of eigenvalue one. We conclude that every eigenvalue of UT is also an eigenvalue of T , or equivalently that

$$[UT, T] = 0. \quad (3.30)$$

Let us analyze what this commutator condition means for the chiral potentials. We already know from the causality compatibility condition that

$$[A_R, T] = 0. \quad (3.31)$$

Hence substituting the definition of U , (3.29), into (3.30) and using that the resulting condition must hold for all x and y , we obtain that $[A_L T, T] = 0$. Subtracting the adjoint and using that T is idempotent gives the stronger statement

$$[A_L, T] = 0. \quad (3.32)$$

From (3.31) and (3.32) we conclude that the chiral potentials must be block diagonal in the sense that $(A_c)_b^a = 0$ if $a = 8$ and $b \neq 8$ or vice versa. Such chiral potentials correspond precisely to the gauge group in (3.21). ■

Remark 3.4 We point out that the subgroup $U(1)_L \otimes U(1)_R$ of the gauge group in (3.21), which acts on the neutrino sector, is not uniquely determined and could be replaced by any other subgroup which contains $U(1)_L$. This can immediately be understood from the fact that the neutrino sector contains only left-handed particles, and thus the form of the potential A_R , which acts on the right-handed component, has no significance. To make this argument rigorous, we consider the Dirac equation (3.5). Since $P = XP$, we may replace the chiral potentials A_c in (3.5) by $A_c X_c$, and this indeed makes the component of A_R in the neutrino sector equal to zero, showing that this component is of no relevance. We conclude that we may arbitrarily change the subgroup $U(1)_R$ in (3.21); e.g. we could replace (3.21) by

$$(U(7) \otimes U(1))_L \otimes U(7)_R \quad \text{or} \quad (U(7)_L \otimes U(7)_R) \otimes U(1). \quad (3.33)$$

We do not write out this obvious arbitrariness in what follows; instead, we will simply give the gauge groups in the most convenient form.

3.2 The Gauge Terms in the Euler-Lagrange Equations

We come to the analysis of the EL equations to the next lower degree 10 on the light cone. According to the formulas of the light-cone expansion [3], the structure of the fermionic projector to the next lower degree is considerably more complicated than (3.6), because in addition to gauge terms, there are also contributions involving the chiral fields and currents as well as the scalar potentials. Fortunately, the following general argument allows us to distinguish these different types of contributions in the EL equations. In the formulas for $P(x, y)$, the gauge terms always involve ordered exponentials of the chiral potentials, integrated along the line segment $\overline{xy} = \{\alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\}$. We refer to such contributions as *line contributions*. The fields, currents, and scalar potentials, however, are in the light-cone expansion evaluated at individual points, namely either at the end points x, y or at an intermediate point $z \in \overline{xy}$; we call the corresponding contributions to the light-cone expansion *point contributions*. In the case of an evaluation at an intermediate point z , the point contributions clearly involves an integral over z along \overline{xy} . But in contrast to the line contribution, where the chiral potentials at different points enter the ordered exponential in a nonlinear way, the line integrals in a point contribution simply takes averages of the potentials, fields, or currents along the line segment. For example by expanding the ordered exponential in a Dyson series (i.e. in an expansion in powers of $y - x$) and considering the higher order terms, one sees immediately that the line and point contributions are independent in the EL equations in the sense that the EL equations must be satisfied separately by the line and point contributions. Moreover, we can distinguish point contributions in the EL equations, provided that their configuration of the tensor indices is different. Therefore, the point contributions involving the scalar potentials, the chiral fields, and the currents are independent in the EL equations as well.

Using the above arguments, we can study the gauge terms and the contributions involving the scalar potentials, the gauge fields, and the currents separately. In the remainder

of this section, we consider only the gauge terms. Thus we restrict attention to chiral potentials, i.e. we consider instead of (3.5) the Dirac equation

$$(i\cancel{\partial} - mY + \chi_L \cancel{A}_R + \chi_R \cancel{A}_L) P(x, y) = 0$$

with Y a fixed matrix. We will return to the general Dirac equation (3.5) in Chapter 4.

Def. 3.5 We introduce for $p \in \{4, \dots, 7\}$ the groups $B_p, F_p \subset U(8)$ by

$$B_p = \{ \underbrace{g \oplus \dots \oplus g}_p \oplus \underbrace{g^{-1} \oplus \dots \oplus g^{-1}}_{7-p} \text{ with } g \in U(1) \} \quad (3.34)$$

$$F_p = U(p) \otimes U(7-p) \otimes U(1) \quad (3.35)$$

and define corresponding subgroups \mathcal{B}_p and \mathcal{F}_p of the dynamical gauge group by

$$\mathcal{B}_p = B_p \otimes \mathbf{1} \subset U(8)_L \otimes U(8)_R, \quad \mathcal{F}_p = \{(g, g) \text{ with } g \in F_p\} \subset \mathcal{F}. \quad (3.36)$$

Their product

$$\mathcal{G}_p = \mathcal{B}_p \cdot \mathcal{F}_p \equiv \{bf \text{ with } b \in \mathcal{B}_p, f \in \mathcal{F}_p\} \subset U(8)_L \otimes U(8)_R \quad (3.37)$$

is called the p^{th} dynamical gauge group.

In block matrix notation, the elements of B_p and F_p can be written as

$$\begin{pmatrix} z \mathbf{1}_p & 0 & 0 \\ 0 & z^{-1} \mathbf{1}_q & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & l \end{pmatrix}, \quad (3.38)$$

respectively, where the first component refers to the first p sectors, the second component to the next $q \equiv 7 - p$ sectors, and the last component to the neutrino sector. Here $z, l \in U(1)$, $g \in U(p)$, and $h \in U(q)$. Clearly, B_p and \mathcal{B}_p are group isomorphic to $U(1)$. Notice that \mathcal{B}_p acts only the left-handed component. The group \mathcal{F}_p transforms the left- and right-handed components in the same way, and so its corresponding gauge potentials are vector potentials. The groups \mathcal{B}_p and \mathcal{F}_p commute, and this ensures that their product (3.37) is again a group. It is easy to verify that \mathcal{F}_p is indeed the largest subgroup of \mathcal{F} which commutes with \mathcal{B}_p ,

$$\mathcal{F}_p = \{f \in \mathcal{F} \mid bfb^{-1} = f \text{ for all } b \in \mathcal{B}_p\}.$$

We introduce an abbreviation for the linear combination of monomials

$$M \equiv T_{[1]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} \overline{T_{[0]}^{(0)}} - T_{[0]}^{(-1)} T_{[2]}^{(1)} \overline{T_{[0]}^{(-1)}} \overline{T_{[0]}^{(0)}}. \quad (3.39)$$

Theorem 3.6 *There are the following possibilities for the choice of the dynamical gauge group.*

- (1) *Without assuming any relations between the basic monomials, the dynamical gauge group must be contained in the free gauge group,*

$$\mathcal{G} \subset \mathcal{F}_0 = U(7) \otimes U(1). \quad (3.40)$$

- (2) If we allow for one relation between the basic monomials, the dynamical gauge group is (possibly after a global gauge transformation) restricted by

$$\mathcal{G} \subset \mathcal{G}_p \quad \text{for some } p \in \{4, \dots, 7\}. \quad (3.41)$$

In this case, the relation between the basic monomials is

$$(M - \overline{M}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} = 0 \quad (3.42)$$

with M according to (3.39).

- (3) If we allow for two relations between the basic monomials, we get no constraints for the dynamical gauge group besides those of Theorem 3.2. The two relations between the basic monomials are (3.42) and

$$M T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} = 0. \quad (3.43)$$

In each of these cases, the gauge terms vanish in the EL equations to degree 10 on the light cone.

Proof. We first bring the EL equations to degree 10 into a more explicit form. Theorem 3.2 implies that the variation of our Lagrangian vanishes to highest degree on the light cone,

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_{ncs}} + (\deg < 9) = 0. \quad (3.44)$$

According to (2.41) and (2.6), it follows that all contributions to the EL equations vanish for which the variation of the Lagrangian is considered to highest degree (even if the spectral projectors or the factors $P(x, y)$ are expanded to lower degree). This means that we only need to compute the Lagrangian to the next lower degree, whereas it suffices to take into account both the spectral projectors and the factors $P(x, y)$ to highest degree.

Since the Lagrangian is a function of the eigenvalues only, our task is to calculate the contribution to the eigenvalues to the next lower degree two, denote by $\Delta \lambda_{ncs}$. This calculation is carried out in a more general context in Appendix A (see Theorems A.1 and A.3). Specializing the obtained results gives

$$\Delta \lambda_{8cs} = 0,$$

whereas for $n = 1, \dots, 7$,

$$\Delta \lambda_{nc-}^{xy} = T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} - T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} + \nu_{nc} \left(T_{[2]}^{(0)} \overline{T_{[0]}^{(0)}} + T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \right) \quad (3.45)$$

$$+ \frac{T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}}}{\lambda_{nc-}^{xy} - \lambda_{nc-}^{xy}} \left(\nu_{nc} T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \overline{\nu_{nc}} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) \quad (3.46)$$

$$\Delta \lambda_{nc+}^{yx} = \Delta \lambda_{n\bar{c}-}^{xy} \quad (3.47)$$

(here λ_{nc-}^{xy} denotes the eigenvalues of A_0 , (3.15)). Using the last equation together with Lemma 3.3, the EL equations take again the form (2.45). Substituting in the asymptotic

formula to highest degree (2.31) and expanding our Lagrangian (2.60) shows that the EL equations become to degree 10,

$$\sum_{nc} \Delta \left(|\lambda_{nc-}|^2 - \frac{1}{28} \sum_{n',c',s'} |\lambda_{n'c's'}|^2 \right) \overline{\lambda_{nc-}} |\nu_{nc}|^2 \chi_c I_{nc} (i\mathcal{G} T_{[0]}^{(-1)}) = 0.$$

Since the eigenvalues appear in complex conjugate pairs, we may replace the sum over s' by a factor two and set $s' = -$. Also, the non-vanishing macroscopic factor \mathcal{G} can be omitted. Furthermore, we use that the spectral projectors I_{nc} are macroscopic and linearly independent, and that $|\nu_{nc}|^2$ vanishes for $n = 8$ and is equal to one otherwise, (3.20). We thus obtain that the EL equations to degree 10 are equivalent to the conditions that for all $n = 1, \dots, 7$,

$$\left[\text{Re}(\overline{\lambda_{nc-}} \Delta \lambda_{nc-}) - \frac{1}{14} \sum_{n',c'} \text{Re}(\overline{\lambda_{n'c'-}} \Delta \lambda_{n'c'-}) \right] T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} = 0.$$

It is more convenient to write this condition in the form that the expression

$$H_{nc} \equiv 2 \text{Re}(\overline{\lambda_{nc-}} \Delta \lambda_{nc-}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}}$$

should be independent of n and c ,

$$H_{nc} = H_{n'c'} \quad \text{for all } n, n' = 1, \dots, 7 \text{ and } c, c' = L, R. \quad (3.48)$$

Next we compute H_{nc} by substituting in the formulas for λ_{nc-} and $\Delta \lambda_{nc-}$, (2.28) and (3.45), (3.46). Since the last summand of $\Delta \lambda_{nc-}$, (3.46) is imaginary, we can use that for $\alpha \in i\mathbb{R}$,

$$2 \text{Re}(\overline{\lambda_{nc-}} \alpha) = \alpha (\overline{\lambda_{nc-}} - \lambda_{nc-}),$$

and the denominator in (3.46) drops out. We thus obtain

$$H_{nc} = (\nu_{nc} M + \overline{\nu_{nc}} \overline{M} + L + \overline{L}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}}, \quad (3.49)$$

where M is the linear combination of monomials (3.39), and L is given by

$$L \equiv T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} \left(T_{[2]}^{(0)} \overline{T_{[0]}^{(0)}} + T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \right). \quad (3.50)$$

We conclude that the EL equations to degree 10 are equivalent to the conditions (3.48) with H_{nc} given by (3.49) and (3.39), (3.50).

Let us analyze what the conditions (3.48) mean. First of all, the contributions to (3.49) which involve L or \overline{L} are clearly independent of n, c and thus drop out in (3.48). In the case $n' = n$ and $c' = \overline{c}$, we can in (3.48) apply the first part of (3.10) to obtain the necessary conditions

$$(\nu_{nc} - \overline{\nu_{nc}}) (M - \overline{M}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} = 0. \quad (3.51)$$

If we assume no relations between the basic monomials, this implies that $\nu_{nc} = \overline{\nu_{nc}}$, and thus

$$\nu_{nc} = \pm 1 \quad \text{for all } n = 1, \dots, 7 \text{ and } c = L, R. \quad (3.52)$$

For $x = y$, the matrix W_c becomes $W_c = X_c X_{\bar{c}}$, and thus the eigenvalues in (3.52) are equal to one. Since these eigenvalues depend smoothly on x and y , we conclude that $\nu_{nc} = 1$ for all x and y . This means in the block matrix representation for W_c , (3.22), that the unitary matrix U is equal to the identity. Thus, according to (3.9),

$$X_c \int_c^y \int_{\bar{c}}^x X_{\bar{c}} = W_c = X_c X_{\bar{c}}. \quad (3.53)$$

Differentiating with respect to y and setting $y = x$ gives

$$X_c (A_c - A_{\bar{c}}) X_{\bar{c}} = 0.$$

Hence the left- and right-handed potentials must coincide on the massive sectors. Using the argument in Remark 3.4, we can arrange the same in the neutrino sector. This gives the dynamical gauge group in (3.40). Conversely, if (3.40) is satisfied, then the matrices W_c are of the form (3.53). It follows that $\nu_{nc} = 1$ for all $n = 1, \dots, 7$ and $c = L, R$, and thus (3.48) holds.

We next consider the case when we allow for relations between the basic monomials. The only way to avoid the conditions (3.6) (which lead to the dynamical gauge group (3.40)) is to assume that the factor $M - \overline{M}$ in (3.51) vanishes. This gives precisely the relation (3.42). If (3.42) holds, H_{nc} simplifies to

$$H_{nc} = 2 \operatorname{Re}(\nu_{nc}) M + L + \overline{L}. \quad (3.54)$$

If we assume no further relations between the basic monomials, the conditions (3.48) are equivalent to

$$\operatorname{Re}(\nu_{nc}) = \operatorname{Re}(\nu_{n'c'}) \quad \text{for all } n = 1, \dots, 7 \text{ and } c = L, R. \quad (3.55)$$

The only way to avoid these conditions is to impose in addition that (3.43) holds. If this is done, all terms involving ν_{nc} or $\overline{\nu_n c}$ drop out in (3.49), and (3.48) is satisfied.

It remains to show that the conditions (3.55) are equivalent to (3.41). Again using the argument in Remark 3.4, it is obvious that the dynamical gauge group has the required form on the neutrino sector, and thus we can in what follows restrict attention to the seven massive sectors. Then \mathcal{G} is a subgroup of $U(7)_L \otimes U(7)_R$, and the matrices W_c are unitary and have the spectral representation

$$W_c(x, y) = \int_c^y \int_{\bar{c}}^x = \sum_{n=1}^7 \nu_{nc} I_{nc}. \quad (3.56)$$

Suppose that (3.41) is satisfied. Then

$$\mathcal{G} \ni \left(\int_x^y \int_{\bar{L}}^x, \int_y^x \int_{\bar{R}}^y \right) = (bf, f)$$

with $b \in \mathcal{B}_p$ and $f \in \mathcal{F}_p$, and thus

$$W_c = bf f^{-1} = b.$$

As one sees immediately from (3.38), the eigenvalues ν_{nc} of b are equal to z and \bar{z} with $z \in U(1) \subset \mathbb{C}$. Thus the conditions (3.55) are satisfied.

Suppose conversely that the conditions (3.55) hold. We denote the Lie algebra of the dynamical gauge group by \mathfrak{g} ; it is a subalgebra of $su(7) \oplus su(7)$. Let π be the projection onto the axial part,

$$\pi : g \mapsto su(7) : (A_L, A_R) \rightarrow A_L - A_R. \quad (3.57)$$

Its image $\pi(\mathfrak{g})$ is a subspace of $su(7)$ (but it is in general no subalgebra). For the first part of our argument, we consider the situation “locally” for x near y . Expanding the ordered exponentials in (3.56) in x and setting $y = x$ yields according to (3.7) that

$$W_{L/R}(x + \varepsilon u, x) = \mathbf{1} \pm i\varepsilon(A_j^L(x) - A_j^R(x)) u^j + O(\varepsilon^2). \quad (3.58)$$

Since the gauge potentials at x can be chosen freely with values in the dynamical gauge algebra, the term $A \equiv (A_j^L(x) - A_j^R(x)) u^j$ can take any value in $\pi(\mathfrak{g})$. The eigenvalues of (3.9) have the expansion $\nu_{nc} = 1 \pm i\varepsilon\lambda_n + o(\varepsilon)$, where λ_n are the eigenvalues of A . We conclude from (3.55) that

$$\sigma(A) = \{\pm\lambda \text{ with } \lambda = \lambda(A) \in \mathbb{R}\} \quad \text{for all } A \in \pi(\mathfrak{g}). \quad (3.59)$$

We can assume in what follows that $\pi(\mathfrak{g})$ is non-trivial, $\pi(\mathfrak{g}) \neq 0$, because otherwise the dynamical gauge potentials are according to (3.57) pure vector potentials, giving rise to (3.40).

Next we consider the eigenvalues of W_c^{xy} “globally” for y away from x . Expanding the ordered exponentials in (3.56) along the line $x + \varepsilon\xi$, $\xi \equiv y - x$ gives

$$W_L(x + \varepsilon\xi, y) = W_L(x, y) + i\varepsilon(A_j^L(x) W_L(x, y) - W_L(x, y) A_j^R(y)) \xi^j + O(\varepsilon^2). \quad (3.60)$$

It would be nicer to have the potentials A^L and A^R on the same side of the factor W_L . Therefore, we perform a unitary transformation with $U_\varepsilon = \mathbf{1} - i\varepsilon A_j^L + O(\varepsilon^2)$ to obtain

$$U_\varepsilon W_L(x + \varepsilon\xi, y) U_\varepsilon^{-1} = W_L(x, y) + i\varepsilon W_L(x, y) A + O(\varepsilon^2), \quad (3.61)$$

where we set $A \equiv (A_j^L - A_j^R) \xi^j$. Let us analyze what (3.55) and our information on A , (3.59), tell us about the form of W_L ; for simplicity, we work rather elementary with matrices. As explained before (3.59), we are free to choose $A \in \pi(\mathfrak{g})$; we fix any $A \neq 0$. We diagonalize the matrix W_L for given x and y . This gives according to (3.55),

$$W_L(x, y) = \begin{pmatrix} z \mathbf{1}_p & 0 \\ 0 & \bar{z} \mathbf{1}_q \end{pmatrix} \quad (3.62)$$

with $z \in U(1)$, where we used a block matrix notation similar to that in (3.38) and again set $q = 7 - p$. We can without loss of generality assume that $p \in \{4, 5, 6\}$. We first consider the case $z \neq \bar{z}$. Computing the eigenvalues of (3.61) in first order perturbation theory, the conditions (3.55) yield that A must be of the form

$$A = \begin{pmatrix} \lambda \mathbf{1}_p & C^* \\ C & -\lambda \mathbf{1}_q \end{pmatrix}$$

with a $q \times p$ matrix C and $\lambda \in \mathbb{R}$. By changing the basis on the eigenspaces of $W_L(x, y)$, we can even arrange that

$$A = \begin{pmatrix} \lambda \mathbf{1}_p & C^* & 0 \\ C & -\lambda \mathbf{1}_p & 0 \\ 0 & 0 & -\lambda \mathbf{1}_{7-2p} \end{pmatrix} \quad (3.63)$$

with a $p \times p$ matrix C . Thus $-\lambda$ is an eigenvalue of A . According to (3.59), the eigenvalues of A are precisely $\pm\lambda$. Since $A \neq 0$, we know furthermore that $\lambda \neq 0$. It is a general result on self-adjoint matrices that if the expectation value of a unit vector coincides with the largest eigenvalue of the matrix, then this vector must be an eigenvector. Applied to (3.63), this result shows that the submatrix C is zero. Thus

$$A = \begin{pmatrix} \lambda \mathbf{1}_p & 0 \\ 0 & -\lambda \mathbf{1}_q \end{pmatrix} \quad \text{with } p \in \{4, 5, 6\}, \lambda \neq 0.$$

This means that W_L and A have the same eigenspaces. Repeating the above construction for general x and y while keeping A fixed, one sees that the matrices $W_L(x, y)$ all have the same eigenspaces as A (and this is trivially true even when W_L degenerates to a multiple of the identity matrix). This shows that in our basis, (3.62) holds even for all x and y . In the case $z = \bar{z}$ for our original matrix W_L (chosen before (3.62)), W_L is a multiple of the identity matrix. If this is true for all x and y , then (3.62) holds for $p = 0$. Otherwise, we choose x and y such that $W_L(x, y)$ is not a multiple of the identity matrix and repeat the above argument. We conclude that for some $p \in \{4, \dots, 7\}$ and possibly after a global gauge transformation, the matrix W_L has for all x and y the form (3.62).

Let us show that the representation (3.62) is surjective in the sense that for every $z \in U(1)$ we can choose the dynamical gauge potentials on the line segment \overline{xy} such that W_L is of the form (3.62) for this given z . To this end, we take the determinant of (3.62) and (3.56),

$$\bar{z}^{7-2p} = \det W_L = \det \left(\int_x^y \mathbf{L} \int_y^x \mathbf{R} \right).$$

Using that the determinant is multiplicative, we obtain from (3.7) that

$$\bar{z}^{7-p} = \exp \left(-i \int_0^1 \text{Tr} (A(\tau y + (1 - \tau)x) d\tau) \right),$$

where we again set $A = (A_j^L - A_j^R) \xi^j$. This shows that the phase of z is simply additive along the line segment \overline{xy} . It follows immediately that this phase can take arbitrary values, provided that there is $A \in \pi(\mathfrak{g})$ with non-zero trace. Indeed, it follows from (3.59) and the fact that A is an odd-dimensional matrix that $\text{Tr}(A) \neq 0$ for all $A \neq 0$.

We finally return to the expansion (3.60). Writing the chiral potentials as block matrices,

$$A_j^c \xi^j = \begin{pmatrix} a_{11}^c & a_{12}^c \\ a_{21}^c & a_{22}^c \end{pmatrix}$$

and using that both $W_L(x + \varepsilon \xi, y)$ and $W_L(x, y)$ are of the form (3.62) with phases denoted by $z = z_\varepsilon$ and $z = z_0$, respectively, we obtain

$$\begin{pmatrix} z_\varepsilon \mathbf{1}_p & 0 \\ 0 & \bar{z}_\varepsilon \mathbf{1}_q \end{pmatrix} = \begin{pmatrix} z_0 \mathbf{1}_p & 0 \\ 0 & \bar{z}_0 \mathbf{1}_q \end{pmatrix} + i\varepsilon \begin{pmatrix} z_0 (a_{11}^L - a_{11}^R) & \bar{z}_0 a_{12}^L - z_0 a_{12}^R \\ z_0 a_{21}^L - \bar{z}_0 a_{21}^R & \bar{z}_0 (a_{22}^L - a_{22}^R) \end{pmatrix} + O(\varepsilon^2).$$

Since $z_0 \in U(1)$ can take arbitrary values, it follows that

$$a_{11}^L - a_{11}^R = \lambda \mathbf{1}_p, \quad a_{22}^L - a_{22}^R = -\lambda \mathbf{1}_q, \quad a_{12}^c = 0 = a_{21}^c.$$

Chiral potentials of this form correspond precisely to the dynamical gauge group \mathcal{G}_p in (3.18). ■

We finally make three remarks which give a better justification of the ansatz for the vector/axial potentials in (3.1), of the formalism used, and of the variational principles to which this formalism applies.

Remark 3.7 (*The chiral potentials on the generations*) Compared to the most general ansatz for the vector and axial potentials,

$$C = C_{(b\beta)}^{(a\alpha)}, \quad E = E_{(b\beta)}^{(a\alpha)}, \quad (3.64)$$

the potentials in (3.2) are restricted in that they must be the same for the three generations. We shall now justify the ansatz in (3.2) from the form of the gauge terms.

Recall that in Section 2.1, we combined the regularization functions of the three generations to new “effective” regularization functions in each sector (2.27). Here we write this procedure symbolically as

$$T_{\bullet}^{(n)} = \sum_{\alpha=1}^3 T_{\alpha\bullet}^{(n)},$$

where $T_{\alpha\bullet}^{(n)}$ involves the regularization functions for a single Dirac sea in the generation α . Let us consider how the analogue of the gauge term (3.6) looks like. In the case when the potentials are diagonal on the generations, i.e.

$$C = (C^\alpha)_b^a \delta_\beta^\alpha, \quad E = (E^\alpha)_b^a \delta_\beta^\alpha, \quad (3.65)$$

the generalization of (3.6) is straightforward, namely

$$P_0(x, y) = \sum_{\alpha=1}^3 \left(\chi_L X_L \int_x^y \mathbb{L} + \chi_R X_R \int_x^y \mathbb{R} \right)_\alpha \frac{i}{2} \not{g} T_{\alpha[0]}^{(-1)}(x, y), \quad (3.66)$$

where the index α of the brackets means that we take the ordered exponentials of the chiral potentials in the corresponding generation. This gauge term involves relative phase shifts of the individual Dirac seas. If we substitute it into the EL equations, we get many contributions involving these relative phases, and if we want these contributions to drop out, we must introduce additional regularization conditions for certain polynomials in $T_{\alpha\bullet}^{(n)}$, $\alpha = 1, 2, 3$. Thus unless we impose very strong additional conditions on the regularization, the only way to fulfill the EL equations is to set all the relative phases to zero. This gives precisely our ansatz (3.2).

If the potentials C and E are not diagonal on the generations, the form of the gauge terms is not obvious because there is no longer a canonical way to put in the factors $T_{\alpha\bullet}^{(-1)}$. This point could be clarified by generalizing the regularized causal perturbation expansion [1, Appendix C] to the case of systems of Dirac seas involving different regularizations, but we do not want to enter these technical details here. Qualitatively speaking, it is clear that if already the potentials (3.65) lead to strong additional conditions in the EL equations, this will even more be the case for the general ansatz (3.64).

Remark 3.8 (*The vector component is null on the light cone*) In [1, Section 3.4], we introduced the regularization condition that the vector component should be null on the light cone. We remarked that this condition need not be imposed ad hoc, but that it

actually follows from the equations of discrete space-time. We are now in a position to justify this regularization condition from the EL equations.

In our formula for the perturbation of the eigenvalues (3.45),(3.46), we omitted all contributions involving factors $T_{\{\cdot\}}^{(n)}$, assuming that they are of lower degree on the light cone. This is the only point where we used that the vector component is null on the light cone. Without imposing this regularization conditions, we get for $\Delta\lambda_{nc-}^{xy}$ the additional contributions

$$-\nu_{nc} \left(T_{[2]}^{(0)} \overline{T_{\{0\}}^{(1)}} + T_{[0]}^{(-1)} \overline{T_{\{2\}}^{(2)}} \right) + \frac{T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}}}{\overline{\lambda_{nc-}^{xy}} - \lambda_{nc-}^{xy}} \left(\nu_{nc} T_{[1]}^{(0)} \overline{T_{\{0\}}^{(1)}} - \overline{\nu_{nc}} T_{\{0\}}^{(1)} \overline{T_{[1]}^{(0)}} \right).$$

This leads to an additional contribution to H_{nc} , (3.49), of the form

$$(\nu_{nc} K + \overline{\nu_{nc}} \overline{K} + L + \overline{L}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}}$$

with polynomials K and L , where K is given explicitly by

$$K = T_{[1]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)} T_{\{0\}}^{(1)}} - T_{[0]}^{(-1)} T_{\{0\}}^{(1)} \overline{T_{[0]}^{(0)} T_{[1]}^{(0)}}.$$

The monomials appearing here have a different homogeneity in the “large” light-cone coordinate l than those in (3.39) (more precisely, they involve an additional factor of l ; note that the scaling in l is given by the upper index of $T_{\bullet}^{(n)}$, see [1, eqn (3.77)–(3.79)]). Using the different scaling behavior in l , we can distinguish the contributions involving K and L in the EL equations in the sense that both of these contributions must vanish separately. But this means that we can just as well omit K in the EL equations, exactly as it was done in (3.45),(3.46) under the assumption that the vector component is null on the light cone. This argument applies similar to other contributions to the EL equations, to every degree on the light cone.

Remark 3.9 (*n-point actions*) We now discuss some difficulties which arise in the study of actions other than two-point actions. These difficulties are the reason why we do not consider such actions here. Let S be a general n -point action

$$S = \sum_{x_1, \dots, x_n \in M} \mathcal{L}[P(x_1, x_2) \cdots P(x_{n-1}, x_n) P(x_n, x_1)] , \quad n \geq 1.$$

If $n = 1$, the corresponding EL equations are of the form (2.8) with

$$Q(x, y) = \delta_{xy} f[P(x, x)] , \quad (3.67)$$

where f is a functional depending only on $P(x, x)$. Expressions like (3.67) do not have a well-defined continuum limit because the methods of [1, Chapter 3] apply to composite expressions only away from the origin (i.e. for $x \neq y$). Even if one succeeded in giving (3.67) a mathematical meaning, this expression is local and does not involve any ordered exponentials of the chiral potentials. As a consequence, we would have no gauge terms, and the only constraint for the chiral potentials would be the causality compatibility condition. The resulting dynamical gauge group $\mathcal{G} = U(8)_L \otimes U(7)_R$ would be too large for physical applications. For these reasons, one-point actions do not seem worth considering.

If on the other hand $n > 2$, the operator Q in the Euler-Lagrange equations takes the form

$$Q(x_1, x_2) = \sum_{x_2, \dots, x_{n-1} \in M} f[P(x_1, x_2) \cdots P(x_{n-1}, x_n) P(x_n, x_1)] P(x_2, x_2) \cdots P(x_{n-1}, x_n) .$$

where f is a functional on the closed chain. Again, it is not clear how to make mathematical sense of this expression in the continuum limit, but in contrast to (3.67) it now seems possible in principle to adapt the methods of [1, Chapter 3]. We disregard these technical difficulties here and merely discuss the form of the gauge terms in the simplest example of a single Dirac sea and a $U(1)$ vector potential A . It might be that the only relevant contributions to the EL equations comes about when the points x_1, \dots, x_n all lie on a straight line. Generally speaking, the situation in this case would be quite similar to that for a two-point action, and does not seem to give anything essentially new (although the quantitative details would clearly be different). In particular, the gauge terms of type (3.6) drop out in the closed chain, in agreement with the fact that the $U(1)$ corresponds to an unbroken local gauge symmetry. However, the situation is much different if we assume that the points x_1, \dots, x_n do not necessarily lie on a straight line. Namely, in this case the phase shifts in the closed chain add up to an integral along the polygon C with vertices x_1, \dots, x_n ,

$$e^{-i \int_{x_1}^{x_2} A_j (x_2 - x_1)^j} \cdots e^{-i \int_{x_n}^{x_{n-1}} A_k (x_n - x_{n-1})^k} e^{-i \int_{x_n}^{x_1} A_l (x_1 - x_n)^l} = \exp \left(-i \oint_C A_i ds^i \right) .$$

Stokes' theorem allows us to write this line integral as a surface integral. More precisely, choosing a two-dimensional surface S with $\partial S = C$,

$$\exp \left(-i \oint_C A_i ds^i \right) = \exp \left(-i \int_S F_{ij} d\sigma^{ij} \right) ,$$

where $F = dA$ is the field tensor and $d\sigma$ is the area form on S . This simple consideration shows that the phase shift in the closed chain is in general not zero; indeed, it is zero for any position of the points x_k if and only if the field tensor vanishes identically. Thus in the Euler-Lagrange we now expect additional contributions which involve surface integrals of the gauge field tensor; we refer to such contributions as *surface terms*. The appearance of surface terms seems a problem because they give constraints even for those gauge potentials which correspond to a local symmetry of the system.

4 Spontaneous Block Formation

The dynamical gauge group introduced in the previous chapter cannot be identified with the physical gauge group, because the results of Theorem 3.6 are not compatible with the gauge groups in the standard model. Namely, if we allow for two relations between the monomials (case **(3)**), the resulting dynamical gauge group $(U(7) \otimes U(1))_L \otimes (U(7) \otimes U(1))_R$ is too large. The cases **(1)** and **(2)**, on the other hand, seem too restrictive because either no chiral gauge fields are allowed (3.40), or else the chiral gauge fields must be Abelian and are diagonal on the sectors (3.41), in contrast to the weak $SU(2)$ gauge fields in the standard model. Fortunately, these seeming inconsistencies disappear when scalar potentials are taken into account, and it is indeed possible in case **(2)** to model an interaction involving non-Abelian chiral gauge fields. The point is that if scalar potentials are included, the dynamical mass matrices $Y_{L/R}$, (3.4), are in general not diagonal on the sectors. Using a local transformation of the fermionic projector, one can reformulate the interaction such that the dynamical mass matrices become diagonal, but then the resulting chiral fields have off-diagonal contributions and can be identified with so-called “effective” non-Abelian gauge fields. In this chapter, we study the EL equations for an interaction involving both chiral and scalar potentials. After the preparations of Section 4.1, in Section 4.2 we show that the EL equations imply that the fermionic projector splits globally into four so-called blocks, which interact with each other only via free gauge fields. We can distinguish between three quark blocks and one lepton block; these will be analyzed in more detail in Section 4.3. In Chapter 5, we finally give the transformation to the corresponding effective interaction.

Since including the scalar potentials may give further constraints for the dynamical gauge potentials, we cannot expect that the dynamical gauge potentials of the previous chapter will all be admissible here. Therefore, we merely assume that the dynamical gauge potentials present in the system correspond to a subgroup of the dynamical gauge group of Theorem 3.6. In order not to get lost in analytical details which are of no physical relevance, we make the following additional assumptions.

(I) The system should contain chiral dynamical gauge fields.

(II) The chiral Dirac particles should enter the EL equations.

From the physical point of view, the last assumption is trivial because otherwise the chiral Dirac particles (=neutrinos) would be unobservable. Furthermore, we need to assume that our system involves several gauge fields which are sufficiently “independent” from each other. This assumption could be made precise in many different ways; our formulation seems particularly convenient.

(III) The free gauge fields should distinguish the chiral and massive Dirac particles in the sense that for every pair of a chiral and a massive Dirac particle there is a free dynamical gauge field which couples to the two particles differently.

For the interactions in the standard model, the last assumption is clearly satisfied because the electromagnetic field couples to all massive Dirac particles, but not to the neutrinos. Assumption **(III)** could be weakened, but this would make it necessary to rule out a number of exceptional cases in the analysis, and we do not want to go into this here. We finally give our guideline for dealing with the regularization.

(IV) Impose as few relations between the basic monomials as possible such that (I)–(III) can be fulfilled.

This method will uniquely determine all relations between the basic monomials.

4.1 The Partial Trace and the Dynamical Mass Matrices

We want to analyze the EL equations in the presence of chiral and scalar potentials (3.5) to the degree 10 on the light cone. Thus the only difference to the setting of Theorem 3.6 is that, instead of a constant matrix Y , we now allow more generally for dynamical mass matrices $Y_L(x)$ and $Y_R(x)$. One difficulty is that the scalar potentials may depend in a complicated way on the generation index (in contrast to the chiral potentials, which we assumed to be constant on the generations; see (3.2)). In particular, the partial trace (1.12) becomes a non-trivial operation when dynamical mass matrices are present. In this section, we give a few general considerations on the partial trace of the dynamical mass matrices.

We first introduce a convenient notation. In our calculations so far, we omitted the mass matrix Y in all contributions to the fermionic projector. Now that we are working with the dynamical mass matrices $Y_{L/R}$, these clearly have to be written out. In composite expressions, we need to make clear how the partial traces are to be taken. To this end, we denote the sums over the upper and lower generation index by the tildes $\acute{}$ and $\grave{}$, respectively. Thus we introduce the matrices

$$\begin{aligned}\acute{Y}_{L/R} : \mathbb{C}^{8 \times 3} &\rightarrow \mathbb{C}^8, & (\acute{Y}_{L/R})_{(b\beta)}^{a\alpha} &= \sum_{\alpha=1}^3 (Y_{L/R})_{(b\beta)}^{(a\alpha)} \\ \grave{Y}_{L/R} : \mathbb{C}^8 &\rightarrow \mathbb{C}^{8 \times 3}, & (\grave{Y}_{L/R})_b^{(a\alpha)} &= \sum_{\beta=1}^3 (Y_{L/R})_{(b\beta)}^{(a\alpha)}.\end{aligned}$$

Similarly, we denote the sum over both generation indices by the accent $\hat{}$,

$$\hat{Y}_{L/R} : \mathbb{C}^8 \rightarrow \mathbb{C}^8, \quad (\hat{Y}_{L/R})_{(b\beta)}^{a\alpha} = \sum_{\alpha,\beta=1}^3 (Y_{L/R})_{(b\beta)}^{(a\alpha)}.$$

Clearly, $(\acute{Y}_{L/R})^* = \grave{Y}_{R/L}$ and $(\hat{Y}_{L/R})^* = \hat{Y}_{R/L}$. In a contribution to the fermionic projector which involves a product of dynamical mass matrices, the partial trace leads us to label the first and last factor $Y_{L/R}$ by $\acute{}$ and $\grave{}$, respectively. For example, in the presence of a homogeneous scalar potential, we write the light-cone expansion of the left-handed component of the fermionic projector in analogy to (2.24) as

$$\chi_L P(x, y) = \chi_L \left(X \frac{i\cancel{g}}{2} T_{[0]}^{(-1)} + \hat{Y}_L T_{[1]}^{(0)} + \frac{i\cancel{g}}{2} \acute{Y}_L \grave{Y}_R T_{[2]}^{(0)} + \dots \right). \quad (4.1)$$

Furthermore, we denote the contraction in the sector index by Tr_S ,

$$\text{Tr}_S B \equiv \sum_{n=1}^8 B_n^n.$$

One should keep in mind that the partial trace is *not* cyclic, because we sum over the upper and lower index independently. For example,

$$\text{Tr}_S \acute{Y}_L \grave{Y}_R \stackrel{\text{in general}}{\neq} \text{Tr}_S \acute{Y}_R \grave{Y}_L. \quad (4.2)$$

But both terms are clearly real and ≥ 0 .

The EL equations are formulated in terms of the fermionic projector, which is obtained from the auxiliary fermionic projector by taking the partial trace (1.12). Therefore, we regard the fermionic projector as a physical object only after the partial trace has been taken. Thus it is a reasonable point of view that we do not need to worry about noncausal line integrals in the light-cone expansion as long as the corresponding contributions to the auxiliary fermionic projector drop out when the partial trace is taken. This leads us to weaken the causality compatibility condition (1.10) by imposing a condition only on the partial trace of the system of interacting Dirac seas \tilde{t} .

Def. 4.1 *The Dirac operator is **weakly causality compatible** if*

$$\sum_{\alpha,\beta=1}^3 (X \tilde{t})_{(b\beta)}^{(a\alpha)} = \sum_{\alpha,\beta=1}^3 (\tilde{t} X^*)_{(b\beta)}^{(a\alpha)} .$$

Under this assumption, the fermionic projector is defined by

$$P_b^a(x, y) = \sum_{\alpha,\beta=1}^3 (X \tilde{t})_{(b\beta)}^{(a\alpha)}(x, y) . \quad (4.3)$$

In what follows, we shall assume that the weak causality compatibility condition is satisfied for all contributions to the fermionic projector which are of relevance to the degree on the light cone under consideration.

Our point of view that the fermionic projector has a physical meaning only after taking the partial trace also implies that we should consider different choices of dynamical mass matrices as being equivalent if taking the partial trace gives the same fermionic projector (1.12). Furthermore, for this equivalence it is not necessary that the fermionic projectors are identical, but it suffices that all contributions to the fermionic projector which enter the EL equations are the same. More specifically, to the degree 10 on the light cone the EL equations will involve at most quadratic terms in m , and so every factor $Y_{L/R}$ carries an accent. Thus two dynamical mass matrices can be considered as being equivalent if their partial traces coincide. In other words, the dynamical mass matrices are determined only modulo the equivalence relation

$$B_1 \simeq B_2 \quad \text{if} \quad \dot{B}_1 = \dot{B}_2 \text{ and } \dot{B}_1 = \dot{B}_2 .$$

This arbitrariness in choosing the dynamical mass matrices can be used to simplify these matrices. For example, we will set the matrix entries to zero whenever possible by applying for every $a, b \in \{1, \dots, 8\}$ and $c \in \{L, R\}$ the rule

$$(\dot{Y}_c)_{(b.)}^a = 0 = (\dot{Y}_c)_b^{(a.)} \implies (Y_c)_{(b.)}^{(a.)} = 0 .$$

Here the dot means that we are using a matrix notation in the generation index, i.e. $(\dot{Y}_c)_{(b.)}^a$ is a 3-vector and $(Y_c)_{(b.)}^{(a.)}$ (for fixed a, b) a 3×3 matrix. We refer to the just-described method of simplifying the dynamical mass matrices that we *choose a convenient representation* of Y_c .

In order to rule out pathological cases, we need to impose a condition on the dynamical mass matrices. Note that in the vacuum the mass matrices are block diagonal in the sense that $(Y_{L/R})_{(b.)}^{(a.)} = \delta_b^a Y_{L/R}^a$ for suitable 3×3 matrices $Y_{L/R}^a$. Thus the off-diagonal elements

$(Y_{L/R})_{(b.)}^{(a.)}$, $a \neq b$, contain scalar potentials. It would be too restrictive to assume that there are no cancellations when the partial trace is taken; i.e. we do want to allow for the possibility that $(\dot{Y}_c)_{(b.)}^a = 0$ or $(\dot{Y}_c)_b^{(a.)} = 0$ although $(Y_c)_{(b.)}^{(a.)} \neq 0$ (for some $a \neq b$). But such cancellations should occur only with a special purpose, for example in order to ensure that the Dirac operator be weakly causality compatible or in order to arrange that certain terms drop out of the EL equations. For such a purpose, it is not sufficient that one off-diagonal element of \dot{Y}_c vanishes, but all the off-diagonal elements in the same row should be zero. This is the motivation for the following definition.

Def. 4.2 *The dynamical mass matrices are **non-degenerate** if for all $a, b \in \{1, \dots, 8\}$, $a \neq b$, and $c \in \{L, R\}$,*

$$(\dot{Y}_c)_b^{(a.)} \neq 0 \text{ and } (\dot{Y}_c)_{(b.)}^a = 0 \implies (\dot{Y}_c)_{(d.)}^a = 0 \text{ for all } d \neq a.$$

The freedom to choose a convenient representation of the dynamical mass matrices reduces our problem to revealing the structure of the matrices \dot{Y}_c and $\dot{Y}_{\bar{c}}$. One difficulty is that the EL equations involve these matrices only in products of the form $\dot{Y}_{L/R}(y)\dot{Y}_{L/R}(x)$. The following elementary lemma will allow us to use information on the matrix product to derive properties for the individual factors.

Lemma 4.3 (uniform splitting lemma) *Let $\mathcal{B} \subset Gl(\mathbb{C}^{p_1}, \mathbb{C}^{p_2})$ be a set of $(p_2 \times p_1)$ matrices with the property that for all $B_1, B_2 \in \mathcal{B}$ there is $\lambda \in \mathbb{C}$ such that*

$$B_1^* B_2 = \lambda \mathbf{1}_{\mathbb{C}^{p_1}}. \quad (4.4)$$

Then there is a unitary $(p_2 \times p_2)$ matrix U and an integer $r \geq 0$ with $rp_1 \leq p_2$ such that every $B \in \mathcal{B}$ can be written in the form

$$B = U \begin{pmatrix} \overbrace{b \oplus \dots \oplus b}^{p \text{ summands}} \\ 0 \end{pmatrix} \begin{matrix} \} rp_1 \text{ rows} \\ \} p_2 - rp_1 \text{ rows} \end{matrix} \quad (4.5)$$

for a suitable $(r \times 1)$ matrix b .

We mention for clarity that $b \oplus \dots \oplus b$ is a $(rp_1 \times p_1)$ matrix; it could also be written as a block matrix with diagonal entries b . The word “uniform” in the name of the lemma refers to the fact that the unitary transformation U is independent of $B \in \mathcal{B}$. In our applications, this will mean that U is constant in space-time. Such constant unitary transformations are irrelevant (e.g. they could be absorbed into a more general definition of the partial trace), and we can often simply ignore them.

Proof of Lemma 4.3. Let (e_1, \dots, e_{p_1}) be an orthonormal basis of \mathbb{C}^{p_1} . We introduce the subspaces

$$E_i = \langle \{B e_i \text{ with } B \in \mathcal{B}\} \rangle \subset \mathbb{C}^{p_2}$$

and the mappings

$$\pi_i : \mathcal{B} \mapsto E_i : B \mapsto B e_i.$$

The property (4.4) implies that for all $B_1, B_2 \in \mathcal{B}$,

$$\langle B_2 e_i, B_1 e_j \rangle = \langle B_1^* B_2 e_i, e_j \rangle = \lambda(B_1, B_2) \delta_{ij}. \quad (4.6)$$

If $i \neq j$, this relation shows that the subspaces $(E_i)_{i=1, \dots, p_1}$ are orthogonal. In the case $i = j$, (4.6) yields that the inner product $\langle \pi_i(B_1), \pi_i(B_2) \rangle$ is independent of i . Thus the

mappings π_i are unitarily equivalent, and so we can arrange by a unitary transformation that the π_i all have the same matrix representation $\pi(B) = b$. \blacksquare

4.2 Analysis of Degeneracies

The operator Q corresponding to the Dirac operator (3.5) is again given by (2.6) and (2.41). But we must be careful because we no longer have simple relations between the spectral decompositions of A_{xy} and A_{yx} (like Lemma 2.1 or Lemma 3.3). Thus

$$Q(x, y) = \frac{1}{4} \left(\sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}} F_k^{xy} P(x, y) + P(x, y) \sum_{k=1}^{K_{yx}} \frac{\partial \mathcal{L}(\lambda^{yx})}{\partial \lambda_k^{yx}} F_k^{yx} \right). \quad (4.7)$$

Following **(I)** and **(IV)**, we can restrict attention to case **(2)** of Theorem 3.6. In this case, the eigenvalues of A are highly degenerate. We must take into account that these degeneracies will in general be removed by the scalar perturbation. This subtle problem is treated in a more general context in Appendix A. We now specialize the obtained results using a notation which is adapted to the dynamical gauge group \mathcal{G}_p , (3.41). We let \uparrow and \downarrow be the sets

$$\uparrow = \{1, \dots, p\}, \quad \downarrow = \{p+1, \dots, 7\}$$

and introduce the corresponding projectors $I_{\uparrow/\downarrow}$ by

$$I_{\uparrow} = \sum_{n \in \uparrow} I_n, \quad I_{\downarrow} = \sum_{n \in \downarrow} I_n,$$

where $(I_n)_b^a = \delta_b^a \delta_n^a$ are the projectors on the sectors (in the case $p = 7$, we set $\downarrow = \emptyset$ and $P_{\downarrow} = 0$). To the highest degree on the light cone (i.e. if the eigenvalues are treated as in Theorem 3.2), the chiral gauge fields corresponding to \mathcal{G}_p lead to five distinct eigenvalues of A_{xy} , one of which is zero. The spectral projectors corresponding to the kernel and the non-zero eigenvalues are given by I_8 and

$$(\chi_c I_{\uparrow} + \chi_{\bar{c}} I_{\downarrow}) F_s \quad \text{with } c = L/R, s = \pm, \quad (4.8)$$

respectively. To the next lower degree on the light cone, we need to take into account the perturbation of A by gauge terms and the scalar potentials. Theorem A.3 shows that the dimension of the kernel of A is not affected by the perturbation, and thus it suffices to consider the non-zero eigenvalues. According to Theorem A.1, the degeneracy of the non-zero eigenvalues is in general removed. In order to describe the splitting of the eigenvalues in the massive sectors, we first associate to each spectral projector (4.8) a projector on an invariant subspace of A (which is no longer necessarily an eigenspace), and the perturbed eigenvalues are then obtained by diagonalizing A on these invariant subspaces (see Sections A.1 and A.4 for details). It is the main result of Theorem A.1 that the perturbation is block diagonal on the left- and right-handed components of the invariant subspaces. This means more precisely that the left- and right-handed components of $(F_k)_{k=2, \dots, K}$, i.e. the image of the eight projectors

$$\chi_c I_{\uparrow} F_s \text{ and } \chi_{\bar{c}} I_{\downarrow} F_s \quad \text{with } c = L/R, s = \pm, \quad (4.9)$$

can be perturbed to obtain invariant subspaces of A , and thus it suffices to analyze A on these smaller subspaces. Since each of these subspaces carries fixed indices (c, s) , a basis on each subspace may be labelled by the sector index n . We choose a basis such that A is diagonal on the invariant spaces. We denote the corresponding eigenvalues (counting multiplicities) by $(\lambda_{ncs} + \Delta\lambda_{ncs})$ and the spectral projectors by $(F_{ncs} + \Delta F_{ncs})$. For clarity, we point out that the unperturbed spectral projectors F_{ncs} appearing here may differ from those in (3.15) in that we are using a different basis on the sectors, which need not be orthogonal and may depend on c, s and x, y . This slight abuse of notation cannot lead to confusion because in (3.15) we are free to choose any basis on the degenerate subspaces.

For our choice of the Lagrangian (2.59) and the dynamical gauge group according to (3.41), the factors $\partial\mathcal{L}/\partial\lambda_k$ in (4.7) vanish identically to the highest degree, see (3.44). Thus it suffices to take into account the perturbation of these factors. Using the above notation, we obtain

$$Q(x, y) = \frac{1}{4} \sum_{n, c, s} \left(\left(\Delta \frac{\partial\mathcal{L}(\lambda^{xy})}{\partial\lambda_{ncs}^{xy}} \right) F_{ncs}^{xy} P(x, y) + P(x, y) \left(\Delta \frac{\partial\mathcal{L}(\lambda^{yx})}{\partial\lambda_{ncs}^{yx}} \right) F_{ncs}^{yx} \right) + (\deg < 10). \quad (4.10)$$

Note that the perturbation of the spectral projectors ΔF_{ncs} does not appear here; this is a major simplification. Computing the perturbation of the Lagrangian and using that the unperturbed eigenvalues satisfy the relations of Lemma 3.3, one sees that the difficult terms to compute are of the form

$$\sum_{n, c, s} \mathcal{P}(\lambda_{ncs}^{xy}, \overline{\lambda_{ncs}^{xy}}) \{ \Delta\lambda_{ncs}^{xy} F_{ncs}^{xy} P(x, y) + P(x, y) \Delta\lambda_{ncs}^{yx} F_{ncs}^{yx} \} \quad (4.11)$$

$$\sum_{n, c, s} \mathcal{P}(\lambda_{ncs}^{xy}, \overline{\lambda_{ncs}^{xy}}) \{ \overline{\Delta\lambda_{ncs}^{xy}} F_{ncs}^{xy} P(x, y) + P(x, y) \overline{\Delta\lambda_{ncs}^{yx}} F_{ncs}^{yx} \}, \quad (4.12)$$

where \mathcal{P} stands for a polynomial in both arguments. The subtle point in computing expressions of the form (4.11), (4.12) is to carry out the sums over $n \in \uparrow$ and $n \in \downarrow$ (for fixed c, s), because the corresponding indices $\{(ncs), n \in \uparrow / \downarrow\}$ label our basis vectors on the invariant subspaces associated to the projectors $\chi_c I_{\uparrow/\downarrow} F_s$. We shall now give a procedure for explicitly computing these sums. First of all, it is helpful that the unperturbed eigenvalues do not depend on $n \in \uparrow$ or \downarrow . Thus the polynomials \mathcal{P} in (4.11), (4.12) are constants, and so we may restrict attention to the terms inside the curly brackets. It is a complication that $\Delta\lambda_{ncs}$ and $P(x, y)$ involve the gauge potentials corresponding to the free gauge group \mathcal{F}_p . To bypass this difficulty, we choose x and y on a fixed null line \mathcal{L} in Minkowski space,

$$x, y \in \mathcal{L} = u + \mathbb{R} v \quad \text{with } v^2 = 0 \quad (4.13)$$

and arrange by a gauge transformation that the free gauge potentials vanish identically on \mathcal{L} (this is possible because free gauge transformations are local unitary transformations; see page 22). After this transformation the chiral potentials are on \overline{xy} Abelian and diagonal in the sector index.

We first state the formulas for the perturbation of the eigenvalues in full generality; we shall discuss and analyze these formulas afterwards beginning with simple special cases. In order to keep the notation as simple as possible, we restrict attention to the case $c = L$ and $n \in \uparrow$, and shall give symbolic replacement rules with which the analogous formulas are obtained in all other cases.

Def. 4.4 Let ν , μ and ν_8 , μ_8 be the phase factors

$$\nu = \text{Tr}_S \left(I_7 \int_x^y \int_y^x \right), \quad \mu = \text{Tr}_S \left(I_1 \int_x^y \right) \text{Tr}_S \left(I_7 \int_y^x \right) \quad (4.14)$$

$$\nu_8 = \text{Tr}_S \left(I_8 \int_x^y \int_y^x \right), \quad \mu_8 = \text{Tr}_S \left(I_1 \int_x^y \right) \text{Tr}_S \left(I_8 \int_y^x \right). \quad (4.15)$$

We introduce the $p \times p$ matrices Λ_- and Λ_+ by

$$\Lambda_- = \nu \int_x^y dz I_\uparrow \dot{Y}_L \dot{Y}_R I_\uparrow T_{[2]}^{(0)} \overline{T_{[0]}^{(0)}} \quad (4.16)$$

$$+ \nu \int_y^x dz I_\uparrow \dot{Y}_R \dot{Y}_L I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \quad (4.17)$$

$$\left. \begin{aligned} &+ I_\uparrow \hat{Y}_L(y) I_\uparrow \hat{Y}_L(x) I_\uparrow T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \\ &- I_\uparrow \dot{Y}_R(y) I_\uparrow \dot{Y}_L(x) I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \\ &- \frac{1}{\nu\lambda_- - \bar{\nu}\lambda_+} I_\uparrow \left(\hat{Y}_L(y) T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_R(y) T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) \\ &\quad \times I_\uparrow \left(\nu \hat{Y}_R(x) T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \bar{\nu} \hat{Y}_L(x) T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) I_\uparrow \end{aligned} \right\} \quad (4.18)$$

$$\left. \begin{aligned} &+ \mu\nu I_\uparrow \hat{Y}_L(y) I_\downarrow \hat{Y}_L(x) I_\uparrow T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \\ &- \mu\nu I_\uparrow \dot{Y}_R(y) I_\downarrow \dot{Y}_L(x) I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \\ &- \frac{\mu\nu}{\lambda_- - \lambda_+} I_\uparrow \left(\hat{Y}_L(y) T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_R(y) T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) \\ &\quad \times I_\downarrow \left(\hat{Y}_R(x) T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \hat{Y}_L(x) T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) I_\uparrow \end{aligned} \right\} \quad (4.19)$$

$$- \mu_8 \nu_8 I_\uparrow \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \quad (4.20)$$

$$\Lambda_+ = \nu \int_x^y dz I_\uparrow \dot{Y}_L \dot{Y}_R I_\uparrow T_{[2]}^{(0)} \overline{T_{[0]}^{(0)}} \quad (4.21)$$

$$+ \nu \int_y^x dz I_\uparrow \dot{Y}_R \dot{Y}_L I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \quad (4.22)$$

$$\left. \begin{aligned} &+ I_\uparrow \hat{Y}_R(y) I_\uparrow \hat{Y}_R(x) I_\uparrow T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \\ &- I_\uparrow \dot{Y}_R(y) I_\uparrow \dot{Y}_L(x) I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \\ &- \frac{1}{\nu\lambda_- - \bar{\nu}\lambda_+} I_\uparrow \left(\nu \hat{Y}_L(y) T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \bar{\nu} \hat{Y}_R(y) T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) \\ &\quad \times I_\uparrow \left(\hat{Y}_R(x) T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_L(x) T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) I_\uparrow \end{aligned} \right\} \quad (4.23)$$

$$\left. \begin{aligned} & +\mu\nu I_{\uparrow} \hat{Y}_R(y) I_{\downarrow} \hat{Y}_R(x) I_{\uparrow} T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \\ & -\mu\nu I_{\uparrow} \hat{Y}_R(y) I_{\downarrow} \hat{Y}_L(x) I_{\uparrow} T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \\ & -\frac{\mu\nu}{\lambda_- - \lambda_+} I_{\uparrow} \left(\hat{Y}_L(y) T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \hat{Y}_R(y) T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) \\ & \quad \times I_{\downarrow} \left(\hat{Y}_R(x) T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_L(x) T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) I_{\uparrow} \end{aligned} \right\} \quad (4.24)$$

$$-\mu_8\nu_8 I_{\uparrow} \hat{Y}_R(y) I_8 \hat{Y}_L(x) I_{\uparrow} T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \quad (4.25)$$

We denote the spectral adjoint of Λ_{\mp} (defined in analogy to (2.63)) by $\overline{\Lambda_{\mp}}$.

Lemma 4.5 Up to contributions of degree < 4 ,

$$\sum_{n \in \uparrow} \Delta \lambda_{nL+}^{xy} F_{nL+}^{xy} P(x, y) = 0 = \sum_{n \in \uparrow} \overline{\Delta \lambda_{nL+}^{xy}} F_{nL+}^{xy} P(x, y) \quad (4.26)$$

$$\sum_{n \in \uparrow} \Delta \lambda_{nL-}^{xy} F_{nL-}^{xy} P(x, y) = \Lambda_- P(x, y) \quad (4.27)$$

$$\sum_{n \in \uparrow} \overline{\Delta \lambda_{nL-}^{xy}} F_{nL-}^{xy} P(x, y) = \overline{\Lambda_-} P(x, y) \quad (4.28)$$

$$\sum_{n \in \uparrow} P(x, y) \Delta \lambda_{nR-}^{yx} F_{nR-}^{yx} = 0 = \sum_{n \in \uparrow} P(x, y) \overline{\Delta \lambda_{nR-}^{yx}} F_{nR-}^{yx} \quad (4.29)$$

$$\sum_{n \in \uparrow} P(x, y) \Delta \lambda_{nR+}^{yx} F_{nR+}^{yx} = \Lambda_+ P(x, y) \quad (4.30)$$

$$\sum_{n \in \uparrow} P(x, y) \overline{\Delta \lambda_{nR+}^{yx}} F_{nR+}^{yx} = \overline{\Lambda_+} P(x, y). \quad (4.31)$$

The corresponding formulas for the opposite chirality are obtained by the symbolic replacements

$$L \longleftrightarrow R, \quad \nu \longleftrightarrow \overline{\nu}, \quad \nu_8 \longleftrightarrow \overline{\nu_8}, \quad \mu_8 \longleftrightarrow \overline{\nu_8} \mu_8. \quad (4.32)$$

In the case $p < 7$, we may furthermore perform the replacements

$$\uparrow \longleftrightarrow \downarrow, \quad \nu \longleftrightarrow \overline{\nu}, \quad \mu \longleftrightarrow \overline{\mu} \quad \text{and} \quad \mu_8 \longleftrightarrow \overline{\mu_8} \mu_8. \quad (4.33)$$

Proof. According to (2.31), to the highest degree on the light cone we have the identities $F_{nc+}P(x, y) = 0$ and $F_{nc-}P(x, y) = \chi_c I_n P(x, y)$. This gives (4.26), and (4.27) follows directly from Theorem A.1 and the results of Section A.3. In a basis where Λ_- is diagonal, (4.28) is an immediate consequence of (4.27). The relations (4.29)–(4.31) are obtained similarly using the identities $P(x, y) F_{nc-} = 0$ and $P(x, y) F_{nc+} = P(x, y) \chi_c I_n$ as well as the formulas of Appendix A. Alternatively, they can be deduced from (4.26)–(4.28) by taking the adjoint and interchanging x with y (note that it follows from Lemma 3.3 and the fact that the eigenvalues appear in complex conjugate pairs that $\overline{\Delta \lambda_{nL+}^{xy}} = \Delta \lambda_{nR-}^{xy}$).

To derive the replacement rule (4.32), we first note that in the case $p = 7$, the projector I_{\downarrow} vanishes, and thus all contributions to Λ_{\pm} involving μ are equal to zero. In the case $p < 7$,

$$\begin{aligned} \mu & \longrightarrow \text{Tr}_S \left(I_1 \int_x^y \mathbf{R} \right) \text{Tr}_S \left(I_7 \int_y^x \mathbf{L} \right) = \overline{\nu} \text{Tr}_S \left(I_1 \int_x^y \mathbf{L} \right) \nu \text{Tr}_S \left(I_7 \int_y^x \mathbf{R} \right) = \mu \\ \mu_8 & \longrightarrow \text{Tr}_S \left(I_1 \int_x^y \mathbf{R} \right) \text{Tr}_S \left(I_8 \int_y^x \mathbf{L} \right) = \overline{\nu} \text{Tr}_S \left(I_1 \int_x^y \mathbf{L} \right) \nu_8 \text{Tr}_S \left(I_8 \int_y^x \mathbf{R} \right) = \overline{\nu_8} \mu. \end{aligned}$$

Using the relations

$$\mathrm{Tr}_S \left(I_7 \int_x^y \right) \mathrm{Tr}_S \left(I_1 \int_y^x \right) = \bar{\mu} \quad \text{and} \quad \mathrm{Tr}_S \left(I_7 \int_x^y \right) \mathrm{Tr}_S \left(I_8 \int_y^x \right) = \bar{\mu\nu} \mu_8 ,$$

the replacement rule (4.33) is straightforward. \blacksquare

We note for clarity that, after multiplying out the formulas of Def. 4.4, Λ_+ and Λ_- are obtained from each other by replacing the chiral indices according to $\hat{Y}_L \hat{Y}_L \longleftrightarrow \hat{Y}_R \hat{Y}_R$. But all other products of the dynamical mass matrices as well as the phase factors ν , μ , and δ are the same in Λ_- and Λ_+ . In contrast, the replacement rule (3.9) flips the chiral indices of all mass matrices, and also reverses the chirality of the gauge fields.

A straightforward calculation using (4.10), Lemma 3.3, and Lemma 4.5 shows that for our Lagrangian (2.59), the EL equations yield the conditions

$$\left[\overline{\lambda_{\uparrow L-}^{xy}} (\Lambda_- + \Lambda_+) + \lambda_{\uparrow L-}^{xy} (\overline{\Lambda_-} + \overline{\Lambda_+}) \right] \overline{\lambda_{\uparrow L-}^{xy}} P(x, y) = f(x, y) I_{\uparrow} P(x, y) , \quad (4.34)$$

where we set $\lambda_{\uparrow cs} = \lambda_{ncs}$, $n \in \uparrow$. Here $f(x, y)$ can be any scalar function; it takes into account that the average of all eigenvalues drops out in the EL equations when we take the difference of the contributions resulting from the two terms in (2.59). Similar conditions for the opposite chirality and for \uparrow replaced by \downarrow are obtained from (4.34) by applying the rules (4.33) and (4.32). We point out that the resulting four equations must clearly be satisfied for the same function $f(x, y)$. These four equations together are even equivalent to the EL equations to degree 10.

The remaining problem is to analyze the obtained equations of types (4.34). At first sight, this seems a difficult problem because the matrices Λ_{\pm} have a complicated explicit form (see Def. 4.4) and because taking the spectral adjoints makes it necessary to diagonalize these matrices. Fortunately, the requirement that the EL equations be mathematically consistent will give us strong restrictions on the form of Λ_{\pm} , and will indeed make it possible to reveal a relatively simple global structure of the admissible interactions. In order to explain how the mathematical consistency conditions come about, we first recall that for polynomial Lagrangians (2.47) we saw after (2.50) that the resulting operator Q is a polynomial in the fermionic projector and is thus well-defined within the formalism of the continuum limit. However, the situation is different for our Lagrangian (2.60), because the spectral weight is an operation which does not necessarily make sense in the continuum limit. Using the relation (2.64), we could show that to the highest degree on the light cone, the operator Q is a linear combination of monomials, but we cannot expect that the same is true to the next lower degree.

More specifically, the mathematical problem in (4.34) is to make sense of the spectral adjoint. For clarity, we explain the difficulty and our basic argument in the simple example

$$\overline{B_1 M_1 + B_2 M_2} , \quad (4.35)$$

where B_1 and B_2 are matrices depending on the macroscopic potentials, and $M_{1/2}$ are two monomials. The monomials can be considered as scalar functions which are highly singular on the light cone, and which we can control in the continuum limit only in the weak sense. To form the spectral adjoint in (4.35), we need to know the eigenvalues and spectral projectors of the matrix $B_1 M_1 + B_2 M_2$. In general, the spectral decomposition of

this matrix will depend nonlinearly on M_1 and M_2 , because the zeros of the characteristic polynomials involve roots of the monomials. In this generic situation, the spectral adjoint is ill-defined in the formalism of the continuum limit. The only case in which the eigenvalues are linear in M_1 and M_2 is when the eigenvectors can be chosen independent of the monomials. This is possible iff the matrices B_1 and B_2 have a common eigenvector basis, or equivalently if they commute,

$$[B_1, B_2] = 0 .$$

This simple argument shows that the requirement that the spectral adjoint be well-defined leads to commutator relations for the macroscopic potentials. In the next lemma we apply this argument to the matrices Λ_- and Λ_+ . By the *contributions to Λ_-* (and similarly for Λ_+) we mean the individual summands obtained by multiplying out all the terms in (4.16)–(4.20).

Lemma 4.6 *For any $x, y \in \mathcal{L}$ there is a basis on the sectors such that the contributions to Λ_- are all diagonal matrices. In this basis, also all contributions to Λ_+ are diagonal.*

Proof. Clearly, our argument after (4.35) applies in the same way to the spectral adjoint of a finite sum. Thus in order to make mathematical sense of the spectral adjoint $\overline{\Lambda_-}$ we need to assume that the contributions to Λ_- all commute with each other. Hence we can choose a basis such that these contributions are all diagonal. In particular, one sees that in this basis the matrix products $\hat{Y}_{c_1}(x)$ and $\hat{Y}_{c_2}(y)$ are diagonal for all $c_1, c_2 \in \{L, R\}$. Since Λ_+ is obtained from Λ_- by suitably changing the chiral indices in these matrix products, it follows immediately that the contributions to Λ_+ are also diagonal. ■

For clarity, we point out that it is a-priori not clear whether the basis of the above lemma is orthogonal. Thus at the moment we do not know if the matrices Λ_{\mp} are normal. But Lemma 4.6 yields that Λ_- and Λ_+ commute with each other (and also these matrices commute by definition with their spectral adjoints). For commuting matrices, the spectral adjoint satisfies the relation

$$\overline{A + B} = \overline{A} + \overline{B} \quad \text{if } [A, B] = 0.$$

This relation is very useful in the EL equations. Namely, setting

$$\Lambda = \Lambda_- + \Lambda_+ ,$$

we can write (4.34) as

$$[\overline{\lambda_{\uparrow L-}} \Lambda + \lambda_{\uparrow L-} \overline{\Lambda}] \overline{\lambda_{\uparrow L-}} P(x, y) = f(x, y) I_{\uparrow} P(x, y) + (\deg < 10) , \quad (4.36)$$

where for simplicity we omitted the indices “ xy ”. The other three EL equations are obtained from (4.36) again by applying the replacement rules (4.32) and (4.33).

We proceed by analyzing the EL equations (4.36) for special choices of x and y , for which the matrix Λ becomes particularly simple. We begin with the situation where we choose x such that the scalar potentials vanish at x , i.e.

$$Y_L(x) = Y = Y_R(x) \quad (4.37)$$

with Y the mass matrix of the vacuum (for example, we can choose x close to infinity). Then the matrices $Y_{L/R}(x)$ are diagonal in the sector index, and are on the massive sectors a multiple of the identity. Thus the “off-diagonal” contributions (4.19),(4.20) to Λ_- vanish

(and similarly for Λ_+). In the “diagonal” contributions (4.16)–(4.18), on the other hand, we can simplify our notation by omitting the factors $\hat{Y}_{L/R}(x)$. Then the matrix Λ takes the form

$$\begin{aligned}\Lambda = & 2\nu \int_x^y dz I_\uparrow \dot{Y}_L \dot{Y}_R I_\uparrow T_{[2]}^{(0)} \overline{T_{[0]}^{(0)}} \\ & + 2\nu \int_y^x dz I_\uparrow \dot{Y}_R \dot{Y}_L I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \\ & + I_\uparrow (\hat{Y}_L(y) + \hat{Y}_R(y)) I_\uparrow T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \\ & - 2 I_\uparrow \dot{Y}_R(y) \dot{Y}_L I_\uparrow T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \\ & - \frac{\nu T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \bar{\nu} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}}}{\nu \lambda_- - \bar{\nu} \lambda_+} I_\uparrow \left(\hat{Y}_L(y) T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_R(y) T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) I_\uparrow \\ & - \frac{T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}}}{\nu \lambda_- - \bar{\nu} \lambda_+} I_\uparrow \left(\nu \hat{Y}_L(y) T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \bar{\nu} \hat{Y}_R(y) T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) I_\uparrow\end{aligned}$$

Evaluating the EL equations (4.36) for this choice of Λ yields the following result.

Lemma 4.7 *Suppose that (I) holds. Without introducing any relations between the basic monomials (besides those of Theorem 3.6), we can choose for any $y \in \mathcal{L}$ suitable parameters $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$ such that at y ,*

$$I_\uparrow \dot{Y}_L \dot{Y}_R I_\uparrow = a I_\uparrow, \quad I_\uparrow \dot{Y}_R \dot{Y}_L I_\uparrow = b I_\uparrow \quad (4.38)$$

$$I_\uparrow \hat{Y}_L(y) I_\uparrow = c I_\uparrow, \quad I_\uparrow \hat{Y}_R(y) I_\uparrow = \bar{c} I_\uparrow. \quad (4.39)$$

The analogous formulas for I_\uparrow interchanged by I_\downarrow are obtained by the replacements

$$\uparrow \longleftrightarrow \downarrow \quad \text{and} \quad L \longleftrightarrow R \quad (4.40)$$

with the parameters a, b , and c unchanged.

Proof. The above Λ contains contributions which are scalar multiples of the matrices $I_\uparrow \hat{Y}_L(y) I_\uparrow$ and $I_\uparrow \hat{Y}_R(y) I_\uparrow$. Thus in the basis of Lemma 4.6, these matrices are both diagonal. Since one is the adjoint of the other, we conclude that these matrices are normal, and thus their spectral adjoints coincide with the usual adjoints,

$$\overline{I_\uparrow \hat{Y}_L I_\uparrow} = I_\uparrow \hat{Y}_R I_\uparrow, \quad \overline{I_\uparrow \hat{Y}_R I_\uparrow} = I_\uparrow \hat{Y}_L I_\uparrow. \quad (4.41)$$

The matrices $\dot{Y}_L \dot{Y}_R$ and $\dot{Y}_R \dot{Y}_L$, on the other hand, are Hermitian and thus spectrally self-adjoint,

$$\overline{\dot{Y}_L \dot{Y}_R} = \dot{Y}_L \dot{Y}_R, \quad \overline{\dot{Y}_R \dot{Y}_L} = \dot{Y}_R \dot{Y}_L. \quad (4.42)$$

Applying the relations (4.41) and (4.42), a straightforward calculation gives

$$\begin{aligned}& \overline{\lambda_{\uparrow L-}} \Lambda + \lambda_{\uparrow L-} \overline{\Lambda} \\ = & 2 \int_x^y dz I_\uparrow \dot{Y}_L \dot{Y}_R I_\uparrow \left(T_{[0]}^{(0)} T_{[2]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} + T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(0)} T_{[2]}^{(0)}} \right) \\ & + 2 \int_y^x dz I_\uparrow \dot{Y}_R \dot{Y}_L I_\uparrow \left(T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[2]}^{(1)}} + T_{[0]}^{(-1)} T_{[2]}^{(1)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} \right)\end{aligned}$$

$$\begin{aligned}
& +\bar{\nu} I_{\uparrow} (\hat{Y}_L + \hat{Y}_R) I_{\uparrow} T_{[0]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[1]}^{(0)}} + \nu I_{\uparrow} (\hat{Y}_L + \hat{Y}_R) I_{\uparrow} T_{[0]}^{(-1)} T_{[1]}^{(0)} \overline{T_{[0]}^{(0)} T_{[1]}^{(0)}} \\
& -2\bar{\nu} I_{\uparrow} \hat{Y}_R \hat{Y} I_{\uparrow} T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[2]}^{(1)}} - 2\nu \overline{I_{\uparrow} \hat{Y}_R \hat{Y} I_{\uparrow}} T_{[0]}^{(-1)} T_{[2]}^{(1)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} \\
& + \left(\nu T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \bar{\nu} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) I_{\uparrow} \left(\hat{Y}_L T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_R T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) I_{\uparrow} \\
& + \left(T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) I_{\uparrow} \left(\nu \hat{Y}_L T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \bar{\nu} \hat{Y}_R T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) I_{\uparrow} \\
= & 2 \int_x^y dz I_{\uparrow} \hat{Y}_L \hat{Y}_R I_{\uparrow} \left(T_{[0]}^{(0)} T_{[2]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} + T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(0)} T_{[2]}^{(0)}} \right) \\
& +2 \int_y^x dz I_{\uparrow} \hat{Y}_R \hat{Y}_L I_{\uparrow} \left(T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[2]}^{(1)}} + T_{[0]}^{(-1)} T_{[2]}^{(1)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} \right) \\
& +2\nu \left(I_{\uparrow} \hat{Y}_L I_{\uparrow} T_{[1]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} - \overline{I_{\uparrow} \hat{Y}_R \hat{Y} I_{\uparrow}} T_{[0]}^{(-1)} T_{[2]}^{(1)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}} \right) \\
& +2\bar{\nu} \left(I_{\uparrow} \hat{Y}_R I_{\uparrow} T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)} T_{[1]}^{(0)}} - I_{\uparrow} \hat{Y}_R \hat{Y} I_{\uparrow} T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)} T_{[2]}^{(1)}} \right),
\end{aligned}$$

where for simplicity the arguments y were omitted. We substitute this formula into (4.36). Since we do not allow for additional relations between the basic monomials, we can simplify the resulting monomials only by applying (3.42). From this one sees that (4.36) is satisfied for suitable $f(x, y)$ if and only if the following five matrices are multiples of I_{\uparrow} ,

$$\int_x^y I_{\uparrow} \hat{Y}_L \hat{Y}_R I_{\uparrow}, \quad \int_y^x I_{\uparrow} \hat{Y}_R \hat{Y}_L I_{\uparrow} \quad (4.43)$$

$$\bar{\nu} \left(I_{\uparrow} \hat{Y}_R I_{\uparrow} - I_{\uparrow} \hat{Y}_R \hat{Y} I_{\uparrow} \right), \quad \nu I_{\uparrow} \hat{Y}_L I_{\uparrow} - \bar{\nu} I_{\uparrow} \hat{Y}_R \hat{Y} I_{\uparrow}, \quad \nu \overline{I_{\uparrow} \hat{Y}_L I_{\uparrow}} - \bar{\nu} I_{\uparrow} \hat{Y}_R I_{\uparrow}. \quad (4.44)$$

We can assume that $y \neq x$, because otherwise (4.38) and (4.39) follow immediately from (4.37). Differentiating (4.43) with respect to y along the line \mathcal{L} gives (4.38) (a and b are real because the matrices on the left of (4.38) are Hermitian). According to **(I)**, the phase factor ν can take any value on the unit circle. Thus in (4.44) the contributions involving ν and $\bar{\nu}$ must separately be multiples of I_{\uparrow} . This gives the left relation in (4.39), and the relation on the right is obtained by taking the adjoint.

The analogous relations for I_{\uparrow} replaced by I_{\downarrow} are derived in the same way. The replacements (4.40) leave the phase factor ν unchanged (see (4.14) and (4.15)). Thus the EL equation (4.34) remains valid under (4.40) for the same function f only if the parameters a , b , and c are unchanged. \blacksquare

Next we consider the degeneracies in the limit $y \rightarrow x$. In this case, the formulas of Def. 4.4 simplify in that all phase factors drop out. We obtain the following result.

Lemma 4.8 *Without introducing any relations between the basic monomials (besides those of Theorem 3.6), the dynamical mass matrices must satisfy the relations*

$$I_{\uparrow} \hat{Y}_L I_{\downarrow} = 0 = I_{\uparrow} \hat{Y}_R I_{\downarrow}. \quad (4.45)$$

Proof. According to the replacement rule (4.32), it suffices to derive the second part of (4.45). We compute the matrix Λ modulo scalar multiples of I_{\uparrow} . Using (4.38) and (2.28),

we obtain

$$\begin{aligned}
\Lambda &= \left(I_{\uparrow} \hat{Y}_L I_{\downarrow} \hat{Y}_L I_{\uparrow} + I_{\uparrow} \hat{Y}_R I_{\downarrow} \hat{Y}_R I_{\uparrow} \right) T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \\
&\quad - \frac{1}{\lambda_- - \lambda_+} I_{\uparrow} \left(\hat{Y}_L T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_R T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) I_{\downarrow} \left(\hat{Y}_R T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \hat{Y}_L T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) I_{\uparrow} \\
&\quad - \frac{1}{\lambda_- - \lambda_+} I_{\uparrow} \left(\hat{Y}_L T_{[1]}^{(0)} \overline{T_{[0]}^{(0)}} - \hat{Y}_R T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \right) I_{\downarrow} \left(\hat{Y}_R T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - \hat{Y}_L T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) I_{\uparrow} \\
&= 2 \left(I_{\uparrow} \hat{Y}_L I_{\downarrow} \hat{Y}_L I_{\uparrow} + I_{\uparrow} \hat{Y}_R I_{\downarrow} \hat{Y}_R I_{\uparrow} \right) \frac{\lambda_-}{\lambda_- - \lambda_+} T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \\
&\quad - 2 I_{\uparrow} \hat{Y}_L I_{\downarrow} \hat{Y}_R I_{\uparrow} \frac{1}{\lambda_- - \lambda_+} T_{[1]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} \overline{T_{[0]}^{(0)}} \\
&\quad - 2 I_{\uparrow} \hat{Y}_R I_{\downarrow} \hat{Y}_L I_{\uparrow} \frac{1}{\lambda_- - \lambda_+} T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \overline{T_{[1]}^{(0)}}.
\end{aligned}$$

Next we compute the square bracket in (4.36),

$$\begin{aligned}
&\overline{\lambda_{\uparrow L-}} \Lambda + \lambda_{\uparrow L-} \overline{\Lambda} \\
&= -2 I_{\uparrow} \hat{Y}_L I_{\downarrow} \hat{Y}_R I_{\uparrow} \frac{1}{\lambda_- - \lambda_+} \left(\lambda_+ T_{[1]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} \overline{T_{[0]}^{(0)}} - \lambda_- T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \overline{T_{[1]}^{(0)}} \right) \\
&\quad + 2 I_{\uparrow} \hat{Y}_R I_{\downarrow} \hat{Y}_L I_{\uparrow} \frac{1}{\lambda_- - \lambda_+} \left(\lambda_- T_{[1]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} \overline{T_{[0]}^{(0)}} - \lambda_+ T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \overline{T_{[1]}^{(0)}} \right)
\end{aligned}$$

Since I_{\downarrow} projects onto a subspace of dimension $7 - p < p$, the rank of the matrices $I_{\uparrow} \hat{Y}_R I_{\downarrow} \hat{Y}_L I_{\uparrow}$ and $I_{\uparrow} \hat{Y}_L I_{\downarrow} \hat{Y}_R I_{\uparrow}$ is smaller than p , and therefore these matrices cannot be scalar multiples of I_{\uparrow} . Thus the EL equations have a well-defined continuum limit only if the factors $(\lambda_- - \lambda_+)^{-1}$ in the above expression drop out. This is the case only if

$$I_{\uparrow} \hat{Y}_R I_{\downarrow} \hat{Y}_L I_{\uparrow} = I_{\uparrow} \hat{Y}_L I_{\downarrow} \hat{Y}_R I_{\uparrow}.$$

If these necessary conditions are satisfied, the above formula simplifies to

$$\overline{\lambda_{\uparrow L-}} \Lambda + \lambda_{\uparrow L-} \overline{\Lambda} = 2 I_{\uparrow} \hat{Y}_L I_{\downarrow} \hat{Y}_R I_{\uparrow} \left(T_{[1]}^{(0)} T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} \overline{T_{[0]}^{(0)}} + T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[1]}^{(0)}} \overline{T_{[1]}^{(0)}} \right).$$

Now the EL equations have a well-defined continuum limit, and assuming for the regularization parameters only the relation (3.42), we conclude that

$$I_{\uparrow} \hat{Y}_R I_{\downarrow} \hat{Y}_L I_{\uparrow} = 0. \quad (4.46)$$

The matrix product in this equation can be written in the form BB^* with $B \equiv I_{\uparrow} \hat{Y}_R I_{\downarrow}$. Hence (4.46) implies that $B = 0$. \blacksquare

The previous two lemmas simplify the structure of the perturbation on the degenerate subspaces considerably. Namely, we can write Λ in the form

$$\Lambda = \rho(\nu, \bar{\nu}) I_{\uparrow} - 2 I_{\uparrow} \dot{Y}_R(y) (I_{\uparrow} + \mu\nu I_{\downarrow} + \mu_8 \nu_8 I_8) \dot{Y}_L(x) I_{\uparrow} T_{[0]}^{(-1)} \overline{T_{[0]}^{(-1)}},$$

where ρ is a complex function which is invariant under the replacements (4.40). A short calculation yields

$$\frac{1}{2} (\overline{\lambda_{\uparrow L-}} \Lambda + \lambda_{\uparrow L-} \overline{\Lambda}) = (a + \nu b + \overline{\nu} \overline{b}) I_{\uparrow} \quad (4.47)$$

$$-\overline{\nu} I_{\uparrow} \dot{Y}_R(y) I_{\uparrow} \dot{Y}_L(x) I_{\uparrow} \overline{N} - \nu \overline{I_{\uparrow} \dot{Y}_R(y) I_{\uparrow} \dot{Y}_L(x) I_{\uparrow} N} \quad (4.48)$$

$$-\mu I_{\uparrow} \dot{Y}_R(y) I_{\downarrow} \dot{Y}_L(x) I_{\uparrow} \overline{N} - \overline{\mu} \overline{I_{\uparrow} \dot{Y}_R(y) I_{\downarrow} \dot{Y}_L(x) I_{\uparrow} N} \quad (4.49)$$

$$-\mu_8 \overline{\nu} \nu_8 I_{\uparrow} \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_{\uparrow} \overline{N} - \overline{\mu_8 \nu \nu_8} \overline{I_{\uparrow} \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_{\uparrow} N}, \quad (4.50)$$

where the complex functions a and b are invariant under (4.40), and N is the monomial

$$N = T_{[0]}^{(-1)} T_{[2]}^{(1)} \overline{T_{[0]}^{(-1)} T_{[0]}^{(0)}}. \quad (4.51)$$

We split up the analysis of the EL equations corresponding to (4.47)–(4.50) into several lemmas. We say that the summands (4.49) or (4.50) are *non-trivial* if there are admissible dynamical mass matrices such that this summand or one of the expressions obtained by applying the replacements (4.32) and/or (4.33) are non-zero. Furthermore, we refer to two phase functions $\alpha, \beta \in S^1$ as being *independent* if for α fixed, β can take any value in S^1 and vice versa.

Lemma 4.9 *Under the assumptions (I)–(III), ν is independent of the phase functions μ , μ_8 , ν_8 , and $\overline{\mu}\mu_8$. The term (4.50) is non-trivial.*

Proof. Suppose that the dynamical mass matrices were zero in the neutrino sector, i.e.

$$Y_L I_8 \equiv 0 \equiv Y_R I_8 \quad (4.52)$$

Then the Dirac operator, and thus also the fermionic projector, would be invariant on the neutrino sector. As a consequence, the chiral Dirac particles would drop out of all composite expressions due to chiral cancellations, in contradiction to (II). We conclude that (4.52) is false. Since we are free to choose a convenient representation of the dynamical mass matrices, we can assume that the matrices

$$\dot{Y}_L I_8, \quad \dot{Y}_R I_8, \quad \dot{Y}_L I_8, \quad \dot{Y}_R I_8 \quad (4.53)$$

do not all vanish identically. The contributions to the fermionic projector which involve the matrix products $I_8 \dot{Y}_{L/R}$ or $\dot{Y}_{L/R} I_8$ enter only the perturbation calculation for the kernel of $P(x, y) P(y, x)$, and according to Theorem A.3 they drop out of the EL equations. Thus (II) is satisfied only if

$$(I_{\uparrow} + I_{\downarrow}) \dot{Y}_L I_8 \neq 0 \quad \text{or} \quad (I_{\uparrow} + I_{\downarrow}) \dot{Y}_R I_8 \neq 0.$$

This shows that (4.50) is non-trivial.

According to (III), there is a free dynamical gauge field which couples differently to the Dirac particles in the sectors $n = 1$ and $n = 8$. The corresponding free gauge potentials describe relative phase shifts of the fermionic projector on $\text{Im } I_1$ and $\text{Im } I_8$. These relative phases are captured by μ_8 and $\mu_8 \nu_8$ (see (4.15)). Since the free gauge potentials on the line segment \overline{xy} can be chosen arbitrarily, it follows that ν is independent of μ_8 and $\mu_8 \nu_8$. A similar argument for I_7 instead of I_1 shows that ν and $\overline{\mu}\mu_8$ are independent. ■

Lemma 4.10 *Imposing at most one additional relation between the basic monomials (besides those of Theorem 3.6), ν and μ are independent. The term (4.49) is non-trivial.*

Proof. Assume to the contrary that ν and μ are dependent or that (4.49) is trivial. Then the phases in (4.47)–(4.49) are all dependent on ν . The independence of the phases established in Lemma 4.9 yields that the EL equations must be satisfied separately for (4.50). Imposing at most one additional relation between the basic monomials, we cannot arrange that (4.50) drops out of the EL equations. We thus obtain that for a suitable complex κ ,

$$I_{\uparrow} \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_{\uparrow} = \kappa(x, y) I_{\uparrow}, \quad (4.54)$$

and this condition must also be satisfied after the replacements (4.32) and/or (4.33) for the same κ . Since the rank of I_8 is smaller than that of I_{\uparrow} , the lhs of (4.54) is a singular matrix, and thus κ vanishes identically. This implies that the lhs of (4.54) is trivial (i.e. vanishes also after the replacements (4.32),(4.33)), in contradiction to Lemma 4.9. ■

Having established that the phases in (4.47) and (4.48) are independent of those in (4.49) and (4.50), we can now apply the uniform splitting lemma to (4.48).

Lemma 4.11 *Imposing at most one additional relation between the basic monomials, we can arrange by a constant unitary transformation that for all $a, b = 1, \dots, p$ and $c, d = p+1, \dots, 7$,*

$$(I_{\uparrow} \dot{Y}_{L/R} I_{\uparrow})_b^{(a\alpha)} = \delta_b^a u_{L/R}^\alpha, \quad (I_{\downarrow} \dot{Y}_{R/L} I_{\downarrow})_d^{(c\alpha)} = \delta_d^c u_{L/R}^\alpha \quad (4.55)$$

with $u_{L/R}(x) \in \mathbb{C}^3$.

Proof. It clearly suffices to consider one chirality. Since ν is independent of μ and $\mu_8 \nu_8$, the EL equations imply that

$$I_{\uparrow} \dot{Y}_R(y) I_{\uparrow} \dot{Y}_L(x) I_{\uparrow} = \lambda(x, y) I_{\uparrow}. \quad (4.56)$$

The dynamical mass matrices can be chosen independently at x and y . Denoting the class of admissible matrices $I_{\uparrow} \dot{Y}_L I_{\uparrow}$ by \mathcal{B} , we are in the setting of Lemma 4.3 with $p_1 = p$ and $p_2 = 3p$. Since p_2 is divisible by p_1 , we can, possibly after increasing r , assume that $p_2 - rp_1 = 0$, and thus

$$I_{\uparrow} \dot{Y}_L I_{\uparrow} = U \underbrace{(u_L \oplus \dots \oplus u_L)}_{p \text{ summands}}$$

with $u_L(x) \in \mathbb{C}^3$. Omitting the constant unitary transformation and writing out the components, this is just the lhs of (4.55). Under the replacement (4.40), ν as well as α and β are unchanged. As a consequence, also the function λ in (4.56) is invariant under (4.40), and this implies that the mappings π_i of Lemma 4.3 obtained for $B = I_{\uparrow} \dot{Y}_L I_{\uparrow}$ and $B = I_{\downarrow} \dot{Y}_R I_{\downarrow}$ are all unitarily equivalent. This proves the rhs of (4.55). ■

It remains to analyze (4.49) and (4.50).

Lemma 4.12 *The EL equations to degree 10 can be satisfied only if we impose at least one additional relation between the basic monomials.*

Proof. In the limit $y \rightarrow x$, the matrices $I_\uparrow \dot{Y}_R(y) I_\downarrow \dot{Y}_L(x) I_\uparrow$ can be written in the form $B^* B$ with $B = I_\downarrow \dot{Y}_L I_\uparrow$ and are therefore Hermitian and positive semidefinite. This shows that (4.49) and (4.50) cannot cancel each other identically. According to Lemma 4.10, (4.49) is non-trivial. It suffices to consider the case that (4.49) does not vanish identically (in the other cases when (4.49) is non-zero after applying (4.32), (4.33) the argument is analogous). Then we can arrange a contribution to (4.48)–(4.50) of the form $(\mu AN + \overline{\mu AN})$ with a matrix $A \neq 0$. The same contribution must be present after performing the replacements (4.40). Since these replacements transform μ into $\overline{\mu}$ (see (4.32) and (4.33)), we obtain a condition of the form

$$\mu AN + \overline{\mu AN} = \overline{\mu} BN + \mu \overline{BN} \quad \text{for all } \mu \in S^1 \quad (4.57)$$

with B a matrix. Without introducing an additional relation between the basic monomials, we must treat N and \overline{N} as being independent, and thus (4.57) has no solution. \blacksquare

Using that A and B go over to positive matrices as $y \rightarrow x$, one sees that in order to arrange that (4.57) has a solution, we need to impose that N and \overline{N} coincide in the EL equations, i.e.

$$(N - \overline{N}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} = 0. \quad (4.58)$$

The next lemma is again an application of the uniform splitting lemma and uses the non-degeneracy assumption of Def. 4.2.

Lemma 4.13 *Suppose that the basic monomials satisfy (besides the conditions of Theorem 3.6) the relation (4.58) with N according to (4.51). Then the parameter p in (3.41) is equal to 4. The phase factors in the neutrino sector are determined by*

$$\nu_8 = \nu \quad \text{and} \quad \mu_8 = \mu \text{ or } \overline{\mu}. \quad (4.59)$$

We can arrange by constant unitary transformations that for $a, b = 1, 2, 3$,

$$(I_\downarrow \dot{Y}_{L/R} I_\uparrow)_b^{(a+4\alpha)} = \delta_b^a v_{L/R}^\alpha, \quad (I_\uparrow \dot{Y}_{R/L} I_\downarrow)_{b+4}^{(a\alpha)} = \delta_b^a \overline{v_{L/R}^\alpha} \quad (4.60)$$

with $v_{L/R}(x) \in \mathbb{C}^3$. In the two cases for μ_8 in (4.59),

$$(I_8 \dot{Y}_{L/R} I_\uparrow)_4^{(8\alpha)} = v_{L/R}^\alpha \text{ or } \overline{v_{L/R}^\alpha}, \quad (4.61)$$

respectively. Furthermore,

$$I_8 \dot{Y}_{R/L} I_\downarrow = 0. \quad (4.62)$$

Proof. Imposing (4.58) and using (4.55), the EL equations (4.36) reduce to the conditions

$$\begin{aligned} & \mu I_\uparrow \dot{Y}_R(y) I_\downarrow \dot{Y}_L(x) I_\uparrow + \overline{\mu I_\uparrow \dot{Y}_R(y) I_\downarrow \dot{Y}_L(x) I_\uparrow} \\ & + \mu_8 \overline{\nu} \nu_8 I_\uparrow \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_\uparrow + \overline{\mu_8 \nu \overline{\nu}_8} I_\uparrow \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_\uparrow = \lambda(x, y) I_\uparrow. \end{aligned} \quad (4.63)$$

We first prove that the phase factors must be dependent in the sense that

$$\mu_8 \overline{\nu} \nu_8 = \mu \text{ or } \overline{\mu}. \quad (4.64)$$

Assuming the contrary, we must treat the four summands in (4.63) as being independent, and thus

$$I_\uparrow \dot{Y}_R(y) I_\downarrow \dot{Y}_L(x) I_\uparrow = \kappa(x, y) I_\uparrow. \quad (4.65)$$

Performing the replacement (4.40) and using that μ transforms to $\bar{\mu}$, we obtain furthermore that

$$I_{\downarrow} \dot{Y}_L(y) I_{\uparrow} \dot{Y}_R(x) I_{\downarrow} = \kappa(x, y) I_{\downarrow} \quad (4.66)$$

with the same κ as in (4.65). We apply Lemma 4.3 to (4.65) (with $p_1 = p$ and $p_2 = 3(7-p)$) and to (4.66) (with $p_1 = 7-p$ and $p_2 = p$). Leaving out the constant unitary transformations, we obtain the representations

$$I_{\downarrow} \dot{Y}_L I_{\uparrow} = \begin{pmatrix} \overbrace{b \oplus \cdots \oplus b}^{p \text{ summands}} \\ 0 \end{pmatrix}, \quad I_{\uparrow} \dot{Y}_R I_{\downarrow} = \begin{pmatrix} \overbrace{\bar{b} \oplus \cdots \oplus \bar{b}}^{7-p \text{ summands}} \\ 0 \end{pmatrix}, \quad (4.67)$$

where \bar{b} is the complex conjugate of the vector $b \in \mathbb{C}^3$. According to Lemma 4.10, (4.49) is non-trivial. Since the contributions to the EL equations involving μ are unchanged when applying the replacements (4.32) and (4.33), we can arrange that (4.65) does not vanish, and thus $b \neq 0$. On the lhs of (4.67), the inequality $rp_1 \leq p_2$ implies that $r < 3$. Thus on the rhs of (4.67), the number of zero rows is $3p - r(7-p) > 3$. Therefore, $I_p \dot{Y}_R I_{\downarrow} = 0$, or, equivalently, by taking the adjoint and in components,

$$(\dot{Y}_L)_{(d.)}^a = 0 \quad \text{for } d = p \text{ and } a = p+1, \dots, 7.$$

On the other hand, the lhs of (4.67) implies that

$$(\dot{Y}_L)_{(d.)}^a \neq 0 \quad \text{for } d = p \text{ and } a = p+1, \dots, 7.$$

The non-degeneracy assumption of Def. 4.2 allows us to conclude that

$$(\dot{Y}_L)_{(d.)}^a = 0 \quad \text{for all } a = p+1, \dots, 7 \text{ and } d \neq a.$$

This implies that $I_{\downarrow} \dot{Y}_L I_{\uparrow} = 0$, in contradiction to the rhs of (4.67) and the fact that $b \neq 0$.

Repeating the above argument for the opposite chirality gives in analogy to (4.64) that

$$\mu_8 = \mu \text{ or } \bar{\mu}. \quad (4.68)$$

Using that μ and ν are independent according to Lemma 4.10, (4.64) and (4.68) are equivalent to (4.59).

In the case $\mu_8 = \mu$, the EL equations (4.36) reduce to the conditions

$$I_{\uparrow} \dot{Y}_R(y) (I_{\downarrow} + I_8) \dot{Y}_L(x) I_{\uparrow} = \kappa(x, y) I_{\uparrow}. \quad (4.69)$$

After the replacement (4.40), the phase factors in (4.63) are no longer dependent (cf. (4.32) and (4.33)), and thus we get the conditions

$$I_{\downarrow} \dot{Y}_L(y) I_{\uparrow} \dot{Y}_R(x) I_{\downarrow} = \overline{\kappa(x, y)} I_{\downarrow} \quad (4.70)$$

$$I_{\downarrow} \dot{Y}_L(y) I_8 \dot{Y}_R(x) I_{\downarrow} = 0. \quad (4.71)$$

The last relation implies (4.62). Applying the above argument for (4.65) and (4.66) to (4.69) and (4.70), we again get a contradiction unless $\text{Rg } I_{\uparrow} = \text{Rg } (I_{\downarrow} + I_8)$. This shows that $p = 4$. Possibly after increasing r , we obtain in analogy to (4.67) the representations

$$(I_{\downarrow} + I_8) \dot{Y}_L I_{\uparrow} = b \oplus b \oplus b \oplus b, \quad I_{\uparrow} \dot{Y}_R I_{\downarrow} = \bar{b} \oplus \bar{b} \oplus \bar{b}. \quad (4.72)$$

Writing these relations in components gives (4.60) and (4.61).

In the case $\mu_8 = \overline{\mu}$, we obtain in analogy to (4.69) the condition

$$I_{\uparrow} \dot{Y}_R(y) I_{\downarrow} \dot{Y}_L(x) I_{\uparrow} + \overline{I_{\uparrow} \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_{\uparrow}} = \kappa(x, y) I_{\uparrow}, \quad (4.73)$$

and after the replacement (4.40) again the conditions (4.70) and (4.71). The lhs of (4.70) can be split into a product of matrices of the form $A(y) B(x)$. Since (4.70) and (4.73) involve the same function $\kappa(x, y)$, the matrices on the lhs of (4.73) must split in the same way. To this end, the matrix $I_{\uparrow} \dot{Y}_R(y) I_8 \dot{Y}_L(x) I_{\uparrow}$ must (possibly after a constant unitary transformation) be diagonal for all x and y , so that the spectral adjoint reduces to the complex conjugate (i.e. to taking the complex conjugate of all matrix entries). After taking this complex conjugate, we can proceed exactly as in the case $\mu_8 = \mu$ above. The only difference is that we obtain a representation not for the matrix $I_8 \dot{Y}_L I_{\uparrow}$ but for its complex conjugate, and this leads to the complex conjugate in (4.61). ■

We remark that the fact that the partial trace is non-cyclic (4.2) is essential for the above construction to work. Namely, according to the lhs of (4.60),

$$I_{\uparrow} \dot{Y}_L I_8 \dot{Y}_R I_{\uparrow} \stackrel{\text{in general}}{\neq} 0. \quad (4.74)$$

On the other hand, the weak causality compatibility condition, Def. 4.1, implies that

$$I_8 \dot{Y}_R I_{\uparrow} \dot{Y}_L I_8 = X_R I_8 \dot{Y}_R I_{\uparrow} \dot{Y}_L I_8 = 0. \quad (4.75)$$

If the partial trace were cyclic, (4.74) and (4.75) would be inconsistent.

Combining the previous lemmas and choosing a convenient representation for the dynamical mass matrices gives the main result of this section.

Theorem 4.14 (spontaneous block formation) *We consider the EL equations corresponding to the Lagrangian (2.61) in the presence of chiral and scalar potentials (3.1)–(3.5) to the degree 10 on the light cone. We assume that the Dirac operator is weakly causality compatible and that the dynamical mass matrices are non-degenerate (see Defs. 4.1 and 4.2). Then, following (IV), we need to introduce two relations between the basic monomials. Imposing that*

$$(M - \overline{M}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} = 0 = (N - \overline{N}) T_{[0]}^{(-1)} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} \quad (4.76)$$

with M, N according to (3.42) and (4.51), we can arrange by constant unitary transformations that the Dirac operator is of the following form,

$$i\partial - m \chi_L \left(Y_R^q \oplus Y_R^q \oplus Y_R^q \oplus Y_R^l \right) - m \chi_R \left(Y_L^q \oplus Y_L^q \oplus Y_L^q \oplus Y_L^l \right) \quad (4.77)$$

$$+ (\chi_R \mathcal{A}_L + \mathcal{A}_V) (\sigma^3 \oplus \sigma^3 \oplus \sigma^3 \oplus \sigma^3) \quad (4.78)$$

$$+ (\mathcal{A}^q \mathbf{1}) \oplus 0_{\mathbf{C}^2} + 0_{\mathbf{C}^6} \oplus (\mathcal{A}^l \mathbf{1} + \mathcal{A}^s \sigma^3). \quad (4.79)$$

Here $Y_{L/R}^{q/l}$ are 2×2 matrices on the sectors which depend also on the generations, i.e. in components

$$Y_c^{q/l} = (Y_c^{q/l})_{(b\beta)}^{(a\alpha)} \quad \text{with } a, b = 1, 2, \alpha, \beta = 1, 2, 3, c = L/R.$$

The chiral and vector potentials are trivial on the generations and depend only on the sector index. A_L , A_V , and A^l are vector fields, and A^q is a 3×3 matrix potential ($\mathbf{1}$ and σ^3 are Pauli matrices). The vector field A^s is a function of A_L and A_R ; the two possible choices are

$$A^s \equiv 0 \quad \text{or} \quad A^s \equiv -A_L - 2A_V. \quad (4.80)$$

The dynamical gauge groups (see Def. 3.1) are given by

$$\mathcal{G} = U(1)_L \otimes \mathcal{F}, \quad \mathcal{F} = U(1)_V \otimes U(3)^q \otimes U(1)^l, \quad (4.81)$$

where the indices clarify to which potentials in the Dirac operator the groups correspond.

Proof. The Lemmas 4.7 and 4.8 do not immediately imply here because they are based on the assumption that we have only one relation between the basic monomials. But it is straightforward to check that if in these lemmas we allowed for an additional relation between the basic monomials, the argument of Lemma 4.12 would still go through, thus making it necessary to introduce a third relation between the basic monomials.

Collecting the results of Lemmas 4.7–4.13 and choosing a convenient representation for the dynamical mass matrices, we obtain that the dynamical mass matrices are block diagonal as in (4.77). Thus it remains to derive the dynamical gauge group and the form of the corresponding gauge potentials. Possibly after reordering the sectors, the $U(1)_L$ is precisely the group \mathcal{B}_4 in Def. 3.5. The free gauge group is obtained by taking the maximal subgroup of \mathcal{F}_4 for which the gauge potentials respect the phase conditions (4.59). In the two cases in (4.80), the $U(1)_V$ shifts the phases of μ and μ_8 by the same or the opposite amount, respectively. This corresponds to the two cases in (4.59). The other potentials must leave the phase functions unchanged, and to this end they must coincide on the sectors which are mapped into each other by the dynamical mass matrices (4.60). This gives the group $U(3)^q \otimes U(1)^l$. ■

We point out that, except for the potentials A^q , the Dirac operator splits into four direct summands. The first three summands are identical and involve massive Dirac particles, whereas the chiral Dirac particles are contained in the last summand. The gauge potentials A^q describe an interaction between the Dirac particles in the three identical summands. In analogy to the standard model, it is natural to identify the fermions in the first three and the last summands with the quarks and leptons, respectively. In order to make these notions precise, we first observe that for the fermionic projector, the above splitting means that for all contributions considered so far¹,

$$P(x, y) = U(x, y) \left(P^q \oplus P^q \oplus P^q \oplus P^l \right), \quad (4.82)$$

where U is the generalized phase transformation by the potentials A^q ,

$$U(x, y) = \text{Pexp} \left(-i \int_0^1 d\tau A_j^q(\tau y + (1 - \tau)x) (y - x)^j \right). \quad (4.83)$$

The unitary transformation (4.83) clearly commutes with the direct sum and thus drops out of the closed chain,

$$A_{xy} = A_{xy}^q \oplus A_{xy}^q \oplus A_{xy}^q \oplus A_{xy}^l \quad \text{with} \quad A_{xy}^{q/l} \equiv P^{q/l}(x, y) P^{q/l}(y, x). \quad (4.84)$$

¹We remark for clarity that the contributions to the fermionic projector which involve the $U(3)^q$ gauge fields or currents, which have not been considered so far, do not split in the form (4.82). This will be of relevance later when gauge fields and currents are analyzed [6].

Def. 4.15 *The first three direct summands in (4.82) and (4.84) are referred to as the quark blocks. The last direct summand is the lepton block.*

4.3 The Dynamical Mass Matrices in the Quark and Neutrino Blocks

We now specify the dynamical mass matrices in the quark and neutrino blocks.

Theorem 4.16 *Under the assumptions of Theorem 4.14, the EL equations are satisfied to the degree 10 on the light cone if and only if the matrices $Y_{L/R}^q$ and $Y_{L/R}^l$ in (4.77) have (after suitable constant unitary transformations) at all space-time points the following properties,*

$$\hat{Y}_L^q = (\hat{Y}_R^q)^* = \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix} \quad (4.85)$$

$$\hat{Y}_L^l = (\hat{Y}_R^l)^* = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \quad (4.86)$$

$$\dot{Y}_L^q = \begin{pmatrix} a & \mathcal{V}_L \bar{b} \\ \mathcal{U}_L b & \bar{a} \end{pmatrix}, \quad \dot{Y}_R^q = \begin{pmatrix} \bar{a} & \mathcal{V}_R \bar{b} \\ \mathcal{U}_R b & a \end{pmatrix} \quad (4.87)$$

and in the two cases in (4.80),

$$\dot{Y}_L^l = \begin{pmatrix} a & 0 \\ \mathcal{W}_L b & 0 \end{pmatrix}, \quad \dot{Y}_R^l = \begin{pmatrix} \bar{a} & * \\ \mathcal{W}_R b & * \end{pmatrix} \quad (4.88)$$

and

$$\dot{Y}_L^l = \begin{pmatrix} a & 0 \\ \mathcal{W}_L \bar{b} & 0 \end{pmatrix}, \quad \dot{Y}_R^l = \begin{pmatrix} \bar{a} & * \\ \mathcal{W}_R \bar{b} & * \end{pmatrix}, \quad (4.89)$$

respectively. Here we use a matrix notation in the sector index. In (4.87)–(4.89), the matrix entries are vectors in \mathbb{C}^3 (and this takes into account the dependence on the generations). The parameter c is complex, $a, b \in \mathbb{C}^3$, and the stars stand for any vectors in \mathbb{C}^3 . The off-diagonal elements are non-trivial in the sense that there is a space-time point where $b \neq 0$. The matrices $\mathcal{U}_{L/R}$, $\mathcal{V}_{L/R}$, $\mathcal{W}_{L/R} \in U(3)$ are constant unitary transformations.

Proof. We only consider the first case in (4.80); the second is obtained in the same way keeping track of the complex conjugates. The weak causality condition of Def. 4.1 implies that $\dot{Y}_L I_8 = 0$. On the other hand, we already observed after (4.53) that the matrix product $\dot{Y}_R I_8$ enters only the perturbation calculation for the kernel, which is trivial according to Theorem A.3. This explains the zeros and stars in (4.86) and (4.88). Then (4.85) and (4.86) follow immediately from Lemma 4.7 and Lemma 4.8. A short calculation using (4.85) as well as (4.76) and (4.59) yields that the EL equations to degree 10 reduce to the conditions

$$\int_x^z dz \, I_\uparrow \dot{Y}_L \dot{Y}_R I_\uparrow = \alpha(x, y) I_\uparrow \quad (4.90)$$

$$\bar{\nu} I_\uparrow \dot{Y}_R(y) I_\uparrow \dot{Y}_L(x) I_\uparrow + \nu \overline{I_\uparrow \dot{Y}_R(y) I_\uparrow \dot{Y}_L(x) I_\uparrow} = \beta(x, y) I_\uparrow \quad (4.91)$$

$$\begin{aligned} & \mu I_\uparrow \dot{Y}_R(y) (I_\downarrow + I_8) \dot{Y}_R(y) I_\uparrow \dot{Y}_L(x) I_\uparrow \\ & + \bar{\mu} \overline{I_\uparrow \dot{Y}_R(y) (I_\downarrow + I_8) \dot{Y}_R(y) I_\uparrow \dot{Y}_L(x) I_\uparrow} = \gamma(x, y) I_\uparrow \end{aligned} \quad (4.92)$$

as well as to the conditions obtained by the replacements

$$L \longleftrightarrow R, \quad \nu \longleftrightarrow \bar{\nu} \quad (4.93)$$

and/or

$$I_{\uparrow} + I_8 \longrightarrow I_{\downarrow}, \quad I_{\downarrow} \longrightarrow I_{\uparrow}, \quad \mu \longleftrightarrow \bar{\mu}, \quad \nu \longleftrightarrow \bar{\nu} \quad (4.94)$$

with the complex functions α, β , and γ unchanged. We first substitute (4.55) into (4.91). Comparing with the relation obtained by applying (4.93), one sees that $\langle u_L, u_L \rangle = \overline{\langle u_R, u_R \rangle}$, and thus we can arrange by a constant unitary transformation that $u_L = \overline{u_R}$. This explains the diagonal entries in (4.87) and (4.88). Substituting (4.60) and (4.61) into (4.92) and comparing with the relation obtained by applying (4.94), we obtain similarly that $\langle u_R, u_R \rangle = \overline{\langle u_L, u_L \rangle}$ and thus, up to a constant unitary transformation, $v_L = \overline{v_R}$. Since we have already used the freedom in choosing orthogonal bases in order to arrange that $u_L = \overline{u_R}$, we now need to take into account these unitary transformations. This gives the off-diagonal elements in (4.87) and (4.88). We conclude that (4.85)–(4.88) are necessary conditions. Substituting (4.85)–(4.88) into (4.90)–(4.92) and applying (4.93), (4.94), one verifies immediately that these conditions are also sufficient. The last statement in Lemma 4.10 implies that b is non-trivial. \blacksquare

In Chapters 3 and 4, we always restricted attention to our model variational principle (2.60). We now make a few general comments on how our methods could be extended to other two-point actions and which features of the Lagrangian are important for getting a physically interesting continuum limit.

The methods of Chapter 3 immediately apply to other two-point actions; the only obstruction is that the gauge terms in the operator $Q(x, y)$ must be polynomials in $T_{\bullet}^{(n)}$ and $\overline{T_{\bullet}^{(n)}}$. The general mechanism is that the eigenvalues of A_{xy} are influenced by the gauge terms (cf. (3.15) and (3.45), (3.46)). When analyzed in the EL equations, this leads to conditions for the eigenvalues of the “phase matrices” W_c (see (3.20) or (3.55)), and these conditions can finally be translated into constraints for the dynamical gauge fields. In this last step one uses crucially that the EL equations are nonlocal in the sense that they yield relations between the chiral potentials even at distant points (see e.g. (3.60)). This gives rise to *global constraints*, i.e. conditions which must hold in all of space-time. For example, Theorem 3.6 states that the dynamical gauge group in case **(2)** must be contained in one of the groups $(\mathcal{G}_p)_{p=0,\dots,3}$ in the whole space-time, but it cannot be the group \mathcal{G}_{p_1} in one region of space-time and a different group \mathcal{G}_{p_2} in another region (as one sees by considering line integrals which join the two regions).

For the spontaneous block formation, it is essential that the EL equations are satisfied only if the eigenvalues of A_{xy} are highly degenerate. The requirement that these degeneracies should be respected by the scalar potentials can then be used to show that the potentials must split globally into a direct sum.

While this general mechanism should occur similarly for most other Lagrangians, the details depend sensitively on the particular form of the action. Our model Lagrangian has the special feature that it involves only the absolute squares of the eigenvalues of A_{xy} . This is the reason why Theorem 3.2 involves only the absolute squares of ν_{nc} , (3.20), leading to the relatively weak constraint for the dynamical gauge group (3.21) (if we had, for example, considered instead the polynomial Lagrangian (2.50), the gauge terms to highest degree would have led to conditions also for the phases of ν_{nc} , giving rise to much stronger conditions). To the next lower degree on the light cone, the phases of ν_{nc} do enter the analysis. But since perturbing the absolute square gives rise to a real part, $\Delta|\lambda_{ncs}|^2 = 2\text{Re}(\overline{\lambda_{ncs}} \Delta\lambda_{ncs})$, we can easily arrange that only the real part of ν_{ncs} comes

into play, and so the phases are fixed only up to signs. This is a major advantage of our action over e.g. polynomial actions, where the same flexibility for the phases can be arranged only for a large degree of the polynomial. Another action which has the nice property that it depends only on the absolute squares of the eigenvalues is the determinant action (2.58). Working with the spectral trace leads to the specific problem that one must handle spectral adjoints. This is clearly a technical complication, but we do not consider it to be essential for the spontaneous block formation.

Remark 4.17 (*Massive neutrinos*) In the analysis of Chapters 3 and 4, the structure of the neutrino sector was used several times: In the vacuum, the chiral cancellations were useful because as a consequence, the EL equations were trivially satisfied in the neutrino sector (see (2.10) and (2.44)). In Theorem 3.2, the chiral cancellations in the neutrino sector are the reason why the dynamical gauge fields are not allowed to describe a mixing between the neutrinos and the massive fermions (see the argument leading to (3.30)). In the proof of Theorem 3.6, it was essential that the number of massive sectors is odd (see (3.63)). Finally, in the analysis of the degeneracies we always treated the neutrino sector separately.

Generally speaking, the chiral fermions lead to complications in the case with interaction, because the dynamical gauge fields were not allowed to describe a mixing between the massive and the chiral fermions, and this made it necessary to take scalar potentials into account. Also in view of recent experimental observations, it thus seems tempting to consider a neutrino sector which is built up of massive chiral Dirac seas. This is indeed possible, although we see the following difficulties. First, it is not clear how chiral fermions should be described in Minkowski space. Furthermore, building in massive chiral fermions is certainly not easy. Namely, if the resulting neutrino sector does not give rise to chiral cancellations, we must extend the Lagrangian in order to arrange that the EL equations are satisfied in the vacuum. The analysis of the interaction would also be considerably different. Finally, one should keep in mind that the recent experiments do not measure the mass of the neutrinos directly, but merely observe neutrino oscillations, i.e. a mixing of the neutrinos in different generations. This mixing could also be explained for massless neutrinos if the interaction of the neutrinos were suitably modified. For these reasons, we feel that before moving on to massive neutrinos, one should first get a better understanding of our variational principles for a massless neutrino sector.

5 The Effective Gauge Group

In this chapter, we will reformulate the interaction of the Dirac particles with chiral and scalar fields as specified in Theorems 4.14 and 4.16 as an interaction via “effective” non-Abelian gauge fields. Before working out the details in Sections 5.1 and 5.2, we now give the general construction. Consider the Dirac equation in the presence of chiral and scalar potentials (3.5). Since the dynamical mass matrix $Y_L = (Y_L)_{(b\beta)}^{(a\alpha)}$ (with $a, b = 1, \dots, 8$ and $\alpha, \beta = 1, 2, 3$) need not be Hermitian, we cannot diagonalize it by a unitary transformation. But using the polar decomposition, we can at least represent Y_L in the form

$$Y_L = U_L Y^{\text{eff}} U_R^{-1} \quad (5.1)$$

with two unitary matrices $U_{L/R} \in U(3 \times 8)$ and Y^{eff} a diagonal matrix with real non-negative entries². We introduce the so-called *chiral transformation* V by

$$V = \chi_L U_R + \chi_R U_L. \quad (5.2)$$

Note that the adjoint of V ,

$$V^* = \chi_R U_L^{-1} + \chi_L U_R^{-1},$$

is in general different from its inverse, which we denote by a bar,

$$\bar{V} \equiv V^{-1} = \chi_L U_L^{-1} + \chi_R U_R^{-1}.$$

Thus the chiral transformation need not be unitary. The chiral transformation of the Dirac operator is defined by and computed to be

$$\bar{V}^* (i\partial + \chi_L(A_R - mY_R) + \chi_R(A_L - mY_L)) \bar{V} = i\partial + \chi_L A_R^{\text{eff}} + \chi_R A_L^{\text{eff}} - mY^{\text{eff}}$$

with Y^{eff} as in (5.1) and

$$A_c^{\text{eff}} = U_c^{-1} A_c U_c + i U_c^{-1} (\partial U_c), \quad c \in \{L, R\}. \quad (5.3)$$

Finally, the effective fermionic projector is obtained from the auxiliary fermionic projector by the chiral transformation

$$P^{\text{eff}} = V P V^*. \quad (5.4)$$

It satisfies the effective Dirac equation

$$(i\partial + \chi_L A_R^{\text{eff}} + \chi_R A_L^{\text{eff}} - mY^{\text{eff}}) P = 0. \quad (5.5)$$

Since the chiral transformation is one-to-one, the effective fermionic projector gives an equivalent formulation of the physical system. The advantage of the effective description is

²For the reader not familiar with the polar decomposition we outline the construction. For a matrix $A \in \text{Gl}(\mathbb{C}^n)$ we introduce the Hermitian and positive semidefinite matrix $R = \sqrt{A^* A}$. A short calculation shows that the matrix V defined by

$$V u = \begin{cases} u & \text{for } u \in \text{Ker } R \\ A R^{-1} u & \text{for } u \in (\text{Ker } R)^\perp \end{cases}$$

is unitary and satisfies the relation $A = VR$. Diagonalizing R by a unitary transformation, i.e. $R = WDW^{-1}$ with D diagonal and W unitary, we obtain the desired representation $A = U_1 D U_2^{-1}$ with $U_1 \equiv VW$ and $U_2 \equiv W$.

that the effective mass matrix Y^{eff} is diagonal. This means that if we interpret the sector index after the chiral transformation as labelling the different types of Dirac particles (like u , d , e , ν_e , etc.), the effective scalar potentials describe a dynamical shift of the mass of each type of fermion, whereas the interaction between different types of fermions is described only by the effective chiral potentials. Thus apart from the fact that we allow for dynamical mass shifts, the Dirac particles interact as in the standard model via chiral fields.

In general, the effective potentials have locally the form of non-Abelian gauge potentials. But they cannot be chosen at every point independently, because it must be possible to represent them in the form (5.3) with A_c the Abelian gauge potentials of Theorem 4.14. We refer to (5.3) as the *gauge condition*.

For clarity, we point out that unitary transformations in the polar decomposition (5.1) are not uniquely determined. Thus, similar to the freedom of choosing different gauges, there is a certain arbitrariness in the choice of U_L and U_R . Since at infinity the dynamical mass matrices go over to the mass matrix Y of the vacuum, we can and shall always choose $U_{L/R}$ such that

$$\lim_{x \rightarrow \infty} U_{L/R}(x) = \mathbf{1} . \quad (5.6)$$

5.1 The Chiral Transformation in the Quark Blocks

Using the splitting (4.84), we may disregard the $U(3)^q$ potentials and can analyze the chiral transformation in the quark and lepton blocks separately. In this section, we consider a quark block and for ease in notation omit the superscript q . According to Theorem 4.16, the EL equations to degree 10 give information only on the partial traces of the dynamical mass matrices. Therefore, the dynamical mass matrices, and as a consequence also the chiral transformation and the effective potentials, are not completely determined. This means that we have a certain freedom to arbitrarily change these objects, and we shall use this freedom to make the following assumption on the form of the effective chiral gauge potentials.

Def. 5.1 *The effective chiral potential A_c , $c \in \{L, R\}$, has **unitary mixing** if for every space-time point x there is a unitary matrix $W_c \in U(3)$ and a $U(2)$ potential a_c such that at x ,*

$$A_c^{\text{eff}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W_c \end{pmatrix} a_c \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W_c^{-1} \end{pmatrix} = \begin{pmatrix} a_c^{11} & a_c^{12} W_c^{-1} \\ a_c^{21} W_c & a_c^{22} \end{pmatrix} \quad (5.7)$$

(here we use as in Theorem 4.16 a matrix notation in the sectors). The matrix W_c is referred to as the **mixing matrix**.

Thus we impose that the effective chiral potentials be trivial on the generations except for a unitary mixing of the generations in the off-diagonal matrix elements. This assumption is clearly satisfied for the gauge potentials in the standard model if we choose $W_R \equiv \mathbf{1}$ and W_L equal to the CKM mixing matrix. Our ansatz is more general in that we allow for both left- and right-handed mixing matrices and that $W_c = W_c(x)$ need not be a constant matrix. Ultimately, the assumption of unitary mixing should be justified from the EL equations. But this makes it necessary to consider the EL equations to the degree 9 on the light cone. We postpone this analysis to [6] and here simply take Def. 5.1 as a physically reasonable technical simplification.

Our first lemma characterizes those chiral transformations which respect the condition of Def. 5.1.

Lemma 5.2 *The effective chiral potential has unitary mixing if and only if the unitary transformation U_c in (5.2) is for all x of the form*

$$U_c = \begin{pmatrix} u_c^{11} & u_c^{12} W_c^{-1} \\ u_c^{21} W_c & u_c^{22} \end{pmatrix} \quad \text{with } u_c \in U(2). \quad (5.8)$$

Furthermore, the mixing matrix is constant.

If U_c is of the form (5.8) with W_c a constant matrix, it is obvious that the corresponding effective chiral potential (5.3) has unitary mixing. In order to show that the converse is also true, we must analyze the differential equation for U_c and use the boundary conditions at infinity (5.6).

Proof of Lemma 5.2. It suffices to prove the “only if” part. Thus we assume that A_c^{eff} has unitary mixing and shall derive that U_c is of the form (5.8). For ease in notation we omit the subscript c . According to Theorem 4.14, A is diagonal and can thus be written as $A = \alpha \mathbf{1} + \beta \sigma^3$ with real functions α and β . When substituting into (5.3), α yields a contribution to A^{eff} with unitary mixing, independent of the form of U . Thus α is irrelevant for the following argument, and we can assume that $A \sim \sigma^3$.

Let Ω be the set where the field tensor $F = dA - iA \wedge A$ is non-zero,

$$\Omega = \{x \mid F(x) \neq 0\}.$$

We shall first prove that on each connected component Ω_C of Ω , U is for all $x \in \overline{\Omega_C}$ of the form

$$U(x) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} u(x) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \quad (5.9)$$

with $u \in U(2)$ and constant unitary matrices $V_1, V_2, W \in U(3)$. To this end, we differentiate (5.3) and (5.7) to obtain

$$U^{-1} F U = F^{\text{eff}} = \begin{pmatrix} f^{11} & f^{12} W^{-1} \\ f^{21} W & f^{22} \end{pmatrix} \quad (5.10)$$

with $f = da + a \wedge a$ (these relations can be understood immediately from the behavior of the field tensor under gauge transformations). At $x \in \Omega_c$, $0 \neq F \sim \sigma^3$. Using this fact in (5.10) shows that $U(x)$ must be of the form

$$U = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \quad (5.11)$$

with $B_1, B_2 \in U(3)$ and $\varphi \in \mathbb{R}$. Hence at x , the first summand in (5.3) is of the required form (5.7), and thus the second summand must also be of this form. Computing $iU^{-1}(\partial U)$ for U according to (5.11), one sees that this term is of the form (5.7) only if at x ,

$$\partial W = 0 \quad \text{and} \quad \partial B_{1/2} \sim B_{1/2} \quad (5.12)$$

(in the special case $\sin \varphi = 0$, we merely obtain that $\partial(B_2 W^{-1}) \sim B_2 W^{-1}$, but since in this case U only involves the product $B_2 W^{-1}$, we can arrange that $\partial W = 0$). Integrating (5.12) gives (5.9).

Let $\Lambda = \mathbb{R}^4 \setminus \Omega$ be the set where F vanishes. We next prove that on each connected component Λ_C of Λ , U is of the form

$$U(x) = \begin{pmatrix} e^{i\phi(x)} & 0 \\ 0 & e^{-i\phi(x)} \end{pmatrix} V u_{\text{eff}}(x) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \quad (5.13)$$

with real ϕ , $u_{\text{eff}} \in U(2)$, and constant matrices $V \in U(6)$ and $W \in U(3)$. In order to derive this formula, we first use that $F = 0$ on Λ_C to represent A as a pure gauge potential, i.e.

$$A = iB^{-1}(\partial B) \quad \text{with} \quad B = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \quad (5.14)$$

and a real function ϕ . According to the first part of (5.10), also F^{eff} vanishes. Let us consider what this tells us about the objects in (5.7). Using that the phase factors can be absorbed into a^{21} , we can arrange that $W^{-1}(\partial W)$ is trace-free. Then the contributions to F^{eff} involving ∂W and ∂a are linearly independent. From this we conclude that W is constant on Λ_C and that A^{eff} can be represented as

$$A^{\text{eff}} = i \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W \end{pmatrix} u_{\text{eff}}^{-1} (\partial u_{\text{eff}}) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \quad (5.15)$$

with $u_{\text{eff}} \in U(2)$. On the other hand, substituting (5.14) into (5.3) gives

$$A^{\text{eff}} = i(BU)^{-1} \partial(BU). \quad (5.16)$$

Differentiating the unitary matrix

$$BU \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W \end{pmatrix} u_{\text{eff}}^{-1}$$

and using (5.15) and (5.16), one sees that this matrix is constant on Λ_C , proving (5.13).

Note that the representation (5.13) poses a weaker constraint on U than (5.9). We shall now prove that on Λ_C even (5.9) holds. If Λ_C extends to infinity, we can according to (5.6) assume that

$$\lim_{\Lambda_C \ni x \rightarrow \infty} \phi = 0, \quad \lim_{\Lambda_C \ni x \rightarrow \infty} u_{\text{eff}} = \mathbf{1} \quad \text{and} \quad V = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W \end{pmatrix}.$$

Then (5.13) indeed goes over to (5.9). If conversely Λ_C is compact, we choose $y \in \partial\Lambda_C$. Then at y both (5.9) and (5.13) hold, and comparing these formulas one sees that V must be a diagonal matrix. This implies that on Λ_C , (5.13) reduces to (5.9).

We just showed that U can for all X be represented in the form (5.9), where V is constant on each connected component of Ω and Λ . Possibly after multiplying U by piecewise constant unitary transformations and/or absorbing constant unitary transformation from u into V_1 , V_2 , or W , we can assume that all matrices in (5.9) are continuous. The asymptotics at infinity (5.6) finally yields that $V_1 = \mathbf{1}$ and $V_2 = W$. ■

Using the result of the previous lemma in (5.1), we can now compute the dynamical mass matrices and analyze the conditions of Theorem 4.16. We restrict attention to the special case which will be of relevance later that the right-handed chiral transformation is trivial.

Lemma 5.3 *Assume that $U_R \equiv \mathbf{1}$. If (4.85) and (4.87) are satisfied, the mixing matrix and the potential u_L in (5.7) must have the properties*

$$\dot{Y} \dot{W}_L = 0 = \dot{W}_L \dot{Y} \quad (5.17)$$

$$|Y \dot{W}_L| = |\dot{Y}| = |\dot{W}_L Y| \quad (5.18)$$

$$u_L \in SU(2). \quad (5.19)$$

Furthermore, (4.85) and (4.87) are also satisfied if we leave U_L unchanged and set the effective scalar potentials to zero,

$$Y^{\text{eff}} \equiv Y. \quad (5.20)$$

Conversely, if $U_R \equiv \mathbf{1}$ and (5.17)–(5.20) are satisfied, then (4.85) and (4.87) hold.

Proof. Evaluating (5.1) for U_L according to (5.7) and $U_R \equiv \mathbf{1}$ gives

$$Y_L = \begin{pmatrix} u_L^{11} Y_1^{\text{eff}} & u_L^{12} W_L^{-1} Y_2^{\text{eff}} \\ u_L^{21} W_L Y_1^{\text{eff}} & u_L^{22} Y_2^{\text{eff}} \end{pmatrix}$$

with $Y^{\text{eff}} = \text{diag}(Y_1^{\text{eff}}, Y_2^{\text{eff}})$. Assume that (4.85) and (4.87) are satisfied. Let us evaluate these relations for the off-diagonal elements of Y_L . Since b in (4.87) is non-trivial, the function $u_L^{21}(x)$ does not vanish identically, but clearly it is zero at infinity. As usual, we implicitly assume that u_L^{21} decays asymptotically at infinity, without necessarily being zero outside a compact set. Then we can apply a perturbation argument to the lower left matrix element of Y_L . Namely, (4.85) yields that $\dot{W}_L \dot{Y}_1^{\text{eff}} = 0$, and taking the asymptotic limit gives the rhs of (5.17). The rhs of (5.17) is obtained similarly from the upper right matrix element of Y_L . Applying the above perturbation argument to the off-diagonal terms in (4.87) yields (5.18).

We next evaluate (4.85) for the diagonal elements of Y_L . Since Y^{eff} is a positive matrix, \dot{Y}_1^{eff} and \dot{Y}_2^{eff} are real and ≥ 0 . Furthermore, u_L satisfies as a $U(2)$ matrix the relation $|u_L^{11}| = |u_L^{22}|$. From (4.85) we conclude that $u_L^{11} = \overline{u_L^{22}}$, and thus $u \in SU(2)$.

Finally, it is straightforward to check that (5.17)–(5.20) imply (4.85) and (4.87). \blacksquare

In the remainder of this section, we shall analyze and discuss the gauge condition (5.3). First, we substitute in (5.8) and pull the constant mixing matrix outside,

$$A_c^{\text{eff}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W_c \end{pmatrix} (u_c^{-1} A_c u_c + i u_c^{-1} (\partial u_c)) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W_c^{-1} \end{pmatrix}.$$

Next, we decompose the potential and the unitary transformation into the $U(1)$ and $SU(2)$ parts, i.e.

$$A_c = \alpha \mathbf{1} + a \sigma^3 \quad \text{and} \quad u_c = e^{-i\phi} v$$

with real functions α, a, ϕ and $v \in SU(2)$. This gives

$$A_c^{\text{eff}} = (\alpha + \partial\phi) \mathbf{1} + \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W_c \end{pmatrix} [a v^{-1} \sigma^3 v + i v^{-1} (\partial v)] \begin{pmatrix} \mathbf{1} & 0 \\ 0 & W_c^{-1} \end{pmatrix}. \quad (5.21)$$

Thus ϕ describe a usual $U(1)$ gauge transformation. The square bracket can be regarded as an $SU(2)$ potential, and the matrix W_c introduces a unitary mixing in the off-diagonal elements. The remaining question is in which way the expression in the square brackets gives a constraint for the $SU(2)$ potential.

Def. 5.4 *The field tensor $F = dA - iA \wedge A$ of an $SU(2)$ potential A is **simple** if for every x there is a real-valued 2-form Λ and $s \in su(2)$ such that*

$$F(x) = \Lambda s. \quad (5.22)$$

Lemma 5.5 *An $SU(2)$ potential A can be represented in the form*

$$A = a v^{-1} \sigma^3 v + i v^{-1} (\partial v) \quad (5.23)$$

with $a(x) \in \mathbb{R}$ and $v(x) \in SU(2)$ if and only if its field tensor is simple.

Proof. If A is of the form (5.23), its field tensor is given by

$$F = f v^{-1} \sigma^3 v$$

with $f = da$, and this is obviously simple.

Assume conversely that F is simple. We choose $v_1 \in SU(2)$ such that

$$v_1 s v_1^{-1} = \lambda \sigma^3$$

with $\lambda \in \mathbb{R}$ and introduce the gauge potential \tilde{A} by

$$\tilde{A} = v_1 A v_1^{-1} - i v_1 (\partial v_1^{-1}) . \quad (5.24)$$

From (5.22) one sees that the corresponding field tensor is

$$\tilde{F} = f \sigma^3$$

with the real-valued 2-form $f = \lambda \Lambda$. From the fact that \tilde{F} is closed we conclude that $d\tilde{f} = 0$, and thus there is a 1-form a with $f = da$. By construction, the $SU(2)$ potentials \tilde{A} and $a\sigma^3$ have the same field tensor \tilde{F} . As a consequence, they are related to each other by an $SU(2)$ transformation, i.e.

$$\tilde{A} = a v_2^{-1} \sigma^3 v_2 + i v_2^{-1} (\partial v_2) \quad (5.25)$$

with $v_2 \in SU(2)$. Substituting (5.25) into (5.24) and solving for A gives (5.23) with $v = v_2 v_1$. ■

With this lemma, we have reformulated the gauge condition (5.21) as a structure condition for the effective field tensor. This makes it possible to regard the effective chiral potentials as locally defined objects. More precisely, we shall treat the effective chiral potentials as local gauge potentials, which are constrained only by local conditions like that the effective field tensor be simple, but we shall not consider the corresponding chiral transformation (which involves integrating the effective potentials and is therefore defined in a nonlocal way). In particular, when we have conditions between the effective potentials in the quark and neutrino sectors, we shall always satisfy them by *local relations*, i.e. by algebraic or differential equations involving the effective potentials. This procedure corresponds to the usual requirement of locality in physics. It could be further justified later [6] by the fact that the EL equations yield differential equations for the effective potentials (the “field equations”), and it seems impossible to satisfy these differential equations if the effective potentials obey nonlocal constraints.

5.2 The Chiral Transformation in the Lepton Block

We come to the analysis in the lepton block; for ease in notation the superscript l will be omitted. As a consequence of the chiral massless fermions, the dynamical matrices are different in the lepton and quark blocks. More precisely, Y_L and Y_R must now be of the

form (4.86) and (4.88),(4.89). We shall first show that these conditions are incompatible with a unitary mixing and then resolve this problem by modifying the mixing in the right-handed component. Let us assume that (4.88) or (4.89) are satisfied for dynamical mass matrices of the form (5.1) with $U_{L/R}$ according to Lemma 5.2. Then, choosing a space-time point where $b \neq 0$,

$$(U_L Y^{\text{eff}} \dot{U}_R^{-1}) I_2 = 0 \quad \text{but} \quad I_2 (U_R Y^{\text{eff}} \dot{U}_L^{-1}) \neq 0, \quad (5.26)$$

where $I_{1/2}$ are again the projectors on the two sectors. Introducing the unit vectors $n = 3^{-\frac{1}{2}}(1, 1, 1) \in \mathbb{C}^3$ and $u = (0, n) \in \mathbb{C}^6$, we can write the first condition in (5.26) without a partial trace as

$$U_L Y^{\text{eff}} U_R^{-1} u = 0. \quad (5.27)$$

In the vacuum, the mass matrix $Y^{\text{eff}} = Y$ is in the first sector strictly positive. A perturbation argument yields that, at least for weak fields, the effective mass matrix is of the form $Y^{\text{eff}} = \text{diag}(Y_1^{\text{eff}}, Y_2^{\text{eff}})$ with $Y_1^{\text{eff}} > 0$. Therefore, the condition (5.26) can only be satisfied if $U_R^{-1} u$ vanishes in the first sector. Thus, using (5.8),

$$\begin{pmatrix} v^{11} & v^{12} W_c^{-1} \\ v^{21} W_c & v^{22} \end{pmatrix} \begin{pmatrix} 0 \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}$$

with $v = u_R^{-1}$. This implies that $v^{12} = 0$ and thus $U_R \equiv 1$. Using this result in (5.27), we obtain that $Y^{\text{eff}} u = 0$, and since Y^{eff} is a diagonal matrix with non-negative entries, we conclude that $Y_2^{\text{eff}} = 0$. Finally, the relations $U_R = \mathbb{1}$ and $Y_2^{\text{eff}} = 0$ imply that $I_2 U_R Y^{\text{eff}} = 0$, in contradiction to the rhs of (5.26).

In order to avoid the above contradiction, the vector $U_R^{-1} u$ must vanish identically in the first sector without U_R being trivial. The natural way to arrange this is to replace the unitary matrix W_R in (5.7) by a matrix which is zero on $\langle n \rangle$ and is unitary on $\langle n \rangle^\perp$. In analogy to the procedure in the previous section, we first introduce the corresponding effective potentials and determine U_R afterwards. We let Π be the projector

$$\Pi = |n\rangle\langle n| \quad \text{with} \quad n = \frac{1}{\sqrt{3}}(1, 1, 1). \quad (5.28)$$

Def. 5.6 *The effective potential A_R^{eff} has **projected mixing** if for every space-time point there is a unitary matrix $W_R \in U(3)$ with*

$$W_R n = n$$

as well as real functions b_R^1 and b_R^2 and a $U(2)$ potential a_R such that at x ,

$$A_R^{\text{eff}} = \begin{pmatrix} b_R^1 & 0 \\ 0 & b_R^2 \end{pmatrix} + (1 - \Pi) \begin{pmatrix} a_R^{11} & a_R^{12} W_R^{-1} \\ A_R^{21} W_R & a_R^{22} \end{pmatrix}. \quad (5.29)$$

W_R is the **mixing matrix**.

Lemma 5.7 *The effective potential A_R^{eff} has projected mixing if and only if the unitary transformation U_R in (5.2) is for all x of the form*

$$U_R = \Pi \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} + (\mathbb{1} - \Pi) \begin{pmatrix} u_{11} & u^{12} W_R^{-1} \\ u^{21} W_R & u^{22} \end{pmatrix} \quad (5.30)$$

with $v_{1/2} \in U(1)$ and $u \in U(2)$. Furthermore, the matrix W_R is constant.

Proof. We consider the effective potential on $\langle n \rangle$ and $\langle n \rangle^\perp$ separately. On $\langle n \rangle^\perp$, A_R^{eff} is of the form as in Def. 5.1, and so Lemma 5.3 applies. On $\langle n \rangle$, on the other hand, A_R^{eff} is a diagonal potential, and integrating the differential equation for U_R as in the proof of Lemma 5.2 shows that U_R is also diagonal. ■

The effective potentials in the second summand in (5.29) have different properties than usual gauge fields. First, one should keep in mind that the right-handed potential A_{eff}^R does not couple to the left-handed massless fermions, and therefore the off-diagonal elements in (5.29) cannot be regarded as describing an interaction between the massive leptons and the neutrinos. Indeed, one must be careful about associating any physical interaction to the second summand in (5.29), because the factor $(\mathbb{1} - \Pi)$ gives zero when the partial trace is taken, and also because some degrees of freedom of the corresponding potentials drop out of the fermionic projector when \tilde{t} is multiplied by the chiral asymmetry matrix (4.3). For these reasons, we regard the off-diagonal elements in (5.29) as describing a new type of interaction whose physical significance is not clear at the moment. We refer to an effective potential which involves a factor $(\mathbb{1} - \Pi)$ as a *nil potential*.

The next lemma is very useful because it allows us to compute the vectors a and b in Theorem 4.16 without specifying U_R . In this way, we can get around the detailed analysis of the nil potential.

Lemma 5.8 *Suppose that A_L^{eff} and A_R^{eff} have unitary and projected mixing, respectively. Then U_L and U_R can be chosen such that*

$$\dot{Y}_L = U_L \dot{Y}^{\text{eff}}.$$

Proof. Since $(\mathbb{1} - \Pi)n = 0$, the partial trace of the second summand in (5.4) is zero, whereas in the first summand the factor Π drops out. Thus $\dot{Y}_L = U_L \dot{Y}^{\text{eff}} \hat{V}$ with V a diagonal $U(2)$ matrix. This matrix commutes with Y^{eff} and can thus be absorbed into U_L . ■

5.3 Derivation of the Effective Gauge Group

We are now ready to prove the main result of this chapter.

Theorem 5.9 *We consider the EL equations corresponding to the Lagrangian (2.59) under the assumptions of Theorem 4.14. Assume furthermore that the right-handed effective potentials in the lepton block have projected mixing, and that all other effective potentials have unitary mixing (see Defs. 5.1 and 5.6). Imposing local relations between the effective potentials (see page 65), the right-handed chiral transformation is trivial in the quark blocks, $U_R^q \equiv \mathbb{1}$. The mixing matrices are constant and satisfy the relations*

$$\dot{Z} \dot{W}_L^q = \dot{W}_L^q \dot{Z} = \dot{Z} \dot{W}_L^l = 0 \quad (5.31)$$

$$|Z \dot{W}_L^q| = |\dot{W}_L^q Z| = |Z \dot{W}_L^l| = |\dot{Z}|, \quad (5.32)$$

where $Z = \frac{1}{m} \text{diag}(m_1, m_2, m_3)$ is the mass matrix of the massive fermions. The effective Dirac operator is of the following form,

$$i\partial\!\!\!/ - m \left(Y_q^{\text{eff}} \oplus Y_q^{\text{eff}} \oplus Y_q^{\text{eff}} \oplus Y_l^{\text{eff}} \right) \quad (5.33)$$

$$+ \chi_R \left(A_L^{\text{eff}} \oplus A_L^{\text{eff}} \oplus A_L^{\text{eff}} \oplus A_L^{\text{eff}} \right) + \chi_L A_R (\sigma^3 \oplus \sigma^3 \oplus \sigma^3 \oplus \sigma^3) \quad (5.34)$$

$$+ (A^q \mathbf{1}) \oplus \left(A^l \mathbf{1} + (\mathbf{1} - \Pi) A_R^{\text{nil}} \right). \quad (5.35)$$

Here A_L^{eff} is a 2×2 matrix potential, A^q is a 3×3 matrix potential, A_R and A^l are vector fields, and A^{nil} is a nil potential (see page 67). The effective gauge group is

$$\mathcal{G}^{\text{eff}} = SU(2)_L^{\text{eff}} \otimes U(1)_R \otimes U(3)^q \otimes U(1)^l. \quad (5.36)$$

The only constraint for the chiral potentials is that the field tensor corresponding to A_L^{eff} must be simple (see Def. 5.4). The EL equations to degree 10 are satisfied for the same effective potentials if the effective scalar potentials are set to zero,

$$Y^{\text{eff}} = Y. \quad (5.37)$$

Proof. We rewrite (4.91) and (4.92) in terms of the effective potentials. According to Lemma 5.8, \dot{Y}_L^l is independent of U_R , and thus β and γ can be expressed in terms of Y_L^{eff} , the Abelian potentials A_V , A_s in (4.78), and the non-Abelian effective potential A_L^{eff} in the lepton block. Furthermore, (4.91) and (4.92) can be satisfied by local relations between the effective potentials only if the non-Abelian effective gauge fields coincide in the quark and lepton blocks. This yields the effective chiral gauge group (5.36) and the form of the corresponding potentials in (5.34) and (5.35). Lemma 5.5 shows that the gauge conditions (5.3) are satisfied if and only if the field tensor corresponding to A_L^{eff} is simple.

According to Lemmas 5.2 and 5.7, the mixing matrices are constant. Lemma 5.3 gives (5.31) and (5.32) in the quark blocks. The corresponding relations in the lepton block are obtained similarly from (4.86) and (4.88). Finally, (5.37) follows immediately from Lemma 5.3 and an analogous perturbation argument in the lepton block. ■

Note that (5.31) and (5.32) are not satisfied if W_L is equal to the identity matrix. Thus the EL equations imply that the off-diagonal components of the effective gauge fields involve a non-trivial mixing of the generations. The fact that we may set Y^{eff} equal to Y , (5.37), means that the effective scalar potentials are irrelevant for the derivation of the effective gauge group. But this does not answer the question whether effective scalar potentials may occur in the system or not; to this end one must analyze the EL equations to lower degree on the light cone [6].

We finally point out that Theorem 5.9 only gives necessary conditions for the effective potentials. But it is to be expected that the derivation of the field equations [6] will give further constraints for the effective potentials. Taking this into account, the results of Theorem 5.9 are in perfect agreement with physics: The $SU(3)^q$ and $SU(2)_L^{\text{eff}}$ can be identified with the strong and weak gauge groups, respectively. The coupling of the corresponding gauge potentials to the fermions is exactly as in the standard model. The $SU(3)^q$ is a free gauge group (see page 23), and this implies that the corresponding gauge fields are necessarily massless. However, the $SU(2)_L^{\text{eff}}$ is spontaneously broken. The electromagnetic potential corresponds to a linear combination of the potentials of the subgroup $SU(2)_L^{\text{eff}} \otimes U(1)_R \otimes U(1)^q \otimes U(1)^l \subset \mathcal{G}^{\text{eff}}$, characterized by the property that it is a traceless vector potential. In order to make the connection to the standard model more precise, it remains to explain why only this particular linear combination occurs, and furthermore one must analyze the masses of the spontaneously broken gauge fields in the resulting field equations. This is precisely the aim of [6].

A Perturbation Calculation for the Spectral Decomposition of $P(x, y) P(y, x)$

In this appendix, we shall develop a convenient method for analyzing the eigenvalues and spectral projectors of the matrix $A_{xy} \equiv P(x, y) P(y, x)$ and compute all contributions to the eigenvalues needed for the derivation of the effective gauge group in Chapter 4. Our strategy is as follows. We decompose the fermionic projector as

$$P = P_0 + \Delta P$$

with P_0 according to (3.6). This gives rise to the decomposition of A

$$A = A_0 + \Delta A \tag{A.1}$$

with

$$A_0 = P_0(x, y) P_0(y, x) \tag{A.2}$$

$$\Delta A = \Delta P(x, y) P_0(y, x) + P_0(x, y) \Delta P(y, x) + \Delta P(x, y) \Delta P(y, x). \tag{A.3}$$

The eigenvalues and spectral projectors of A_0 were computed explicitly in Chapter 3, see (3.14) and (3.15). On the light cone, $P_0(x, y)$ has singularities of the order $\mathcal{O}((y-x)^{-4})$, whereas $\Delta P(x, y) = \mathcal{O}((y-x)^{-2})$. Likewise, ΔA is compared to A_0 of lower degree on the light cone. For this reason, ΔA can be treated perturbatively in the sense that the eigenvalues and spectral projectors of A can be expressed to any given degree on the light cone by a finite order perturbation calculation. Apart from the purely computational aspects, the main difficulty is that A_0 may have degenerate eigenvalues, and in this case we need to carefully analyze whether the degeneracy is removed by the perturbation. Our method is to first compute projectors on invariant subspaces of A (Section A.1). Considering the perturbation on these invariant subspaces will then give the spectral decomposition of A (Section A.4).

A.1 Perturbation of Invariant Subspaces

We write the spectral decomposition of A_0 as

$$A_0 = \sum_{k=1}^K \lambda_k F_k$$

with distinct eigenvalues λ_k and corresponding spectral projectors F_k . As in Section 2.1, we use the convention $\lambda_1 = 0$. Clearly, the F_k are the sum of the spectral projectors counting multiplicities,

$$F_k = \sum_{n,c,s \text{ with } \lambda_{ncs} = \lambda_k} F_{ncs} \tag{A.4}$$

with λ_{ncs} and F_{ncs} according to (3.15). Since the perturbation ΔA will in general split up the degenerate eigenvalues, we cannot expect that by perturbing F_k we obtain spectral projectors of the matrix A . But we can form projectors G_k on the space spanned by all eigenvectors of A whose eigenvalues are sufficiently close to λ_k . The G_k are most conveniently introduced using contour integrals. We choose $\varepsilon > 0$ such that

$$|\lambda_i - \lambda_j| < 2\varepsilon \quad \text{for all } i, j = 1, \dots, K \text{ and } i \neq j.$$

Then we set

$$G_k = \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} (z-A)^{-1} dz, \quad (\text{A.5})$$

The Cauchy integral formula shows that G_k is indeed a projector on the desired subspace.

The integral formula (A.5) is very useful for a perturbation expansion. To this end, we substitute (A.1) into (A.5) and compute the inverse with the Neumann series,

$$\begin{aligned} G_k &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} (z-A_0-\Delta A)^{-1} dz \\ &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} (\mathbb{1} - (z-A_0)^{-1} \Delta A)^{-1} (z-A_0)^{-1} dz \\ &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \sum_{n=0}^{\infty} ((z-A_0)^{-1} \Delta A)^n (z-A_0)^{-1} dz. \end{aligned}$$

Interchanging the integral with the infinite sum gives the perturbation expansion,

$$G_k = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} ((z-A_0)^{-1} \Delta A)^n (z-A_0)^{-1} dz, \quad (\text{A.6})$$

where n is the order in perturbation theory. After substituting in the spectral representation for $(z-A_0)^{-1}$,

$$(z-A_0)^{-1} = \sum_{l=1}^K \frac{F_l}{z-\lambda_l}, \quad (\text{A.7})$$

the contour integral in (A.6) can be carried out with residues. For example, we obtain to second order,

$$\begin{aligned} G_k &= F_k + \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} (F_k \Delta A F_l + F_l \Delta A F_k) + \mathcal{O}((\Delta A)^3) \\ &+ \sum_{l, m \neq k} \frac{1}{(\lambda_k - \lambda_l)(\lambda_k - \lambda_m)} (F_k \Delta A F_l \Delta A F_m + F_l \Delta A F_k \Delta A F_m + F_l \Delta A F_m \Delta A F_k) \\ &- \sum_{l \neq k} \frac{1}{(\lambda_k - \lambda_l)^2} (F_k \Delta A F_k \Delta A F_l + F_k \Delta A F_l \Delta A F_k + F_l \Delta A F_k \Delta A F_k). \end{aligned} \quad (\text{A.8})$$

To the order $n > 2$, the corresponding formulas are clearly more complicated, but even then they involve matrix products which are all of the form

$$F_{k_1} \Delta A F_{k_2} \Delta A \cdots F_{k_n} \Delta A F_{k_{n+1}}. \quad (\text{A.9})$$

Substituting in (A.4) and expanding, we can just as well consider matrix products of the form (A.9) with the factors F_k replaced by F_{ncs} . Furthermore, for the computation of the eigenvalues we need to take the expectation values of G_k with certain matrix elements of ΔA . This leads us to traces of matrix products of the form

$$\text{Tr} (F_{n_1 c_1 s_1} \Delta A_1 F_{n_2 c_2 s_2} \Delta A_2 \cdots F_{n_l c_l s_l} \Delta A_l) \quad (\text{A.10})$$

with $l = n + 1$. We refer to a trace of the form (A.10) as a *matrix trace*. Our first task is to develop an efficient method for computing matrix traces (Sections A.2 and A.3); after that we will proceed with the calculation of the eigenvalues of A (Section A.4).

A.2 Factorization of Matrix Traces

If one attempts to calculate a matrix trace (A.10) directly by substituting in the formulas of the light-cone expansion [3], the resulting expressions become so complicated and involve so many Dirac matrices that they are almost impossible to handle. We shall now simplify the situation by giving a procedure which allows us to factor matrix traces into a product of so-called elementary traces, which are much easier to compute. According to (A.3), we can assume that each factor ΔA_j in (A.10) is the product of a contribution to $P(x, y)$ with a contribution to $P(y, x)$. Denoting the contributions to $P(x, y)$ by B_j and using that the corresponding contributions to $P(y, x)$ are obtained by taking the adjoint with respect to the spin scalar product, we can write each A_j in the form

$$\Delta A_j = B_{j_1} B_{j_2}^* .$$

Inserting the completeness relation

$$\sum_{ncs} F_{ncs} = \mathbb{1}$$

and expanding gives for (A.10) a sum of terms of the form

$$\text{Tr} (F_{n_1 c_1 s_1} B_1 F_{n_2 c_2 s_2} B_2^* \cdots F_{n_{k-1} c_{k-1} s_{k-1}} B_{k-1} F_{n_k c_k s_k} B_k^*) \quad (\text{A.11})$$

with indices (n_j, s_j, c_j) (which are in general different from those in (A.10)) and $k = 2l$.

In order to handle the sector indices in (A.11), we introduce operators K_{n_1, n_2} which act on the sector index and map sector n_2 to sector n_1 , i.e. in components

$$(K_{n_1 n_2})_{n'}^n = \delta_{n_1}^n \delta_{n' n_2} . \quad (\text{A.12})$$

Then

$$F_{ncs} = K_{n1} F_{1cs} K_{1n} . \quad (\text{A.13})$$

If we substitute this relation into (A.11) and combine the operators K . and B_j to “new” operators B_j , we obtain a matrix trace again of the form (A.11), but with all indices n_j equal to one. Therefore we can in what follows restrict attention to the case of one sector and omit the sector indices. The generalization to several sectors will be straightforward by inserting operators K . into the end formulas.

We choose a space-like unit vector u which is orthogonal to ξ and $\bar{\xi}$. Then the imaginary vector $v = iu$ satisfies the relations

$$v_j \xi^j = 0 = v_j \bar{\xi}^j , \quad v^2 = 1 , \quad \bar{v} = -v . \quad (\text{A.14})$$

An explicit calculation using (3.15) yields that

$$F_{R+} = \psi F_{L+} \psi , \quad F_{L-} = \frac{1}{z} \not{\xi} \psi F_{L+} \psi \not{\xi} , \quad F_{R-} = \frac{1}{z} \not{\xi} F_{L+} \not{\xi} . \quad (\text{A.15})$$

Substituting these formulas into (A.11), we obtain an expression involving only the spectral projector F_{L+} , namely

$$(A.11) = \text{Tr} (F_{L+} C_1 F_{L+} C_2 \cdots F_{L+} C_k) \quad (\text{A.16})$$

with suitable matrices C_j . Since the F_{L+} are projectors on one-dimensional subspaces,

$$F_{L+} C F_{L+} = \text{Tr}(F_{L+} C) F_{L+} .$$

By iteratively applying this relation in (A.16), we get the product of traces

$$\text{Tr}(F_{L+} C_1) \text{Tr}(F_{L+} C_2) \cdots \text{Tr}(F_{L+} C_k) .$$

If we express the matrices C_j explicitly in terms of B_j and B_j^* , we obtain the following factorization formula,

$$\begin{aligned} & \text{Tr}(F_{c_1 s_1} B_1 F_{c_2 s_2} F_2^* \cdots B_{k-1} F_{c_k s_k} B_k^*) \\ &= F_{s_1 s_2}^{c_1 c_2}(B_1) F_{s_2 s_3}^{c_2 c_3}(B_2^*) \cdots F_{s_{k-1} s_k}^{c_{k-1} c_k}(B_{k-1}) F_{s_k s_1}^{c_k c_1}(B_k^*) , \end{aligned} \quad (\text{A.17})$$

where $F_{s_i s_j}^{c_i c_j}$ are the so-called *elementary traces* defined by

$$\left. \begin{aligned} F_{++}^{LL}(B) &= \text{Tr}(F_+ \chi_L B) & , & & F_{++}^{LR}(B) &= \text{Tr}(F_+ \psi \chi_L B) \\ F_{+-}^{LL}(B) &= \text{Tr}(\not{x} F_+ \psi \chi_L B) & , & & F_{+-}^{LR}(B) &= \text{Tr}(\not{x} F_+ \chi_L B) \\ F_{-+}^{LL}(B) &= \frac{1}{z} \text{Tr}(F_+ \psi \not{x} \chi_L B) & , & & F_{-+}^{LR}(B) &= \frac{1}{z} \text{Tr}(F_+ \not{x} \chi_L B) \\ F_{--}^{LL}(B) &= \frac{1}{z} \text{Tr}(\not{x} F_+ \not{x} \chi_L B) & , & & F_{--}^{LR}(B) &= \frac{1}{z} \text{Tr}(\not{x} F_+ \psi \not{x} \chi_L B) . \end{aligned} \right\} \quad (\text{A.18})$$

These formulas are also valid for the opposite chirality after the replacements $L \leftrightarrow R$. The elementary traces of B^* are obtained by taking the complex conjugate,

$$\left. \begin{aligned} F_{++}^{LL}(B^*) &= \overline{F_{--}^{RR}(B)} & , & & F_{++}^{LR}(B^*) &= \overline{F_{--}^{LR}(B)} \\ F_{+-}^{LL}(B^*) &= \overline{F_{+-}^{RR}(B)} & , & & F_{+-}^{LR}(B^*) &= \overline{F_{+-}^{LR}(B)} \\ F_{-+}^{RR}(B^*) &= \overline{F_{-+}^{LL}(B)} & , & & F_{-+}^{LR}(B^*) &= \overline{F_{-+}^{LR}(B)} \\ F_{--}^{LL}(B^*) &= \overline{F_{++}^{RR}(B)} & , & & F_{--}^{LR}(B^*) &= \overline{F_{++}^{LR}(B)} . \end{aligned} \right\} \quad (\text{A.19})$$

The relations (A.17)–(A.19) are verified by a straightforward calculation using (3.15), (3.13), and (A.14).

To summarize, the above procedure reduces the calculation of the matrix trace (A.10) to the computation of the elementary traces (A.18) for the contributions B to the light-cone expansion of $P(x, y)$. Taking the complex conjugate (A.19), one obtains the elementary traces of the corresponding contributions to $P(y, x)$. By applying (A.17) and, in the case of several sectors, by suitably inserting the operators K , every matrix trace can be written as a linear combination of products of elementary traces.

A.3 Calculation of the Matrix Traces

We decompose $\Delta P(x, y)$ into its odd and even parts, denoted by B_o and B_e ,

$$\Delta P(x, y) = B_o(x, y) + B_e(x, y) .$$

Explicit formulas for the fermionic projector in the presence of chiral and scalar potentials are listed in the appendix of [3]. For the purpose of this paper, only the contributions involving the mass matrices $Y_{L/R}$ and their derivatives are of importance. But for completeness and for later use, we will also compute the contributions which contain the chiral field strength and the chiral currents. However, we will omit all contributions quadratic in the field strength. Namely, these contributions are related to the energy-momentum

tensor of the chiral fields, and it is therefore reasonable to postpone their analysis until when gravity is studied. Thus the phase-free contributions relevant here are

$$\begin{aligned}
\chi_L B_e &= \frac{1}{2} \chi_L m T^{(0)}(x, y) \oint \int_x^y dz \gamma^j (D_j Y_L) \\
&\quad + \chi_L m T^{(0)}(x, y) Y_L(x) + \mathcal{O}(\log |\xi^2| \xi^0) \\
\chi_L B_o &= \frac{i}{2} \chi_L m^2 T^{(0)}(x, y) \oint \int_x^y dz Y_L Y_R \\
&\quad + i \chi_L m^2 T^{(1)}(x, y) \int_x^y dz [0, 1 | 0] Y_L \gamma^j (D_j Y_R) \\
&\quad + i \chi_L m^2 T^{(1)}(x, y) \int_x^y dz [0, 1 | 0] \gamma^j (D_j Y_L) Y_R \\
&\quad - i \chi_L m^2 T^{(1)}(x, y) Y_L \int_x^y dz \gamma^j (D_j Y_R) \\
&\quad + \chi_L T^{(0)}(x, y) \xi^i \int_x^y dz [0, 1 | 0] \gamma^l F_{li}^L \\
&\quad + \frac{1}{4} \chi_L T^{(0)}(x, y) \oint \int_x^y dz \gamma^j \gamma^k F_{jk}^L \\
&\quad - \frac{1}{2} \chi_L T^{(0)}(x, y) \oint \xi^i \int_x^y dz [0, 0 | 1] j_i^L \\
&\quad + \chi_L T^{(1)}(x, y) \xi^i \int_x^y dz [0, 1 | 1] (\not{\partial} j_i^L) \\
&\quad + \chi_L T^{(1)}(x, y) \int_x^y dz [0, 2 | 0] j_k^L \gamma^k \\
&\quad + \oint \mathcal{O}(\xi^{-2}) + \gamma^j F_{jk}^L \xi^k \mathcal{O}(\xi^{-2}) + \mathcal{O}(F_L^2) + \mathcal{O}(\log |\xi^2| \xi^0).
\end{aligned}$$

A straightforward calculation yields for the elementary traces

$$F_{+-}^{LR}(P_0) = (\deg \leq 1) = \frac{i}{2} X_L (z T_{[0]}^{(-1)}) \quad (\text{A.20})$$

$$F_{-+}^{LR}(P_0) = (\deg \leq 2) = \frac{i}{2} X_L T_{[0]}^{(-1)} \quad (\text{A.21})$$

$$F_{++}^{LL}(B_e) = (\deg \leq 1) = Y_L(x) T_{[1]}^{(0)} + (\deg < 1) \quad (\text{A.22})$$

$$F_{+-}^{LL}(B_e) = (\deg \leq 0) \quad (\text{A.23})$$

$$F_{-+}^{LL}(B_e) = (\deg \leq 1) \quad (\text{A.24})$$

$$F_{--}^{LL}(B_e) = (\deg \leq 1) = Y_L(y) T_{[1]}^{(0)} + (\deg < 1) \quad (\text{A.25})$$

$$F_{++}^{LR}(B_o) = (\deg \leq 1) \quad (\text{A.26})$$

$$= v^j \xi^k \int_x^y dz [0, 1 | 0] F_{jk}^L T_{[0]}^{(0)} + (\deg < 1) \quad (\text{A.27})$$

$$+ \frac{2i}{z - \bar{z}} \epsilon_{ijkl} \xi^i \bar{\xi}^j v^k \int_x^y dz [0, 1 | 0] F_L^{lm} (\xi_m T_{[0]}^{(0)}) \quad (\text{A.28})$$

$$+ \frac{i}{z - \bar{z}} \epsilon^{ijkl} (\xi_i \bar{\xi}_j + \bar{\xi}_i \xi_j^{(0)} - \xi_i \xi_j^{(0)}) v_k \int_x^y dz \xi^n F_{nl}^L T_{[0]}^{(0)} \quad (\text{A.29})$$

$$\begin{aligned}
F_{+-}^{LR}(B_o) &= (\deg \leq 0) \\
&= \frac{i}{2} \int_x^y dz Y_L Y_R ((z T_{[2]}^{(0)}) + 4 T_{[2]}^{(1)}) + (\deg < 0)
\end{aligned} \quad (\text{A.30})$$

$$-2i Y_L(x) Y_R(y) T_{[2]}^{(1)} \quad (\text{A.31})$$

$$-\frac{1}{2} \xi_i \int_x^y dz [0, 0 | 1] j_L^i ((z T_{[0]}^{(0)}) + 8 T_{[0]}^{(1)}) \quad (\text{A.32})$$

$$+\frac{i}{2} \epsilon_{ijkl} \frac{z \bar{\xi}^i - \bar{z} \xi^i}{z - \bar{z}} \int_x^y F_L^{jk} (\xi^l T_{[0]}^{(0)}) \quad (\text{A.33})$$

$$\begin{aligned} F_{-+}^{LR}(B_o) &= (\deg \leq 1) \\ &= \frac{i}{2} \int_x^y dz Y_L Y_R T_{[2]}^{(0)} + (\deg < 1) \end{aligned} \quad (\text{A.34})$$

$$-\frac{1}{2} \xi_i \int_x^y dz [0, 0 | 1] j_L^i T_{[0]}^{(0)} \quad (\text{A.35})$$

$$-\frac{i}{2} \epsilon_{ijkl} \frac{\bar{\xi}^i - \xi^i}{z - \bar{z}} \int_x^y F_L^{jk} (\xi^l T_{[0]}^{(0)}) \quad (\text{A.36})$$

$$\begin{aligned} F_{--}^{LR}(B_o) &= (\deg \leq 1) \\ &= v^j \xi^k \int_x^y dz [1, 0 | 0] F_{jk}^L T_{[0]}^{(0)} + (\deg < 1) \end{aligned} \quad (\text{A.37})$$

$$+\frac{i}{2} \epsilon^{ijkl} \xi_i v_j \int_x^y F_{kl}^L T_{[0]}^{(0)} \quad (\text{A.38})$$

$$+\frac{2i}{z - \bar{z}} \epsilon_{ijkl} \xi^i \bar{\xi}^j v^k \int_x^y dz [0, 1 | 0] F_L^{lm} (\xi_m T_{[0]}^{(0)}) \quad (\text{A.39})$$

$$+\frac{i}{z - \bar{z}} \epsilon^{ijkl} (\xi_i \bar{\xi}_j + \bar{\xi}_i \xi_j^{(0)} - \xi_i \xi_j^{(0)}) v_k \int_x^y dz \xi^n F_{nl} T_{[0]}^{(0)} . \quad (\text{A.40})$$

Here the totally anti-symmetric tensor ϵ_{ijkl} appears because we applied the identity

$$\text{Tr}(\chi_{L/R} \not{d} \not{b} \not{c} \not{d}) = 2((ab)(cd) + (da)(bc) - (ac)(bd)) \mp 2i \epsilon_{ijkl} a^i b^j c^k d^l .$$

Therefore, the corresponding formulas for the opposite chirality are now obtained by the replacements

$$L \longleftrightarrow R, \quad \epsilon_{ijkl} \longrightarrow -\epsilon_{ijkl} . \quad (\text{A.41})$$

The elementary traces of the adjoints are computed via (A.19). All other elementary traces vanish.

Applying (A.17) and the degree estimates for the elementary traces and omitting all terms quadratic in the field strength, we can factorize and estimate the following matrix traces,

$$\begin{aligned} \text{Tr}(F_{L+} \Delta A) &= F_{+-}^{LR}(P_0) F_{-+}^{RL}(B_o^*) + F_{+-}^{LR}(B_o) F_{-+}^{RL}(P_0^*) \\ &\quad + F_{++}^{LL}(B_e) F_{++}^{LL}(B_e^*) + (\deg < 2) \end{aligned} \quad (\text{A.42})$$

$$\begin{aligned} \text{Tr}(F_{L-} \Delta A) &= F_{-+}^{LR}(P_0) F_{+-}^{RL}(B_o^*) + F_{-+}^{LR}(B_o) F_{+-}^{RL}(P_0^*) \\ &\quad + F_{--}^{LL}(B_e) F_{--}^{LL}(B_e^*) + (\deg < 2) \end{aligned} \quad (\text{A.43})$$

$$\text{Tr}(F_{Ls} \Delta A F_{Ls} \Delta A) = \text{Tr}(F_{Ls} \Delta A) \text{Tr}(F_{Ls} \Delta A) = (\deg < 5) \quad (\text{A.44})$$

$$\text{Tr}(F_{Ls} \Delta A F_{Rs} \Delta A) = (\deg < 5) \quad (\text{A.45})$$

$$\begin{aligned} \text{Tr}(F_{L+} \Delta A F_{R-} \Delta A) &= (F_{++}^{LL}(B_e) F_{+-}^{LR}(P_0^*) + F_{+-}^{LR}(P_0) F_{--}^{RR}(B_e^*)) \\ &\quad \times (F_{--}^{RR}(B_e) F_{-+}^{RL}(P_0^*) + F_{-+}^{RL}(P_0) F_{++}^{LL}(B_e^*)) + (\deg < 5) \end{aligned} \quad (\text{A.46})$$

$$\text{Tr}(F_{L-} \Delta A F_{R+} \Delta A) = (F_{--}^{LL}(B_e) F_{-+}^{LR}(P_0^*) + F_{-+}^{LR}(P_0) F_{++}^{RR}(B_e^*))$$

$$\times (F_{++}^{RR}(B_e) F_{+-}^{RL}(P_0^*) + F_{+-}^{RL}(P_0) F_{--}^{LL}(B_e^*)) + (\deg < 5) \quad (\text{A.47})$$

$$\text{Tr}(F_{L+} \Delta A F_{L-} \Delta A) = 0 = \text{Tr}(F_{L-} \Delta A F_{L+} \Delta A). \quad (\text{A.48})$$

If we consider more generally the matrix trace of order l , factorization gives a linear combination of products of elementary traces as in (A.17) (with $k = 2l$). Let us estimate the degree of each of these products. Clearly, the number of factors F_{+-} equals the number of factors F_{-+} , we denote the number of such pairs by p . Furthermore, let q be the number of factors $F_{\pm\pm}(\Delta P^*)$ (where ΔP^* stands for either ΔP or ΔP^*). According to (A.3), each ΔA contains at least one factor ΔP^* , hence $q \geq l$. The number of factors F_{++} and F_{--} is $2(l - p)$, and we saw above that each of these factors must involve ΔP^* , thus $q \geq 2(l - p)$. Adding our two upper bounds for q gives the inequality $q + p \geq 3l/2$. To estimate the degrees we first note that the degree of the pair $F_{+-}(P_0) F_{-+}(P_0)$ is three, and is decreased at least by one each time a P_0 is replaced by ΔP^* . The total number of factors $F_{+-}(\Delta P^*)$ and $F_{-+}(\Delta P^*)$ is $q - 2(l - p)$. On the other hand, the degree of each factor F_{++} and F_{--} is at most one. Hence the degree of the matrix is bounded from below by $3p - (q - 2(l - p)) + 2(l - p) = 4l - (q + p)$. Substituting in our above lower bound for $q + p$ gives the degree estimate

$$\text{Tr}(F_{c_1 s_1} \Delta A \cdots F_{c_l s_l} \Delta A) = \left(\deg < \frac{5}{2} l \right) = (\deg < 3l - 1) \quad \text{for } l \geq 3. \quad (\text{A.49})$$

The above formulas are valid in the case $N = 1$ of one sector. The generalization to several sectors is done by inserting suitable operators K into the traces. This has no effect on the degree on the light cone, and thus the estimates of the matrix traces in (A.42)–(A.49) hold in the general case as well. We substitute the above results for the elementary traces (A.20)–(A.40) into (A.42)–(A.47) and insert the operators K to obtain the following explicit formulas:

$$\begin{aligned} \text{Tr}(F_{nL+} \Delta A) &= (\deg < 2) \\ &+ \text{Tr}_S \left\{ I_n \hat{Y}_L(x) \hat{Y}_L(y) \right\} T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \end{aligned} \quad (\text{A.50})$$

$$+ \frac{1}{4} \int_x^y dz \text{Tr}_S \left\{ I_n \dot{Y}_L \dot{Y}_R X_R \right\} ((z T_{[2]}^{(0)}) + 4 T_{[2]}^{(1)}) \overline{T_{[0]}^{(-1)}} \quad (\text{A.51})$$

$$- \text{Tr}_S \left\{ I_n \dot{Y}_L(x) \dot{Y}_R(y) X_R \right\} T_{[2]}^{(1)} \overline{T_{[0]}^{(-1)}} \quad (\text{A.52})$$

$$+ \frac{1}{4} \int_y^x dz \text{Tr}_S \left\{ I_n X_L \dot{Y}_R \dot{Y}_L \right\} (z T_{[0]}^{(-1)}) \overline{T_{[2]}^{(0)}} \quad (\text{A.53})$$

$$+ \frac{i}{4} \xi_i \int_x^y dz [0, 0 | 1] \text{Tr}_S \left\{ I_n j_L^i X_R \right\} ((z T_{[0]}^{(0)}) + 8 T_{[0]}^{(1)}) \overline{T_{[0]}^{(-1)}} \quad (\text{A.54})$$

$$- \frac{i}{4} \xi_i \int_y^x dz [0, 0 | 1] \text{Tr}_S \left\{ I_n X_L j_R^i \right\} (z T_{[0]}^{(-1)}) \overline{T_{[0]}^{(0)}} \quad (\text{A.55})$$

$$+ \frac{1}{4} \epsilon_{ijkl} \frac{z \bar{\xi}^i - \bar{z} \xi^i}{z - \bar{z}} \xi^l \int_x^y \text{Tr}_S \left\{ I_n F_L^{jk} X_R \right\} T_{[0]}^{(0)} \overline{T_{[0]}^{(-1)}} \quad (\text{A.56})$$

$$+ \frac{1}{4} \epsilon_{ijkl} \frac{\bar{\xi}^i - \xi^i}{z - \bar{z}} \xi^l \int_y^x \text{Tr}_S \left\{ I_n X_L F_R^{jk} \right\} (z T_{[0]}^{(-1)}) \overline{T_{[0]}^{(0)}} \quad (\text{A.57})$$

$$\begin{aligned} \text{Tr}(F_{nL-} \Delta A) &= (\deg < 2) \\ &+ \text{Tr}_S \left\{ I_n \hat{Y}_L(y) \hat{Y}_L(x) \right\} T_{[1]}^{(0)} \overline{T_{[1]}^{(0)}} \end{aligned} \quad (\text{A.58})$$

$$+\frac{1}{4} \int_x^y dz \operatorname{Tr}_S \left\{ I_n \dot{Y}_L \dot{Y}_R X_R \right\} T_{[2]}^{(0)} (\overline{z T_{[0]}^{(-1)}}) \quad (\text{A.59})$$

$$+\frac{1}{4} \int_y^x dz \operatorname{Tr}_S \left\{ I_n X_L \dot{Y}_R \dot{Y}_L \right\} T_{[0]}^{(-1)} ((\overline{z T_{[2]}^{(0)}}) + 4 \overline{T_{[2]}^{(1)}}) \quad (\text{A.60})$$

$$-\operatorname{Tr}_S \left\{ I_n X_L \dot{Y}_R(y) \dot{Y}_L(x) \right\} T_{[0]}^{(-1)} \overline{T_{[2]}^{(1)}} \quad (\text{A.61})$$

$$-\frac{i}{4} \xi_i \int_y^x dz [0, 0 | 1] \operatorname{Tr}_S \left\{ I_n X_L j_R^i \right\} T_{[0]}^{(-1)} ((\overline{z T_{[0]}^{(0)}}) + 8 \overline{T_{[0]}^{(1)}}) \quad (\text{A.62})$$

$$+\frac{i}{4} \xi_i \int_x^y dz [0, 0 | 1] \operatorname{Tr}_S \left\{ I_n j_L^i X_R \right\} T_{[0]}^{(0)} (\overline{z T_{[0]}^{(-1)}}) \quad (\text{A.63})$$

$$-\frac{1}{4} \epsilon_{ijkl} \frac{\bar{\xi}^i - \xi^i}{z - \bar{z}} \xi^l \int_x^y \operatorname{Tr}_S \left\{ I_n F_L^{jk} X_R \right\} T_{[0]}^{(0)} (\overline{z T_{[0]}^{(-1)}}) \quad (\text{A.64})$$

$$-\frac{1}{4} \epsilon_{ijkl} \frac{z \bar{\xi}^i - \bar{z} \xi^i}{z - \bar{z}} \xi^l \int_y^x \operatorname{Tr}_S \left\{ I_n X_L F_R^{jk} \right\} T_{[0]}^{(-1)} \overline{T_{[0]}^{(0)}} \quad (\text{A.65})$$

$$\begin{aligned} \operatorname{Tr}(F_{nL+} \Delta A F_{n'R-} \Delta A) &= (\deg < 5) \\ -\frac{1}{4} \operatorname{Tr}_S \left\{ I_n \left(\hat{Y}_L(x) X_L T_{[1]}^{(0)} (\overline{z T_{[0]}^{(-1)}}) - X_L \hat{Y}_R(x) (z T_{[0]}^{(-1)}) \overline{T_{[1]}^{(0)}} \right) \right. \\ &\quad \left. \times I_{n'} \left(\hat{Y}_R(y) X_R T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - X_R \hat{Y}_L(y) T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) \right\} \end{aligned} \quad (\text{A.66})$$

$$\begin{aligned} \operatorname{Tr}(F_{nL-} \Delta A F_{n'R+} \Delta A) &= (\deg < 5) \\ -\frac{1}{4} \operatorname{Tr}_S \left\{ I_n \left(\hat{Y}_L(y) X_L T_{[1]}^{(0)} \overline{T_{[0]}^{(-1)}} - X_L \hat{Y}_R(y) T_{[0]}^{(-1)} \overline{T_{[1]}^{(0)}} \right) \right. \\ &\quad \left. \times I_{n'} \left(\hat{Y}_R(x) X_R T_{[1]}^{(0)} (\overline{z T_{[0]}^{(-1)}}) - X_R \hat{Y}_L(x) (z T_{[0]}^{(-1)}) \overline{T_{[1]}^{(0)}} \right) \right\} \end{aligned} \quad (\text{A.67})$$

A.4 Perturbation of the Non-Zero Eigenvalues

In Section 4 we calculated the eigenvalues λ_{ncs} of A in the presence of chiral and scalar potentials to the leading degree 3, (3.15). Now we shall compute the contributions to the non-zero eigenvalues of degree two, denote by $\Delta\lambda_{ncs}$, $n = 1, \dots, 7$ (the kernel of A will be considered in Section A.5). To this end, we need to analyze the matrix A on the invariant subspaces $\operatorname{Im} G_k$. First, we choose for fixed $k > 1$ a convenient basis of $\operatorname{Im} G_k$ as follows. The degeneracy of the unperturbed eigenspace $\operatorname{Im} F_k$ can be described by the index set I ,

$$I = \{(ncs) \text{ with } \lambda_{ncs} = \lambda_k\}. \quad (\text{A.68})$$

Note that, according to (3.15), s is the same for all elements $(ncs) \in I$, provided that the eigenvalue is non-zero. The index c , however, may take both values L and R , giving rise to the partition of I into I_L and I_R ,

$$I_{L/R} = \{(ncs) \in I \text{ with } c = L/R\}.$$

The set I can be used to index a basis of F_k ; namely we choose

$$(\phi_{ncs})_{(ncs) \in I} \quad \text{with} \quad 0 \neq \phi_{ncs} \in \operatorname{Im} F_{ncs}. \quad (\text{A.69})$$

It is convenient to assume that the basis vectors are related to each other by

$$\phi_{n'cs} = K_{n'n} \phi_{ncs}, \quad \phi_{n'\bar{c}s} = K_{n'n} \psi \phi_{ncs}; \quad (\text{A.70})$$

this can clearly be arranged according to (A.13)–(A.15). Since F_k projects onto a null space, the inner product of any two basis vectors ϕ_{ncs} vanishes. Thus in order to be able to evaluate vectors in $\text{Im}F_k$ using the scalar product, we choose a “dual basis” $(\phi^{ncs})_{(ncs) \in I}$ of $\text{Im}F_k^*$ given by

$$\phi^{ncs} \in \text{Im} F_{ncs}^*, \quad \phi^{n'cs} = K_{n'n} \phi^{ncs}, \quad \phi^{n'\bar{c}s} = K_{n'n} \psi \phi^{ncs}. \quad (\text{A.71})$$

The basis vectors and their duals are orthogonal in the sense that for $(ncs) \neq (n'c's)$,

$$\langle \phi^{ncs} | \phi_{n'c's} \rangle = \langle F_{ncs}^* \phi^{ncs} | F_{n'c's} \phi_{n'c's} \rangle = \langle \phi^{ncs} | F_{ncs} F_{n'c's} \phi_{n'c's} \rangle = 0.$$

We normalize the basis vectors such that

$$\langle \phi^{ncs} | \phi_{n'c's} \rangle = \delta_{n'}^n \delta_{c'}^c \quad \text{for all } (ncs), (n'c's) \in I. \quad (\text{A.72})$$

Next we introduce a basis $(\psi_{ncs})_{(ncs) \in I}$ of the invariant subspace $\text{Im}G_k$ by applying the projector P_k to the ϕ_{ncs} ,

$$\psi_{ncs} = G_k \phi_{ncs}. \quad (\text{A.73})$$

Finally, we introduce a basis $(\psi^{ncs})_{(ncs) \in I}$ which is dual to (ψ_{ncs}) . We must be careful because projecting on $\text{Im}G_k$ and $\text{Im}G_k^*$, respectively, does not preserve the orthonormality; more precisely,

$$\begin{aligned} S_{n'c's}^{ncs} &\equiv \langle G_k^* \phi^{ncs} | \psi_{n'c's} \rangle = \langle G_k^* \phi^{ncs} | G^k \phi_{n'c's} \rangle \\ &= \langle \phi^{ncs} | G^k | \phi_{n'c's} \rangle \stackrel{\text{in general}}{\neq} \delta_{n'}^n \delta_{c'}^c. \end{aligned} \quad (\text{A.74})$$

But S is a perturbation of the identity, and thus it can be inverted within the perturbation expansion by a Neumann series. This makes it possible to introduce $(\psi^{ncs})_{(ncs) \in I}$ by

$$\psi^{ncs} = \sum_{(n'c's) \in I} (S^{-1})_{n'c's}^{ncs} G_k^* \phi^{n'c's}. \quad (\text{A.75})$$

A short calculation shows that this basis of $\text{Im}G_k^*$ is indeed dual to (ψ_{ncs}) in the sense that

$$\langle \psi^{ncs} | \psi_{n'c's} \rangle = \delta_{n'}^n \delta_{c'}^c \quad \text{for all } (ncs), (n'c's) \in I. \quad (\text{A.76})$$

Using the basis (ψ_{ncs}) and its dual (ψ^{ncs}) , we can write down matrix elements of A ,

$$A_{n'c's}^{ncs} = \langle \psi^{ncs} | A | \psi_{n'c's} \rangle \quad \text{for } (ncs), (n'c's) \in I. \quad (\text{A.77})$$

From the orthonormality (A.76) one sees that $A_{n'c's}^{ncs}$ is indeed a matrix representation for A in the basis (ψ_{ncs}) , and thus the eigenvalues of A on the invariant subspace $\text{Im}G_k$ are obtained simply by diagonalizing this matrix. In the unperturbed case (i.e. if $\Delta A = 0$), the matrix $A_{n'c's}^{ncs}$ simplifies to

$$\begin{aligned} A_{n'c's}^{ncs} &= \langle \phi^{ncs} | A_0 | \phi_{n'c's} \rangle = \langle \phi^{ncs} | A_0 F_{n'c's} \phi_{n'c's} \rangle \\ &= \lambda_k \langle \phi^{ncs} | \phi_{n'c's} \rangle = \lambda_k \delta_{n'}^n \delta_{c'}^c, \end{aligned}$$

in agreement with the fact that $\text{Im}F_k$ is an eigenspace of A_0 corresponding to the eigenvalue λ_k . Thus we see that the matrix elements $A_{n'c's}^{ncs}$ are to leading order on the light cone of degree 3. In the following theorem we compute the matrix elements up to contributions of degree < 2 .

Theorem A.1 *We consider the fermionic projector in the presence of chiral and scalar potentials (3.5) and in composite expressions disregard all terms quadratic in the field strength. Then for all $k = 2, \dots, K$ and $(ncs), (n'c's) \in I$,*

$$\begin{aligned} A_{n'c's}^{ncs} &= \lambda_k \delta_{n'}^n \delta_{c'}^c + \delta_{c'}^c \text{Tr}(F_{ncs} \Delta A K_{n'n}) \\ &+ \delta_{c'}^c \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} \text{Tr}(F_{ncs} \Delta A F_l \Delta A K_{n'n}) + (\deg < 2). \end{aligned} \quad (\text{A.78})$$

Proof. We begin by computing the matrix S , (A.74), and its inverse. This calculation will also illustrate how the relations (A.70) and (A.71) make it possible to rewrite expectation values as matrix traces and thus to apply the results of Sections A.2 and A.3. In the case $c = c'$, we obtain from (A.74) and (A.70),

$$\begin{aligned} S_{n'cs}^{ncs} &= \langle \phi^{ncs} | G_k | \phi_{n'cs} \rangle = \langle \phi^{ncs} | G_k | K_{n'n} \phi_{ncs} \rangle \\ &\stackrel{(\text{A.69}), (\text{A.71})}{=} \langle F_{ncs}^* \phi^{ncs} | G_k K_{n'n} | F_{ncs} \phi_{ncs} \rangle \\ &= \langle \phi^{ncs} | F_{ncs} G_k K_{n'n} F_{ncs} | \phi_{ncs} \rangle \\ &\stackrel{(*)}{=} \text{Tr}(F_{ncs} G_k K_{n'n}) \langle \phi^{ncs} | F_{ncs} | \phi_{ncs} \rangle \\ &= \text{Tr}(F_{ncs} G_k K_{n'n}) \langle \phi^{ncs} | \phi_{ncs} \rangle \stackrel{(\text{A.72})}{=} \text{Tr}(F_{ncs} G_k K_{n'n}), \end{aligned}$$

where in $(*)$ we used that F_{ncs} projects on a one-dimensional subspace. If we substitute the perturbation expansion for G_k , (A.6), into the obtained matrix trace, the estimate (A.49) shows that the orders $n > 2$ yield contributions to S of degree < -1 . Thus it suffices to consider for G_k the second order expansion (A.8). This gives

$$S_{n'cs}^{ncs} = \delta_{n'}^n - \sum_{l \neq k} \frac{1}{(\lambda_k - \lambda_l)^2} \text{Tr}(F_{ncs} \Delta A F_l \Delta A K_{n'n}) + (\deg < -1). \quad (\text{A.79})$$

Note that of the matrix trace appearing here we need to take into account only the leading contributions of degree 5; these are easily obtained from (A.66) and (A.67). In the case $c \neq c'$, we obtain similarly

$$S_{n'\bar{c}s}^{ncs} = \langle \phi^{ncs} | G_k K_{n'n} \psi \phi_{ncs} \rangle = \text{Tr}(F_{ncs} G_k K_{n'n} \psi).$$

We again substitute in the expansion for G_k (A.8). As a consequence of the additional factor ψ , the contribution to zeroth order in ΔA now drops out. The first order contribution to $S_{n'\bar{c}s}^{ncs}$ is

$$\begin{aligned} \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} (F_{ncs} \Delta A F_l K_{n'n} \psi) &= \frac{1}{\lambda_{ncs} - \lambda_{n'\bar{c}s}} (F_{ncs} \Delta A F_{n'\bar{c}s} K_{n'n} \psi) \\ &= \frac{1}{\lambda_{ncs} - \lambda_{n'\bar{c}s}} (F_{ncs} \Delta A K_{n'n} \psi) = (\deg < -1), \end{aligned}$$

because according to (A.18) and (A.26) the last matrix trace has degree ≤ 1 . Here we implicitly assumed that $\lambda_{ncs} \neq \lambda_{n'\bar{c}s}$, because otherwise we clearly get zero. A straightforward calculation using the factorization formula (A.17) as well as the estimates for the elementary traces following (A.20) shows that the second order contribution to $S_{n'\bar{c}s}^{ncs}$ also is of degree < -1 . We conclude that

$$S_{n'\bar{c}s}^{ncs} = (\deg < -1). \quad (\text{A.80})$$

Now we can take the inverse of the expansions (A.79) and (A.80). This gives

$$(S^{-1})_{n'c's}^{ncs} = \delta_{n'}^n \delta_{c'}^c + \delta_{c'}^c \sum_{l \neq k} \frac{1}{(\lambda_k - \lambda_l)^2} \text{Tr}(F_{ncs} \Delta A F_l \Delta A K_{n'n}) + (\text{deg} < -1). \quad (\text{A.81})$$

We next compute the expectation values

$$\langle \phi^{ncs} | A G_k | \phi_{n'c's} \rangle$$

up to contributions of degree < 2 . The method is the same as for the above calculation of the matrix S . In the case $c = c'$, we obtain the following matrix trace,

$$\begin{aligned} \langle \phi^{ncs} | A G_k | \phi_{n'cs} \rangle &= \langle \phi^{ncs} | A G_k K_{n'n} | \phi_{ncs} \rangle \\ &= \langle \phi^{ncs} | F_{ncs} A G_k K_{n'n} F_{ncs} | \phi_{ncs} \rangle = \text{Tr}(F_{ncs} A G_k K_{n'n}). \end{aligned}$$

Substituting in (A.1) and (A.6), the estimate (A.49) shows that it suffices to take into account G_k to second order (A.8). We get

$$\begin{aligned} \langle \phi^{ncs} | A G_k | \phi_{n'cs} \rangle &= \lambda_k \delta_{n'}^n \delta_{c'}^c + \text{Tr}(F_{ncs} \Delta A K_{n'n}) \\ &+ \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} \text{Tr}(F_{ncs} \Delta A F_l \Delta A K_{n'n}) \\ &- \sum_{l \neq k} \frac{\lambda_k}{(\lambda_k - \lambda_l)^2} \text{Tr}(F_{ncs} \Delta A F_l \Delta A K_{n'n}) + (\text{deg} < 2). \end{aligned} \quad (\text{A.82})$$

In the case $c \neq c'$, we can rewrite the expectation value as follows,

$$\langle \phi^{ncs} | A G_k | \phi_{n'\bar{c}s} \rangle = \langle \phi^{ncs} | A G_k K_{n'n} \psi | \phi_{ncs} \rangle = \text{Tr}(F_{ncs} A G_k K_{n'n} \psi).$$

If we substitute in (A.1) and (A.8), factor the resulting matrix traces, and use the estimates of the elementary traces of Section A.3, we obtain that

$$\langle \phi^{ncs} | A G_k | \phi_{n'\bar{c}s} \rangle = (\text{deg} < 2). \quad (\text{A.83})$$

In order to bring the matrix elements (A.77) into a suitable form, we substitute the definitions (A.73) and (A.75) into (A.77) to obtain

$$\begin{aligned} A_{n'c's}^{ncs} &= \sum_{(\tilde{n}\tilde{c}s) \in I} (S^{-1})_{\tilde{n}\tilde{c}s}^{ncs} \langle G_k^* \phi^{\tilde{n}\tilde{c}x} | A | G_k \phi_{n'c's} \rangle \\ &= \sum_{(\tilde{n}\tilde{c}s) \in I} (S^{-1})_{\tilde{n}\tilde{c}s}^{ncs} \langle \phi^{\tilde{n}\tilde{c}x} | A G_k | \phi_{n'c's} \rangle, \end{aligned}$$

where in the last step we used that G_k commutes with A (as the projector on an invariant subspace). Putting in the expansions (A.81) and (A.82),(A.83) gives the result. \blacksquare

If there are no degeneracies, the above theorem reduces to the well-known formula of second order perturbation theory. The important result is that to the considered degree on the light cone, the matrix elements $A_{n'c's}^{ncs}$ are all zero if $c \neq c'$. In other words, the left- and right-handed components are invariant subspaces of A . This fact immediately gives rise to the following corollary.

Corollary A.2 *We consider the fermionic projector in the presence of chiral and scalar potentials (3.5) and in composite expressions disregard all terms quadratic in the field strength. Suppose that the matrix $A_{n'c's}^{ncs}$, (A.78), is diagonal in the sector indices n, n' for all $k = 2, \dots, K$. Then for $n = 1, \dots, 7$, the contributions to the eigenvalues of degree two are*

$$\Delta\lambda_{nL+} = \text{Tr}(F_{nL+} \Delta A) + \sum_{n'=1}^8 \frac{1}{\lambda_{nL+} - \lambda_{n'R-}} \text{Tr}(F_{nL+} \Delta A F_{n'R-} \Delta A) \quad (\text{A.84})$$

$$\Delta\lambda_{nL-} = \text{Tr}(F_{nL-} \Delta A) + \sum_{n'=1}^8 \frac{1}{\lambda_{nL-} - \lambda_{n'R+}} \text{Tr}(F_{nL-} \Delta A F_{n'R+} \Delta A) . \quad (\text{A.85})$$

The traces appearing here are given explicitly by (A.50)–(A.67), where the line integrals are in phase-free form (ϵ_{ijkl} is the totally antisymmetric tensor). The corresponding formulas for the opposite chirality are obtained by the replacements (A.41).

Proof. The result is an immediate consequence of Theorem A.1 and the estimates (A.42)–(A.48). ■

A.5 Perturbation of the Kernel

The results of the previous section do not apply to the kernel of A . The reason is that for $k = 1$, the index set I , (A.68), is

$$I = \{(ncs) \text{ with } n = 8, c = L/R, s = \pm\},$$

and this index set contains both elements with $s = +$ and $s = -$, giving rise to different types of matrix elements. On the other hand, the situation for the kernel is easier because the unperturbed spectral projector on the kernel satisfies the relations

$$X^* F_1 X = 0 \quad (\text{A.86})$$

$$\chi_R F_1 X = 0 = X^* F_1 \chi_L, \quad (\text{A.87})$$

and furthermore we can simplify the calculations using that $\lambda_1 = 0$. Using these relations, we follow that, neglecting all contributions of degree < 2 , the dimension of the kernel is not affected by the perturbation.

Theorem A.3 *We consider the fermionic projector in the presence of chiral and scalar potentials (3.5) and assume that the fermionic projector is weakly causality compatible (see Def. 4.1). In composite expressions we disregard all terms quadratic in the field strength. Then*

$$A G_1 = (\deg < 2) .$$

Proof. Using the definition (A.5),

$$\begin{aligned} A G_1 &= \frac{1}{2\pi i} \oint_{|z|=\varepsilon} A (z - A)^{-1} dz = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} (z (z - A)^{-1} - \mathbb{1}) dz \\ &= \frac{1}{2\pi i} \oint_{|z|=\varepsilon} z (z - A)^{-1} dz . \end{aligned}$$

Performing the perturbation expansion gives similar to (A.6),

$$A G_1 = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|z|=\varepsilon} z ((z - A_0)^{-1} \Delta A)^n (z - A_0)^{-1} dz . \quad (\text{A.88})$$

When we substitute in (A.7) and carry out the contour integral with residues, we get zero unless the factor z is differentiated. For this to occur, the pole at $z = 0$ must be at least of order two, and thus we need to take into account only the orders in perturbation theory $n \geq 2$. If on the other hand $n > 2$, we can as in the previous section transform the matrix products into matrix traces, and the estimate (A.49) yields that the resulting contributions to AG_1 are of degree < 2 . Thus it suffices to consider the second order in perturbation theory,

$$\begin{aligned} A G_1 &= \frac{1}{2\pi i} \oint_{|z|=\varepsilon} z (z - A_0)^{-1} \Delta A (z - A_0)^{-1} \Delta A (z - A_0)^{-1} dz + (\text{deg} < 2) \\ &= - \sum_{l=2}^K \frac{1}{\lambda_l} (F_l \Delta A F_1 \Delta A F_1 + F_1 \Delta A F_1 \Delta A F_l + F_1 \Delta A F_l \Delta A F_1) \\ &\quad + (\text{deg} < 2) \end{aligned} \quad (\text{A.89})$$

The weak causality compatibility condition implies that

$$X P(x, y) = P(x, y) = P(x, y) X^* , \quad (\text{A.90})$$

and similarly for composite expressions in the fermionic projector. As a consequence, the first two matrix products in (A.89) vanish; namely,

$$\Delta A F_1 \Delta A = (\Delta A X^*) F_1 (X \Delta A) = \Delta A (X^* F_1 X) \Delta A \stackrel{(\text{A.86})}{=} 0 .$$

In the last matrix product in (A.89) we can apply (A.87),

$$F_1 \Delta A F_l \Delta A F_1 = F_1 (X \Delta A) F_l (\Delta A X^*) F_1 = \chi_L F_1 \Delta A F_l \Delta A F_1 \chi_R . \quad (\text{A.91})$$

Next we substitute in (A.4), rewrite the resulting operator products as matrix traces, factor these matrix traces into elementary traces, and apply the estimates of Section A.3. This straightforward calculation shows that the matrix product (A.91) is on the light cone of degree < 5 . From (A.89) we conclude that AG_1 is of degree < 2 . ■

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