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# Evolving Networks with distance preferences

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## Abstract

We study evolving networks where new nodes when attached to the network form links with other nodes of preferred distances. A particular case is where always the shortest distances are selected (“make friends with the friends of your present friends”). We present simulation results for network parameters like the first eigenvalue of the graph Laplacian (synchronizability), clustering coefficients, average distances, and degree distributions for different distance preferences and compare with the parameter values for random and scale free networks. We find that for the shortest distance rule we obtain a power law degree distribution as in scale free networks, while the other parameters are significantly different, especially the clustering coefficient.

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## 1 Introduction

Graphs can be considered as substrata of dynamic networks, and so, several types of graph models have been proposed for capturing the properties of specific networks [1]. In particular, evolving networks can be modelled through growing graphs, i.e. graphs to which continuously new nodes (vertices) and new links (edges) are added. While regular graphs, i.e. ones where each node has the same connectivity pattern and where consequently interactions are local in nature and progress in a slow and orderly fashion from neighbor to neighbor, can exhibit subtle combinatorial patterns, for a realistic network model typically a certain amount of irregularity or randomness is needed. The prototypes here are the random graphs introduced by Erdős and Rényi where the connections between the nodes are chosen completely randomly [2]. These exhibit quite interesting

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properties, but often real networks are not entirely random in this sense, but show some kind of regularity, not directly in their connectivity pattern, but with respect to some other variable or order parameter. Such a parameter can be a clustering coefficient, the average or maximal distance between nodes in the network (as measured by the minimal number of links separating them), the distribution of the number of links between the nodes, the correlation of such properties between neighboring nodes (i.e. those connected by a link of distance 1), or the first eigenvalue of the graph Laplacian which is relevant for synchronization properties throughout the network of dynamic activities at the individual nodes. Models have been proposed that capture some of these aspects. The small world networks introduced by Watts and Strogatz [3] are constructed from regular graphs by creating additional random links between nodes, with or without deleting some of the existing ones. Once a certain number of such new links has been introduced in proportion to the number of regular ones, distances in the graph get dramatically shortened, and, consequently, activity can spread quite rapidly from a localized source through the entire network. Another interesting model is the one of a scale free network as introduced by Barabasi and Albert [4]. This is a graph where new nodes are added and form a fixed number of links with the existing nodes not completely at random, but with a preference towards those nodes that already have more connections than other ones. More precisely, the probability with which an existing nodes receives a link from a new node is proportional to the number of links it already possesses. The characteristic feature of the emerging graph here is that the number of nodes with a given number of links does not decrease exponentially as a function of the latter as for example in random graphs, but follows a power law – the reason why such a graph is called scale free. Such models can provide valuable insight into existing real networks, for example into patterns of social relations or spreading of diseases in the small world model, or the connection patterns of internet sites or flight connections between airports in the scale free model.

It is then a natural question whether there exists an encompassing scheme that on one hand can put these specific models into a more general perspective and that on the other hand can offer systematic tools for analyzing the dependencies among the various network features listed above. Ideally, these features should depend in an analyzable manner on certain parameters of the network construction, and so their interdependencies could then be studied in terms of relations between the parameters involved.

We attempt here to take a step in this direction by proposing a general scheme for constructing evolving networks. Our model is characterized by a distance preference function. This function specifies the probability in terms of the distance with which an existing node in the network receives a new link from a newly created node that already has formed one random link so as to attach it to the network and to define its distances to the other nodes. The number of links each node is allowed to make can be either fixed – as in our simulation results below – or also follow some random distribution. So, for example, we can stipulate that the shortest distances are always preferred. Thus, a node that is allowed to form a new link does so preferably to another node of distance 2, i.e. to a direct neighbor of a node that it is already attached to. This might constitute a useful model for the formation of social relationships (you want to become a friend of the friends of your present friends as the easiest or safest means of forming new relationships).

Conversely, we might also stipulate that always the most distant nodes are the preferred recipients of new links. Obviously, one then expects that the resulting network has a quite short average distance between any two nodes, as in the small world and scale free models. More interestingly, one may even expect a certain tendency towards the scale free type when shortest distances are preferred. Namely, a node that is highly connected then has a greater chance of receiving a new link than a less well connected one, because the former has a greater chance of being a direct neighbor of another node that has received a previous link from a new node that is attaching itself to the network. Thus, we see the principle that the rich get richer that is characteristic for scale free networks also at work here, although in an indirect and somewhat mediated form. A conceptual advantage of this construction over the scale free one might be that here, for each link, we only need to evaluate local information, namely check those sites in its vicinity. More precisely, if we exclusively select sites of distance two as recipients of new links, then we only have to list all the neighbors of the present neighbors of the link forming node at each step. In contrast to this, for the scale free model, the complete connectivity pattern of any potential recipient anywhere in the network has to be evaluated. In general, in our scheme, whether we give preference to short distances or not, what is crucial for the decision about a new link is not an absolute property of the candidate as in the scale free model, but rather its relation, as expressed by the distance, to the link forming node. This may capture a property that is relevant in some applications.

On the other hand, the scheme where short distances are preferred should lead to more pronounced local clustering effects and larger average distances in the network than the scale free construction model. In this way, we can check that certain network properties are independent of or at least not strongly related to each other.

Of course, our scheme also includes the possibility that all distances are equally preferred. This should generate properties similar to a random network, although the construction is not entirely identical, because for a random graph, all nodes are considered equal, whereas here, only those of the same distance to the node forming links have equal recipient probabilities, because the distances need not be evenly distributed among the nodes.

We could also easily supplement our construction scheme by a rule for the deletion of links and/or nodes according to some criterion to be specified, as a means to stabilize the size of our network. This would allow a comparison of our model with other ones for evolving networks of given size range. Here, however, we do not pursue this aspect systematically.

## 2 Network construction

We start with a small network having  $m_0$  nodes and then let it grow according to the following scheme. We fix a number  $m$  as the number of connections each new node is allowed to establish to other nodes existing in the network; in principle, this number could also be randomly chosen from some distribution instead of being fixed, but, for simplicity, in our simulations, we only work with a fixed  $m = m_0$ , as this will probably not dramatically affect the resulting network properties. The crucial part of our scheme

is the specification of a probability distribution  $p(d)$  for the preferred distance to a node with which a new link is established. So, when a new node  $x_n$  comes in, it is first allowed to make one connection with a randomly chosen node in the network, in order to attach it to the network. (We could also change this rule and let the first connection prefer well connected recipient nodes, as in the scale free model, but in the present paper, we do not perform numerical simulations for that rule.) This leaves us with  $m - 1$  further links that it is allowed to establish. For the formation of any such link, we consider a node  $x$  in the network and select it as the recipient of the new link with a probability given by  $p(d(x_n, x))$ . Of course, the formation of any new link changes the distances in the network and the creation of further links, until the allotted number  $m$  of them has been formed from  $x_n$ , then is governed by the new distance pattern. Once  $x_n$  is connected according to this scheme, we create a new node  $x_{n+1}$  and repeat the procedure.

The distance preference function  $p(d)$  encodes all the features of our construction. An important case is where this function is in fact deterministic, namely where only nodes of distance 2 from  $x_n$  are allowed as link recipients, i.e. the ones that have the smallest possible distance from it (we are not allowing multiple links, and so no further link can be attached to a node at distance 1). Another deterministic choice of  $p(d)$  would be to allow only recipients of maximal distance from  $x_n$ . This obviously makes the scheme computationally much more expensive than the exclusive selection of nodes at distance 2. More generally, we are interested in distance preference functions  $p(d)$  that are decreasing functions of  $d$ , i.e. where short distances are preferred over large ones, but the latter can still be selected with a positive probability.

In our simulations as described in the Table 1, we consider the cases the number of links that each new node is allowed to form is  $m = 2, 3, 4$ , and 5. We let the network grow until its size was 30,000 nodes when we evaluated the various parameters. We considered three different versions of the probability for the distances. In Model 1, we exclusively selected links to nodes of distance 2, i.e. we always formed triangles. In Model 2, we let the probability be proportional to the inverse distance. Thus, there was a (slight) preference for shorter distances over larger ones. In Model 3, in contrast to this, we let the preference function be proportional to the distance itself (scaled with the maximal distance in the network). Thus, there is a preference for larger over shorter distance. Our comparison models are the growing random graph model where all  $m$  links are randomly connected (Model 4) and the scale free or real world model (Model 5).

In Table 1 we give the first eigenvalue  $\lambda_1$ , the clustering coefficient  $C$ , the mean path length  $L$  and the second moment of degrees  $\langle k^2 \rangle$ , for different  $m$  values, for Models 1 to 5. The discussion below will employ the simulation results for  $m = 5$ , as one can see from the table, the results for  $m = 3, 4$  are qualitatively similar but  $m = 2$  is slightly different. The table gives the averages over 10 simulations each; the standard deviations are quite small.

### 3 First eigenvalue

The first (nonzero) eigenvalue of the graph Laplacian is the crucial parameter for the synchronization properties of activities at the network sites as systematically investigated

| $m$     | $\lambda_1$ | $C$     | $L$    | $\langle k^2 \rangle$ |
|---------|-------------|---------|--------|-----------------------|
| Model 1 |             |         |        |                       |
| 2       | .00051      | .692780 | 9.9977 | 28.2986               |
| 3       | .00089      | .658008 | 7.2686 | 72.4940               |
| 4       | .00213      | .620832 | 6.0137 | 140.6150              |
| 5       | .00501      | .586553 | 5.2833 | 236.4537              |
| Model 2 |             |         |        |                       |
| 2       | .13906      | .001011 | 7.0212 | 22.3045               |
| 3       | .25099      | .001002 | 5.6292 | 48.8695               |
| 4       | .32974      | .000986 | 4.9776 | 85.6206               |
| 5       | .38889      | .001058 | 4.5795 | 132.6747              |
| Model 3 |             |         |        |                       |
| 2       | .13872      | .000031 | 7.1207 | 21.7022               |
| 3       | .24933      | .000140 | 5.7022 | 47.1782               |
| 4       | .32844      | .000247 | 5.0324 | 82.4328               |
| 5       | .38688      | .000365 | 4.6203 | 127.5877              |
| Model 4 |             |         |        |                       |
| 2       | .13929      | .000197 | 7.0690 | 21.9742               |
| 3       | .25053      | .000293 | 5.6659 | 47.9818               |
| 4       | .32948      | .000408 | 5.0061 | 83.8960               |
| 5       | .38816      | .000526 | 4.6017 | 129.8109              |
| Model 5 |             |         |        |                       |
| 2       | .15605      | .000696 | 5.8862 | 39.9532               |
| 3       | .27093      | .001276 | 4.8676 | 90.2483               |
| 4       | .35066      | .001688 | 4.3696 | 161.7150              |
| 5       | .40970      | .002076 | 4.0593 | 250.7354              |

Table 1: The first eigenvalue  $\lambda_1$ , the clustering coefficient  $C$ , the mean path length  $L$  and the second moment of degrees  $\langle k^2 \rangle$ , for Models 1-5, for different  $m$  values.

in our previous work [5]. We naturally assume here that the graph  $\Gamma$  under consideration is connected, as are the graphs resulting from our constructive scheme. Moreover they are symmetric because we consider undirected links. We label the nodes of  $\Gamma$  as  $x_1, x_2, \dots, x_n$ , and we let  $n_x$  denote the connectivity, i.e. the number of neighbors of the node  $x$ . The first eigenvalue is then given by

$$\lambda_1 = \inf_{u: \Gamma \rightarrow \mathbb{R}, \sum n_{x_i} u(x_i) = 0} \frac{\sum_{x_i \sim x_j} (u(x_i) - u(x_j))^2}{\sum n_{x_i} u(x_i)^2}, \quad (1)$$

where  $x_i \sim x_j$  denotes that they are neighbors. We can now provide the following heuristic argument how the creation of a new link in the network affects  $\lambda_1$  depending on the distance  $d(x, y)$  between the two nodes  $x, y$  before the link between them is formed. Namely, for any function  $u$  as evaluated for the infimum in (1), the new link only creates an additional summand  $(u(x) - u(y))^2$  in the numerator while the denominator is left unchanged. As the difference in  $u$  between neighbors is minimized for a first eigenfunction, the expected squared difference  $(u(x) - u(y))^2$  should be an increasing function of the distance between  $x$  and  $y$ . Therefore, the value of a typical candidate function  $u$  for the infimum in (1) should increase as a result of the new link in a manner that is positively correlated with the distance  $d(x, y)$ . Thus, if our scheme prefers larger distances the first eigenvalue should get larger than when we select short distances for new links. Of course, this fits well together with the fact that on one hand, a larger  $\lambda_1$  facilitates synchronization across the network, and on the other hand, connecting nodes that had a large distance should have the effect of a more pronounced decrease of the average distance which in turn facilitates synchronization as well.

Our simulations (as described in the Table 1) yield that the first eigenvalue for Model 1 is .005 which is quite close to the value for a regular network. Thus, synchronization is quite difficult in such a network although the average or maximal distance in the network are quite low (as described below) and the degree distribution of the nodes is quite similar to the scale free case. In all the other models,  $\lambda_1$  is substantially larger, namely around .39 for Models 2-4 and .41 for Model 5. It might be of some interest that it appears to be about the same or perhaps even slightly smaller in Model 2, where shorter distances are preferred, than in the random Model 4, which in turn has a smaller value than Model 3 with the preference for larger distances. Thus, the scale free model is the most easily synchronizable of the five, a not always desirable property.

## 4 Clustering

If our distance preference is for the shortest possible distance, namely 2, then the emerging graph will contain many triangles, i.e. triples of nodes of mutual distance 1. As a consequence, we expect that the graph contains highly connected subclusters.

Also, since the creation of any new link increases the first eigenvalue, it has been suggested by Eckmann and Moses [6] to employ the number of triangles for defining some notion of curvature of a graph. This is based on an analogy with Riemannian geometry where the so-called Ricci curvature yields a lower bound for the eigenvalue of the Laplace-Beltrami operator (the Riemannian version of the Laplacian). In other words, the larger



the curvature, the higher the expected value of the first eigenvalue. As our preceding heuristic analysis of the first eigenvalue of the graph Laplacian shows, however, there is a problem with the analogy between the number of triangles and the curvature. Namely, if we add a link to a given graph, then the expected increase in the eigenvalue is the higher, the larger the original distance between the two linked nodes was. In other words, when we select the new link so as to form a new triangle, the expected eigenvalue increase is smallest, or, when trying to pursue the analogy with Riemannian geometry, the additional curvature is least.

The clustering coefficient for a node  $i$  is measured as follows

$$C_i = \frac{2E_i}{k_i(k_i - 1)} \quad (2)$$

where the  $\frac{k_i(k_i-1)}{2}$  is the maximum possible number of connections between the  $k_i$  neighbors of  $i$  if they were fully connected, and  $E_i$  is the actual number of connections between them. The clustering coefficient  $C$  of the network is the average of the coefficients  $C_i$  of the individual nodes.

For our choice  $m = 5$ , for a regular network the value for  $C$  is  $2/3$  (as the number of links of each node is constrained, not all the neighbors of a given node can be connected among each other, and so the value is smaller than 1 in any case). In our Model 1, the value .5851 is quite close to this value, as to be expected, whereas in all other Models, it is dramatically smaller. In fact, for Models 2 and 3 as well as for the random Model 4, it is even smaller than for the scale free Model 5. In particular, the difference between the Models 1 and 2 is striking here.

## 5 Distances

As already explained, the resulting average or maximal distance in our network should be smaller when large distances are preferred for the establishment of new links. However, this is not so easy to support through numerical simulations, as in any case, independently of the preference function adopted, our networks, like the small world and scale free ones, exhibit rather small maximal distances, say around the order of 4 or 5 for networks with ten or twenty thousand sites, and so the difference resulting from the preference function cannot be very pronounced.

There is one observation that can be made here, however. Namely, the direct preference for forming links to nodes at largest distance is not as efficient in reducing the average or maximal distance in the network as the more indirect scheme of preferential attachment to highly connected nodes employed in the Barabasi-Albert model. This demonstrates the virtue of the latter model. In fact, the average distance  $L$  between all possible pairs of nodes is smallest for that model, namely 4.06, around 4.6 for Models 2-4, and about 5.2 for Model 1. Not surprisingly, a preference for short connections leads to a larger average distance although the effect is by no means as pronounced as one might naively expect. It is surprising, however, that  $L$  is slightly larger for Model 3 where large distances are preferred than for the random Model 4, and slightly smaller for Model 2 with its preference for shorter distances.

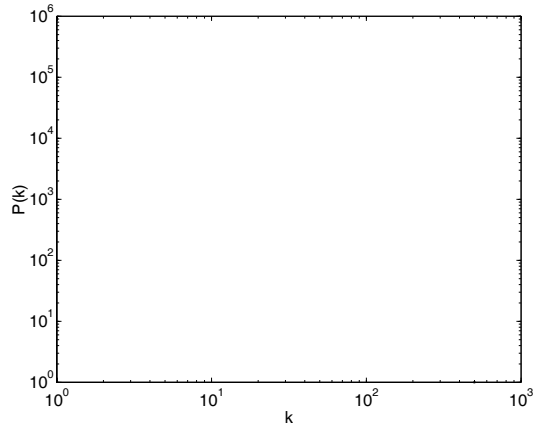


Figure 1: Degree distribution  $P(k)$  for Model 1

## 6 Degree distribution

One of the distinguishing features of the scale free or real world model (Model 5) is that the distribution of the degrees of the nodes decays like a power law in contrast to the exponential of, for example, the random graph model. In Figures (1–5) we give the plot for degree distribution,  $P(k)$  for models 1–5, respectively, with  $m = 4$ . We find that in our Model 1, where exclusively short connections are selected once a node is anchored in the network, the degree distribution likewise follows a power laws, at least over most of its regime. (For  $m = 3$ , we get a power law distribution only for some part of the distribution while the end decays exponentially.) Thus, our mechanism is capable of producing a network that exhibits a power law distribution of the degrees but that differs from the scale free model with respect to a number of distinctive other parameters, like first eigenvalue and synchronizability, clustering, average distance, etc. In particular, this feature is independent of those other features.

Models 2 and 3 show an exponential distribution as in the random model (Model 4). We also find that the distribution of the neighbor degrees (i.e. the sum of the degrees of all the neighbors of a given node,  $P(kk)$ ) also partly follows a power law in our simulations for Models 1 and 5. In Fig. 6 we plot that for Model 1 with  $m = 4$ .

## 7 Correlations

We may ask whether our scheme leads to strong correlations between neighboring sites in the network, with regard to their connectivity. One possible source of such a correlation in connectivity could be a correlation in age. Namely, older nodes in the network have had more chances than younger ones of receiving a random connection from a new node, and so, the connectivity should be positively correlated with the age of a node. However, there is no direct reason why neighboring nodes should exhibit a pronounced age correlation.

Another line of reasoning can go as follows: If  $x_1$  is a neighbor of a site  $x_2$  of connectivity  $l$ , then if distance 2 is selected by our preference function, then  $x_2$  has an  $l$ -fold chance

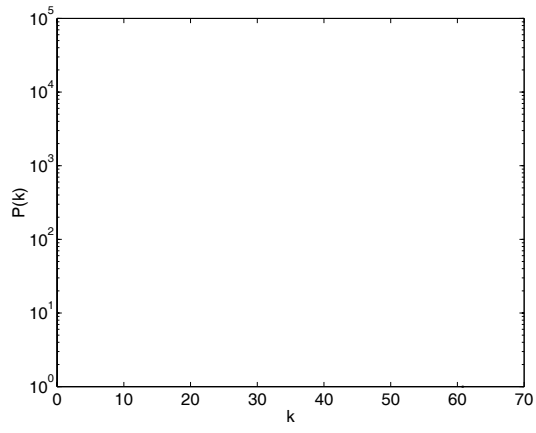


Figure 2: Degree distribution  $P(k)$  for Model 2

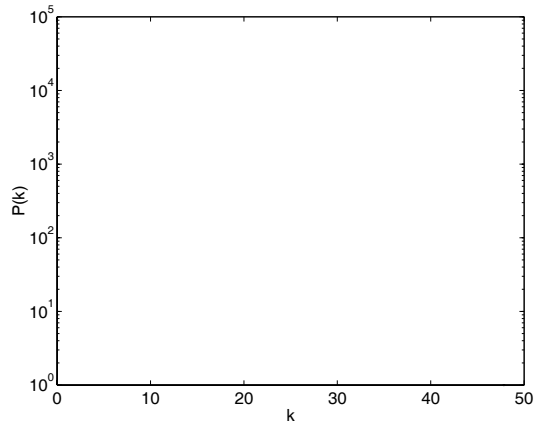


Figure 3: Degree distribution  $P(k)$  for Model 3

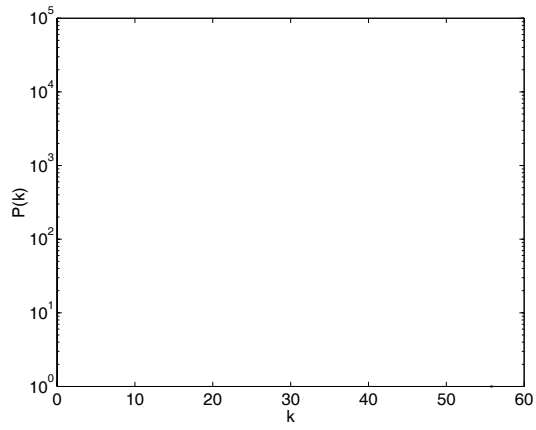


Figure 4: Degree distribution  $P(k)$  for Model 4

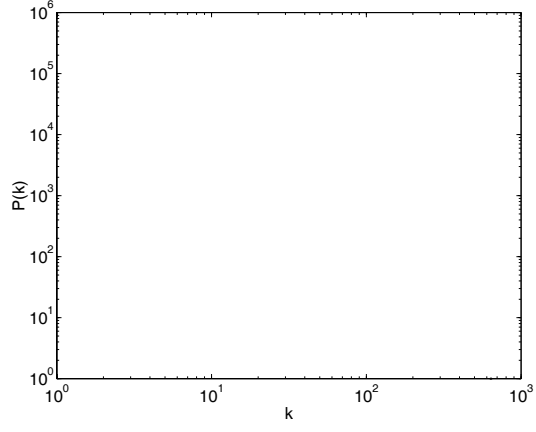


Figure 5: Degree distribution  $P(k)$  for Model 5

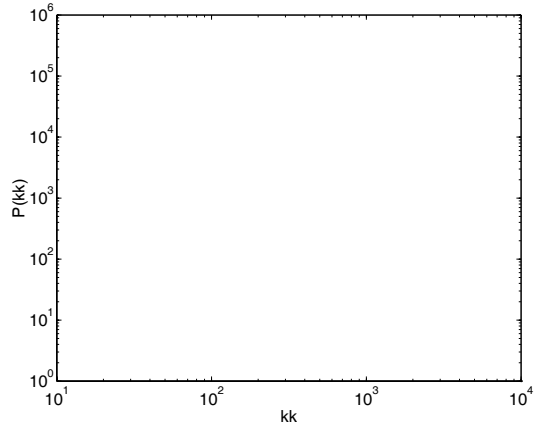


Figure 6: Neighbor Degree distribution  $P(kk)$  for Model 1

of receiving the second connection that a new node  $x_n$  is making, but the chances of  $x_1$  to benefit from this and receive the third connection that  $x_n$  is making is proportional to  $1/l$  as it is facing the competition of the  $l - 1$  other neighbors of  $x_2$ . Thus, the factors cancel, and here, we do not get an advantage for a node from being a neighbor of a well connected node. Of course, this heuristic argument does not take the triangle pattern in the network into account. We calculated the average of the square of the degrees of the nodes (second moment),  $\langle k^2 \rangle$  and is given in the last column of the table. The value of this parameter is around 250 for models 1 and 5 while for models 2, 3 and 4 it is almost half of that value.

## 8 Comparison with other recent network constructions

Holme and Kim [7] introduced a model that in some respects is similar to ours. They let the first connection of a new node form according to preferential attachment as in the scale-free model and then introduce subsequent links that either form triangles or constitute once more preferential attachments, according to some random preference. The resulting network is again scale free. Their main result is that in a scale free network, the clustering coefficient can take different values (according to the strength of the triangle preference).

Klemm and Eguíluz [8] consider a growing network model based on the scale free paradigm, with the distinctive feature that older nodes become inactive at the same rate that new ones are introduced. This is interpreted as a finite memory effect, in the sense that older contributions tend to be forgotten when they are not frequently enough employed. This results in networks that are even more highly clustered than regular ones.

Davidson et al. [9] consider a network that rewires itself through triangle formation. Nodes together with all their links are randomly removed and replaced by new ones with one random link. The resulting network again is highly clustered, has small average distance, and can be tuned towards a scale free behavior.

## 9 Conclusion and discussion

We have introduced a model for evolving networks where each new node, once it is (randomly) anchored to the network, forms further links according to some distance preference function, and we have compared simulation results for the evolved networks with those for two main types previously considered, namely the random graph model and the scale free or real world model of Barabasi-Albert. We found that when always the shortest possible distances are selected for the recipients of new links, we get a highly clustered network which is difficult to synchronize, although it still has a relatively small average distance between nodes. It also exhibits a power law type behavior for the distribution of the degrees of the nodes comparable to the scale free model, although the underlying network forming mechanism is different, and, in particular, there is no direct preference for highly

connected nodes which is considered as the main reason for the power law behavior in the scale free model. As the other network parameters are different from the scale free model, this shows that this feature is independent of clustering or synchronizability properties. For other distance preference functions, we found network parameters that were roughly comparable with the ones for a random graph network, and in fact regardless of whether our preference was proportional or inversely proportional the distance between the link forming node and the potential recipient.

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