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**On the dynamics of cracks in three
dimensions**

by

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On the dynamics of cracks in three dimensions

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Abstract

We introduce a three dimensional dynamic crack propagation law, which is derived from Hamilton's principle. The result is an extension of a previous one obtained, corresponding to the two dimensional case. As a matter of fact, in an orthogonal plane to the crack front, the geometric condition to be satisfied over the path is the same as in two dimensions. The third mode enters only through the energy release rate. The fact that the physics of the problem is locally two dimensional is a consequence of the virtual motions allowed in the set of possible the crack configurations.

Key words: dynamic fracture, variational principles, crack propagation law

1 Introduction

In this note we extend the results obtained in Oleaga (2001) to the dynamic growth of a crack surface in a three dimensional brittle body. The propagation law is obtained by means of the application of Hamilton's principle for

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dissipative forces (cf. (3.7)). It turns out that for smooth propagation the law is given by two scalar equations involving the stress intensity factors. The first one is essentially the equation of energy conservation:

$$\frac{1-\nu^2}{Y} \left[A_I(v) K_I^2 + A_{II}(v) K_{II}^2 \right] + \frac{1}{2\mu} A_{III}(v) K_{III}^2 = 2\gamma.$$

The second one is given by:

$$K_I K_{II} = 0,$$

which implies the co-called symmetry principle (see Cotterel and Rice, 1980, Hodgdon and Sethna, 1993 for the quasi static regime).

The derivation is done under minimal physical assumptions, without imposing any ad-hoc conditions such as the principle of maximum dissipation (see Gurtin and Podio Guidugli, 1998) or the principle of stationary energy density factor (see Sih, 1973). We do not assume a priori balance of configurational forces and pseudomomentum as in Maugin (1993). Adda-Bedia *et al* (1999) computed the components of the so called Eshelby force assuming *a priori* two dimensional physics near the crack front. It turns out that their computation yields the same expression obtained here for the vector ω (cf. (4.14)) defining the crack direction of propagation. Stumpf and Le (1990) performed the analysis of dynamic propagation by means of d'Alembert principle. Their result yields a formally different propagation law for the dynamic regime, even though their expression coincides with the one shown here for a quasi static evolution of the crack. We refer to the conclusions Section in Oleaga (2001) for discussion and comparison of different points of view.

In recent times a great interest has appeared in the expression of the propagation law in terms of the stress intensity factors, due to the discovering of the so called *crack front waves*. These are disturbances of the edge of the crack which propagate along it without attenuation or dispersion. They were found during numerical simulations by Morrissey and Rice (1998). The underlying theory was provided by Willis and Movchan (2000) by means of the solution of a general perturbation of a plane crack in 3D. When this solution is coupled with the fracture criterion it permits the study of the stability of a propagating crack.

As to the plan of this paper, in Sections 2 and 3 we present the basic mathematical and physical assumptions of the problem. In Section 4 we obtain an expression for the vector defining the direction of motion, and finally, in Section 5 we describe the components of this vector, in a local frame, in terms of the stress intensity factors. Some of the computations are done in Appendix A to allow for readability.

2 Kinematics of a 3D cracked body

2.1 The cracked body

Consider an open domain $\mathcal{B} \subset \mathbb{R}^3$ with nonempty interior and smooth boundary $\partial\mathcal{B}$ with normal vector field \mathbf{n} . The motion of a thin macroscopic crack will be defined by a family of non self intersecting surfaces $\mathcal{S}(t) \subset \mathcal{B}$ with smooth boundaries $\mathcal{F}(t) := \partial\mathcal{S}(t)$ such that:

$$\mathcal{S}(t_1) \subset \mathcal{S}(t_2) \quad \text{for } t_1 \leq t_2. \quad (2.1)$$

The cracked body is thus modeled by the set (dependent on time) $\mathcal{B}(t) := \mathcal{B} \setminus \mathcal{S}(t)$. The crack *front* is given by the curves $\mathcal{F}(t)$, and the crack *faces* are denoted by \mathcal{S}^\pm and are given by the set of points $\mathcal{S}(t) \setminus \mathcal{F}(t)$ with corresponding normal vectors \mathbf{n}^\pm attached to each point (see Figure 1). The assumption (2.1) implies that the crack can propagate only through the motion of the front, and no transversal displacement is allowed. We also assume that the crack is far enough from the body surface.

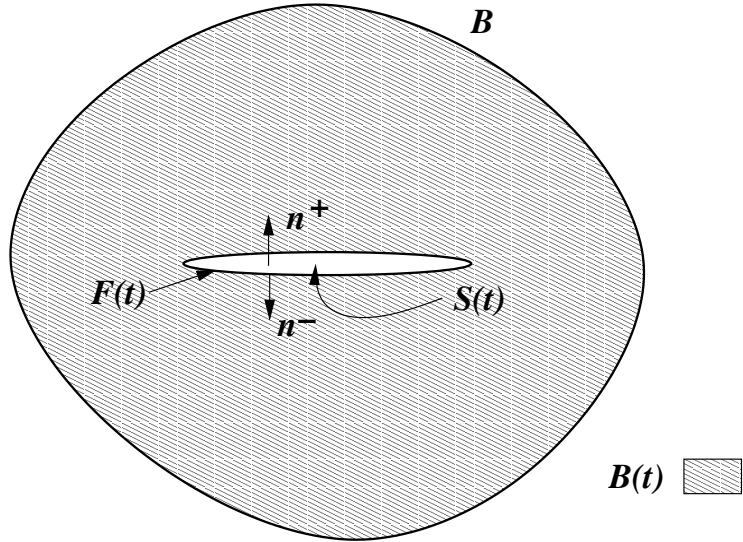


Fig. 1. The crack surface

To each $\mathbf{x} \in \mathcal{F}(t)$ we attach an orthonormal frame $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in such a way that \mathbf{i} is normal to $\mathcal{F}(t)$ and tangent to $\mathcal{S}(t)$; \mathbf{j} is normal to $\mathcal{S}(t)$, and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$ is tangent to $\mathcal{F}(t)$ (see Figure 2). In this way, \mathbf{k} is defined by the current configuration of the crack front, \mathbf{i} is the one that points to the direction of crack front motion, and \mathbf{j} is a normal vector to $\mathcal{S}(t)$ at the boundary $\mathcal{F}(t)$. The velocity of the crack front is denoted by \mathbf{v} .

Now we make some important remarks. The crack front velocity field \mathbf{v} *cannot*

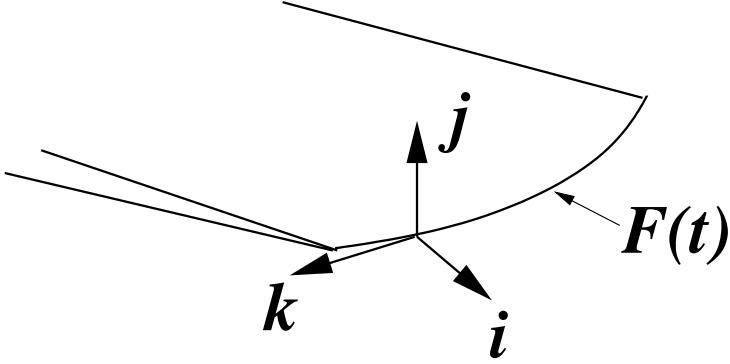


Fig. 2. The local frame

have a component tangential to $\mathcal{F}(t)$. For instance, suppose that the surface $\mathcal{S}(t)$ is not growing in time, so there are no changes in the configuration of the body. In that case there can be no tangential motion along the crack front, because the crack is not a material object inside three dimensional space. This implies that there is no configurational force in the direction of \mathbf{k} , because there are no variations of the energy associated with this non physical displacement. On the other side, the instantaneous change of shape of the set $\mathcal{S}(t)$ can be described by means of the normal velocity of $\mathcal{F}(t)$. It is easy to see that a tangential component for the velocity of the front is superfluous for the description of the evolution of $\mathcal{S}(t)$ under the assumption (2.1). Thus, we assume that $\mathbf{v}(\mathbf{x})$ lies in the plane $\{\mathbf{i}, \mathbf{j}\}$ for each $\mathbf{x} \in \mathcal{F}(t)$. We also have that \mathbf{v} must be tangent to $\mathcal{S}(t)$, so it must lie in the plane $\{\mathbf{i}, \mathbf{k}\}$. Summing up, we have that $\mathbf{v} = v \mathbf{i}$, where $v = |\mathbf{v}|$.

The state of the body is thus determined by the following variables:

- (1) The displacement field of the material particles $\mathbf{u} := (u_1, u_2, u_3)$. This field is defined over $\mathcal{B}(t)$ for each time t and could have a jump discontinuity at $\mathcal{S}(t) \setminus \mathcal{F}(t)$. We call \mathbf{u}^+ and \mathbf{u}^- the values of the displacement over the upper (\mathcal{S}^+) and lower (\mathcal{S}^-) faces of the crack.
- (2) The surface $\mathcal{S}(t)$. This set can also be defined by

$$\mathcal{S}(t) = \mathcal{S}(0) \cup \{\mathcal{F}(t) : 0 < \tau \leq t\},$$

where $\mathcal{S}(0)$ is the initial configuration of the crack.

Boundary data may include displacement conditions (*ie* kinematic constraints). If \mathbf{d} denotes the field of given displacements and $\partial\mathcal{B}_d \subset \partial\mathcal{B}$ denotes the piece of the boundary where they are prescribed, we have $\mathbf{u}(\mathbf{x}, t) := \mathbf{d}(\mathbf{x}, t)$ for $\mathbf{x} \in \partial\mathcal{B}_d$ and all t .

2.2 Virtual variations

The concept of a virtual variation is basic for the expression of the variational principle. As in classical settings (cf. Lanczos, 1949), the virtual variations of the kinematical variables are defined by means of a family of states, indexed by a small parameter $\varepsilon \geq 0$, that can be reached from the actual one in a way compatible with the kinematic constraints. These *variations* are performed for a fixed value of time and are given by functions of the nonnegative parameter ε in the following way:

- $\mathbf{u}_\varepsilon(\mathbf{x}, t) = \mathbf{d}(\mathbf{x}, t)$ for $\mathbf{x} \in \partial\mathcal{B}_d$, and $\mathbf{u}_0 \equiv \mathbf{u}$. We suppose that \mathbf{u}^+ and \mathbf{u}^- can vary freely over the crack faces to avoid contact and friction between \mathcal{S}^+ and \mathcal{S}^- .
- Variations of the crack are given by surfaces \mathcal{S}_ε , such that $\mathcal{S}_0 = \mathcal{S}(t)$ and $\mathcal{S}_{\varepsilon_1} \subset \mathcal{S}_{\varepsilon_2}$ for $0 \leq \varepsilon_1 < \varepsilon_2$.

We omitted the dependence on time in the second item, having in mind that it takes a fixed value. There is a slight difference between the actual motion $\mathcal{S}(t)$ and the virtual motion $\varepsilon \mapsto \mathcal{S}_\varepsilon$ that would be worth to remark. In the first case the surface is smooth, and in the second one \mathcal{S}_ε may have a corner, *i.e.* a discontinuity in its normal vector at the curve $\mathcal{F}(t)$ (see Figure 2). We assume that \mathcal{S}_ε is obtained by gluing a smooth band on $\mathcal{F}(t)$ in the following way:

$$\mathcal{S}_\varepsilon = \mathcal{S}_0 \cup \Delta_\varepsilon .$$

The surface Δ_ε has two boundary curves: $\mathcal{F}(t)$ on one side, and $\mathcal{F}_\varepsilon = \partial\mathcal{S}_\varepsilon$ on the opposite. We also assume that $\mathcal{S}_0 \cap (\Delta_\varepsilon \setminus \mathcal{F}(t)) = \emptyset$, because the virtual states must satisfy the same kinematical conditions imposed to the actual states.

We define now the operator δ as the one side derivative in $\varepsilon = 0$ of any field f :

$$\delta f := \lim_{\varepsilon \rightarrow 0^+} \frac{f_\varepsilon - f_0}{\varepsilon} . \quad (2.2)$$

Notice that f can be also any component of a vector or tensor field.

We need to make an important remark concerning the independence of the kinematical variables. When we perform a variation of the crack geometry, the displacement field is also changed by the virtual motion of the crack due to the variation of domain. In other words, we cannot make variations of \mathcal{S} while keeping constant \mathbf{u} . There is a way to obtain independent quantities, by writing the fields in a fixed domain at time t with the help of a change of variables dependent on ε :

$$\varphi_\varepsilon : \mathcal{B} \mapsto \mathcal{B} , \quad \varphi_0 = \text{Id} , \quad (2.3)$$

such that

$$\varphi_\varepsilon(\mathcal{S}(t)) = \mathcal{S}_\varepsilon \quad \text{and} \quad \varphi_\varepsilon(\mathbf{x}) = \mathbf{x}, \quad \forall \mathbf{x} \in \partial\mathcal{B}. \quad (2.4)$$

Notice that if $\mathbf{x} \in \mathcal{F}(t)$ then $\delta\varphi(\mathbf{x})$ is the virtual velocity of the crack front. It is assumed to be *normal* to the curve $\mathcal{F}(t)$, according to the remarks made about the non-tangential motion of the crack. We must have then the following condition:

$$\delta\varphi(\mathbf{x}) \perp \mathbf{k}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{F}(t). \quad (2.5)$$

Applying now φ_ε

$$\mathbf{u}_\varepsilon(\varphi_\varepsilon(\mathbf{x}), t) = \bar{\mathbf{u}}_\varepsilon(x, t), \quad (2.6)$$

that is, the variation in the “moving” domain is obtained by means of the variation of a displacement field $\bar{\mathbf{u}}$ in a *fixed* domain and composed with the map defining the change of variables. Now we can say that $\bar{\mathbf{u}}$ can be selected independently of the change in domain. We have, according to (2.2), that:

$$\delta\mathbf{u}(\mathbf{x}, t) + \nabla\mathbf{u} \cdot \delta\varphi(\mathbf{x}, t) = \delta\bar{\mathbf{u}}(\mathbf{x}, t)$$

or

$$\delta\mathbf{u}(\mathbf{x}, t) = \delta\bar{\mathbf{u}}(\mathbf{x}, t) - \nabla\mathbf{u} \cdot \delta\varphi(\mathbf{x}, t). \quad (2.7)$$

Moreover, we have that $\delta\bar{\mathbf{u}}(\mathbf{x}, t)$ is not singular near the tip (it has the same behaviour as the field \mathbf{u}). The variables $\bar{\mathbf{u}}$ and φ are independent. The variation $\delta\bar{\mathbf{u}}$ can have any value, except on \mathcal{S}_d where it must be zero, according to the imposed Dirichlet boundary conditions. The variation of domain $\delta\varphi(\mathbf{x})$ for $\mathbf{x} \in \mathcal{F}(t)$ can have any direction inside the plane $\{\mathbf{i}, \mathbf{j}\}$ except of the one given by $-\mathbf{i}$, that is, except of the one that points to the opposite direction of crack motion. The functions $\delta\bar{\mathbf{u}}$ are the classical variations of displacements for a fixed configuration. They are not relevant for the crack propagation law because they do not involve variations of the domain.

3 Physical assumptions

3.1 Small strain elasticity

As already mentioned, displacements \mathbf{d} are prescribed over $\partial\mathcal{B}_d$ and tractions \mathbf{t} are given over $\partial\mathcal{B}_t := \partial\mathcal{B} \setminus \partial\mathcal{B}_d$. For simplicity, we assume that the body undergoes infinitesimal strains, and the corresponding measure of the geometrical deformation is given by the tensor:

$$\mathbf{e} := \frac{1}{2} (\nabla\mathbf{u} + \nabla\mathbf{u}^T) \quad e_{ij} := \frac{1}{2} (\partial_i u_j + \partial_j u_i). \quad (3.1)$$

In this way, we do not need to distinguish between the reference and the deformed configurations, even though our results are easily applied to the more general case of finite deformations of hyperelastic bodies.

The Cauchy stress tensor is denoted by $\boldsymbol{\sigma}(\mathbf{x}, t)$. For simplicity, we assume that no external volumetric forces are applied to the body, and no tractions are applied to the crack faces.

3.2 Energy terms for the cracked body

The internal energy density, under thermodynamic equilibrium, is given by the *strain energy function*:

$$U := U(\mathbf{e}) \quad \text{and} \quad \boldsymbol{\sigma} = \frac{\partial U}{\partial \mathbf{e}_{ij}}. \quad (3.2)$$

The total elastic energy stored on $\mathcal{B}(t)$ can be written as:

$$E(t) := \int_{\mathcal{B}(t)} U(\mathbf{e}(\mathbf{x}, t)) \, dx, \quad (3.3)$$

The total kinetic energy of the body is defined by:

$$T(t) := \int_{\mathcal{B}(t)} k(\dot{\mathbf{u}}(\mathbf{x}, t)) \, dx, \quad (3.4)$$

where $\dot{\mathbf{u}}(\mathbf{x}, t) := \partial \mathbf{u} / \partial t$ is the velocity field of the material particles and

$$k(\dot{\mathbf{u}}) := \frac{1}{2} \rho \dot{u}_i \dot{u}_i = \frac{1}{2} \rho |\dot{\mathbf{u}}|^2, \quad (3.5)$$

is the kinetic energy density of the body, with ρ being the mass density per unit volume.

The presence of the crack adds another energy term that we call the *surface energy*. This is related to the work done by the interatomic forces when the faces are separated. Under the classical Griffith assumption for brittle materials (see Griffith, 1920) we have that the surface energy S is given by:

$$S(t) := 2\gamma \int_{\mathcal{S}(t)} d\mathcal{S}, \quad (3.6)$$

where γ is a material constant and $d\mathcal{S}$ is the area element. In this simple model, crack surface energy is proportional to the crack area.

3.3 Hamilton's Principle

The basic point of departure for the determination of the crack propagation law, is the following inequality:

$$\int_{t_1}^{t_2} \delta W + \delta T \leq 0 , \quad (3.7)$$

where δW represents the (infinitesimal) virtual work of *all* the forces involved in the process, and δT is the variation of the scalar function defining the kinetic energy of the cracked body. In our simple model the work function is given by:

$$\delta W = \delta W_t - \delta E - \delta S \quad (3.8)$$

The forces involved in the crack process are: the traction at the boundary, whose virtual work term is given by δW_t , the variation of elastic energy (related to the work of elastic forces) δE , and the variation of crack surface energy (related to the work of cohesive forces at the crack faces) δS .

A more extensive explanation about the meaning of (3.7) can be found in Vujanovic and Jones, (1989), Section 1.8. We make in the following paragraphs some general remarks concerned with this variational inequality.

First of all, this principle involves only scalar quantities, such as the total elastic and kinetic energies. These are well defined finite quantities obtained by the integration of singular densities over $\mathcal{B}(t)$. Other variational principles, such as D'Alembert's principle, involve the concept of *inertial force*, requiring the acceleration field for its formulation. For virtual variations of the domain, the virtual work of these forces is not well defined over $\mathcal{B}(t)$. The principle must be somehow complemented in order to obtain a crack propagation law (see Stumpf and Le, 1990).

The operator δ cannot be outside the integral for several reasons. On one side not all the forces are derived from energy functions (*ie* from scalar quantities that are functions of state). On the other side, if we take the δ outside we would change the physics of the problem allowing inadmissible transversal motion of the crack. Each virtual variation must be done for a single value of time without changing the previous shape of $\mathcal{S}(t)$.

The appearance of a “ \leq ” symbol instead of the “ $=$ ” one in (3.7) is due to the fact that the crack cannot close (cf. (2.1)), so we have a kinematic condition of irreversibility over its motion. The inequality must be applied because we are not allowed to compare the actual configuration with all the possible neighbours.

4 The propagation law

4.1 Computation of the energy variations

We compute the variation of the different terms arising in the integrand of Hamilton's inequality (3.7). When we perform variations over the domain by means of the field φ (see (2.3)), the field \mathbf{u} is subjected to variations according to equation (2.6) and so does the velocity $\dot{\mathbf{u}}$ and the strain tensor. Thus, for fixed time t , the strain and kinetic energy densities are subjected to variations.

In the case of the infinitesimal work of the traction field \mathbf{t} at the boundary, the operator δ is not applied to a scalar function of state, and the variation has its own physical definition, given by:

$$\delta W_{\mathbf{t}} := \int_{\partial\mathcal{B}_t} \mathbf{t} \cdot \delta \mathbf{u}. \quad (4.1)$$

Now we compute the expression for the variation of surface energy for fixed time t using (2.2):

$$S_{\varepsilon}(t) - S(t) \equiv 2\gamma \int_{\mathcal{S}_{\varepsilon} \setminus \mathcal{S}} d\mathcal{S} \quad (4.2)$$

In this way, for a small ε , we can identify the added surface with the following set (see also (2.3)-(2.5)):

$$\mathcal{S}_{\varepsilon} \setminus \mathcal{S} \sim \{\mathbf{x} + \zeta \delta \varphi(\mathbf{x}) : \mathbf{x} \in \mathcal{F}(t), 0 < \zeta \leq \varepsilon\}$$

The virtual crack front is thus:

$$\mathcal{F}_{\varepsilon} \sim \{\mathbf{x} + \varepsilon \delta \varphi(\mathbf{x}) : \mathbf{x} \in \mathcal{F}(t)\}.$$

If we consider the parametrization of $\mathcal{S}_{\varepsilon} \setminus \mathcal{S}$ defined by:

$$(s, \zeta) \mapsto \mathbf{x}(s) + \zeta \delta \varphi(s),$$

where s is the length parameter along $\mathcal{F}(t)$, we have that the normal vector to the surface is given by:

$$(\mathbf{k} + \zeta \frac{d}{ds} \delta \varphi) \times \delta \varphi.$$

The surface energy of the added piece can be computed now easily:

$$S_{\varepsilon}(t) = S(t) + 2\gamma \int_{\mathcal{F}(t)} \int_0^{\varepsilon} |(\mathbf{k} + \zeta \frac{d}{ds} \delta \varphi) \times \delta \varphi| d\zeta ds,$$

Taking the derivative with respect to ε in $\varepsilon = 0$ we find:

$$\delta S = 2\gamma \int_{\mathcal{F}(t)} |\mathbf{k} \times \delta \varphi| ds.$$

Taking into account that $|\mathbf{k}| = 1$ and that $\delta\varphi \perp \mathbf{k}$ (see the discussion in section 2.2 and its consequence (2.5)) we have that $|\mathbf{k}(\mathbf{x}) \times \delta\varphi(\mathbf{x})| = |\delta\varphi(\mathbf{x})|$, and thus:

$$\delta S = 2\gamma \int_{\mathcal{F}(t)} |\delta\varphi| \, ds. \quad (4.3)$$

Notice that there is no contribution of the *curvature* of the front $\mathcal{F}(t)$.

In the following we give expressions for δE and δT . These quantities are derivatives of integrals of scalar singular functions over moving domains. We assume the validity of the following transport lemma, whose proof can be easily extended from the two dimensional case given in Oleaga (2001).

Lemma 1 Suppose that $f_\varepsilon(\mathbf{x})$ is a scalar function defined on \mathcal{B}_ε with an integrable singularity over \mathcal{F}_ε . We want to compute:

$$\delta F, \quad \text{where } F_\varepsilon := \int_{\mathcal{B}_\varepsilon} f_\varepsilon.$$

If $\mathcal{R}^\eta(t) \subset \mathcal{B}(t)$ is a small cylindrical region of “radius” $\eta > 0$ around $\mathcal{F}(t)$, and $\mathcal{R}_\varepsilon^\eta := \varphi_\varepsilon(\mathcal{R}^\eta(t))$ isolates \mathcal{F}_ε (see 2.3), we have that:

$$\delta F = \delta \left(\int_{\mathcal{R}_\varepsilon^\eta} f_\varepsilon \right) - \int_{\partial\mathcal{R}^\eta} f \mathbf{n} \cdot \delta\varphi + \int_{\mathcal{B}^\eta(t)} \delta f. \quad (4.4)$$

In (4.4) \mathbf{n} is the normal vector field to $\partial\mathcal{R}^\eta$ (pointing inside the body), $\mathcal{B}^\eta(t) := \mathcal{B}(t) \setminus \mathcal{R}^\eta$ and the δ operator is defined in (2.2).

Remark The left side of (4.4) exists and is independent of η .

Applying Lemma 1 to $f \equiv U, k$ we obtain the following expresions:

$$\begin{aligned} \delta E = \delta \left(\int_{\mathcal{R}_\varepsilon^\eta} U_\varepsilon \right) - \int_{\partial\mathcal{R}^\eta(t)} [(\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta\mathbf{u} + U \mathbf{n} \cdot \delta\varphi] + \int_{\partial B_t} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta\mathbf{u} \\ - \int_{\mathcal{S}^\pm \setminus \mathcal{R}^\eta} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta\mathbf{u} - \int_{B_\eta(t)} \operatorname{div} \boldsymbol{\sigma} \cdot \delta\mathbf{u} \end{aligned} \quad (4.5)$$

$$\int_{t_1}^{t_2} \delta T = \int_{t_1}^{t_2} \left[\delta \left(\int_{\mathcal{R}_\varepsilon^\eta} k_\varepsilon \right) + \int_{\partial\mathcal{R}_\varepsilon^\eta(t)} \{ \rho \mathbf{v} (\dot{\mathbf{u}} \cdot \delta\mathbf{u}) - k \delta\varphi \} \cdot \mathbf{n} - \int_{B_\eta(t)} \rho \ddot{\mathbf{u}} \cdot \delta\mathbf{u} \right], \quad (4.6)$$

where $\mathcal{B}_\eta(t) := \mathcal{B}(t) \setminus \mathcal{R}_\eta(t)$. A basic derivation of these results is given in Appendix A.

4.2 The crack propagation law

We gather the expressions (4.1), (4.3), (4.5) and (4.6) in (3.7) and we obtain:

$$\begin{aligned} & \int_{t_1}^{t_2} \left\{ \delta \left(\int_{\mathcal{R}_\varepsilon^\eta} k_\varepsilon - U_\varepsilon \right) + \int_{B^\eta(t)} (\operatorname{div} \boldsymbol{\sigma} - \rho \ddot{\mathbf{u}}) \cdot \delta \mathbf{u} \right. \\ & \quad + \int_{\partial \mathcal{R}^\eta(t)} (U - k) \mathbf{n} \cdot \delta \boldsymbol{\varphi} + (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta \mathbf{u} + \rho (\mathbf{v} \cdot \mathbf{n}) (\dot{\mathbf{u}} \cdot \delta \mathbf{u}) \\ & \quad \left. + \int_{\partial B_t} (\mathbf{t} - \boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta \mathbf{u} + \int_{\mathcal{S}^\pm \setminus \mathcal{R}^\eta} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta \mathbf{u} - 2\gamma \int_{\mathcal{F}(t)} |\delta \boldsymbol{\varphi}| \right\} \leq 0. \quad (4.7) \end{aligned}$$

Consider now regions $\mathcal{R}^\eta(t)$ satisfying:

$$\lim_{\eta \rightarrow 0} \operatorname{Volume}\{\mathcal{R}^\eta(t)\} = 0,$$

uniformly in t .

We take first $\delta \boldsymbol{\varphi} \equiv \mathbf{0}$ (no variation of domain). We have that $\delta \mathbf{u}$ is equal to $\delta \bar{\mathbf{u}}$ in (2.7) and has no singular behaviour around the crack front. The integrand over $\partial \mathcal{R}^\eta$ is bounded, then taking limit in (4.7) for $\eta \rightarrow 0$ and considering the reversibility of $\delta \bar{\mathbf{u}}$ we obtain the following equality:

$$\int_{t_1}^{t_2} \left\{ \int_{B(t)} (\operatorname{div} \boldsymbol{\sigma} - \rho \ddot{\mathbf{u}}) \cdot \delta \mathbf{u} + \int_{\partial B_t} (\mathbf{t} - \boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta \mathbf{u} + \int_{\mathcal{S}^\pm} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta \mathbf{u} \right\} = 0.$$

This leads immediately to the balance equations for elastic media with boundary conditions over a moving surface $\mathcal{S}(t)$ in the absence of body forces:

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} - \rho \ddot{\mathbf{u}} &= \mathbf{0} \quad \mathbf{x} \in \mathcal{B}(t), \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t} \quad \mathbf{x} \in \mathcal{B}_t, \\ \boldsymbol{\sigma} \cdot \mathbf{n}^\pm &= \mathbf{0} \quad \mathbf{x} \in \mathcal{S}^\pm(t). \end{aligned} \quad (4.8)$$

Now we return to (4.7) assuming $\delta \bar{\mathbf{u}} \equiv 0$ in (2.7), that is:

$$\delta \mathbf{u}(\mathbf{x}, t) = -\nabla \mathbf{u} \cdot \delta \boldsymbol{\varphi}(\mathbf{x}, t), \quad (4.9)$$

and assuming (4.8) and taking limit for $\eta \rightarrow 0$ we obtain the following inequality:

$$\begin{aligned} & \int_{t_1}^{t_2} \lim_{\eta \rightarrow 0} \left\{ \int_{\partial \mathcal{R}^\eta(t)} \left[(U - k) \mathbf{n} - \nabla \mathbf{u}^T (\boldsymbol{\sigma} \cdot \mathbf{n} + \rho (\mathbf{v} \cdot \mathbf{n}) \dot{\mathbf{u}}) \right] \cdot \delta \boldsymbol{\varphi} \right. \\ & \quad \left. - 2\gamma \int_{\mathcal{F}(t)} |\delta \boldsymbol{\varphi}| \right\} \leq 0. \quad (4.10) \end{aligned}$$

We apply now the following relationship:

$$\int_{\mathcal{R}^\eta(t)} d\mathcal{R} \sim \int_{\mathcal{F}(t)} d\mathcal{F} \int_{\mathcal{C}^\eta(t)} d\mathcal{C} \quad \eta \rightarrow 0, \quad (4.11)$$

where $\mathcal{C}^\eta(t)$ is the curve obtained by the intersection of the surface $\partial\mathcal{R}^\eta(t)$ and the plane normal to the front \mathcal{F} . Both integrals are taken with respect to the length parameter of each curve. Relation (4.11) means that for small η the area element of the surface \mathcal{R}^η is the same as the product of the individual length elements of the curves. We rewrite (4.10) in the following way:

$$\int_{t_1}^{t_2} \left(\int_{\mathcal{F}(t)} \left\{ \lim_{\eta \rightarrow 0} \int_{\mathcal{C}^\eta(t)} [(U - k) \mathbf{n} - \nabla \mathbf{u}^T (\boldsymbol{\sigma} \cdot \mathbf{n} + \rho (\mathbf{v} \cdot \mathbf{n}) \dot{\mathbf{u}})] \cdot \delta\boldsymbol{\varphi} \right\} - 2\gamma |\delta\boldsymbol{\varphi}| \right) dt \leq 0. \quad (4.12)$$

Taking arbitrary variations of $\delta\boldsymbol{\varphi}$ compatible with kinematic restrictions we find that:

$$\lim_{\eta \rightarrow 0} \int_{\mathcal{C}^\eta(t)} [(U - k) \mathbf{n} - \nabla \mathbf{u}^T (\boldsymbol{\sigma} \cdot \mathbf{n} + \rho (\mathbf{v} \cdot \mathbf{n}) \dot{\mathbf{u}})] \cdot \delta\boldsymbol{\varphi} \leq 2\gamma |\delta\boldsymbol{\varphi}(\mathbf{x})|, \quad (4.13)$$

for $\mathbf{x} \in \mathcal{F}(t)$. Now, as $\mathcal{C}^\eta(t)$ shrinks to the point \mathbf{x} , $\delta\boldsymbol{\varphi} \rightarrow \delta\boldsymbol{\varphi}(\mathbf{x})$ and we define the vector field $\boldsymbol{\omega}$ over $\mathcal{F}(t)$ as:

$$\boldsymbol{\omega}(\mathbf{x}) := \lim_{\eta \rightarrow 0} \int_{\mathcal{C}^\eta(t)} [(U - k) \mathbf{n} - \nabla \mathbf{u}^T (\boldsymbol{\sigma} \cdot \mathbf{n} + \rho (\mathbf{v} \cdot \mathbf{n}) \dot{\mathbf{u}})] \quad (4.14)$$

so we can write (4.13) as:

$$\boldsymbol{\omega} \cdot \delta\boldsymbol{\varphi}(\mathbf{x}) - 2\gamma |\delta\boldsymbol{\varphi}(\mathbf{x})| \leq 0, \quad (4.15)$$

for every admissible direction $\delta\boldsymbol{\varphi}(\mathbf{x})$ at $\mathbf{x} \in \mathcal{F}(t)$. If we select $\delta\boldsymbol{\varphi} \equiv \mathbf{v}$, that is, the actual direction of motion, we must have that (4.15) becomes an equality representing the conservation of energy, that is, the sum of all the works involved (including inertial) must be zero for the actual motion. So we have:

$$\boldsymbol{\omega} \cdot \delta\mathbf{i} \leq 2\gamma \quad \text{for } |\delta\mathbf{i}| = 1, \quad (4.16)$$

$$\boldsymbol{\omega} \cdot \mathbf{i} = 2\gamma \quad \text{for } |\mathbf{v}| \neq 0. \quad (4.17)$$

This implies in particular, that for $v \neq 0$ we have the following propagation law:

$$\frac{\mathbf{v}}{v} = \frac{\boldsymbol{\omega}}{2\gamma} \quad \text{for each } \mathbf{x} \in \mathcal{F}(t). \quad (4.18)$$

In this way, the direction of crack propagation is defined by the vector $\boldsymbol{\omega}$ given in (4.14). Equation (4.18) also implies that for $v \neq 0$ we must have $|\boldsymbol{\omega}| = 2\gamma$. When $|\boldsymbol{\omega}| < 2\gamma$ the crack cannot propagate at that point, because if $v \neq 0$ we would have that (4.17) holds, which then implies that $|\boldsymbol{\omega}| = 2\gamma$.

5 The crack law in terms of K_I , K_{II} and K_{III}

Let us assume that a smooth crack front is in motion according to (4.18). We proceed to compute the vector ω defined in (4.14) in terms of the stress intensity factors. For $v \neq 0$ we have by (4.18) that the crack tip velocity v is parallel to ω . The asymptotic fields are obtained assuming that locally the crack surface is a plane and the crack front is a straight line (see Freund, 1990). We use an orthogonal coordinate system such that x_1 points in the direction of ω , x_2 is the axis orthogonal to the plane containing the crack, and x_3 points in the direction of the crack front (cf. Figure 3). In this case we have that

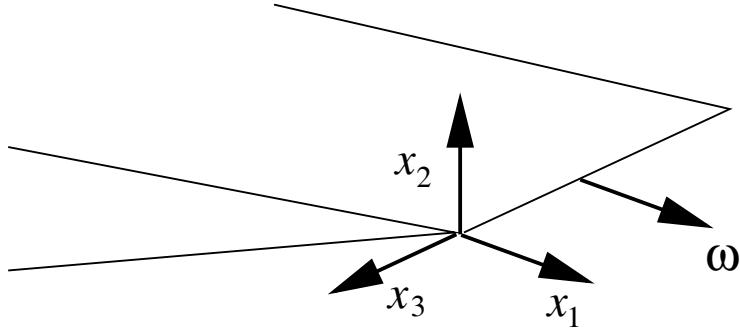


Fig. 3. Axis near the crack front

$$\omega \equiv (2\gamma, 0, 0).$$

We consider now each component in (4.14).

Taking into account that $\omega_1 = \omega \cdot \frac{v}{v}$ we find:

$$\begin{aligned} \omega(\mathbf{x}) \cdot \frac{\mathbf{v}}{v} &= \frac{1}{v} \lim_{\eta \rightarrow 0} \int_{C^\eta(t)} [(U - k) n_j v_j - (\sigma_{ij} n_j + \rho(v_j n_j) \dot{u}_i) u_{i,k} v_k] \\ &= \frac{1}{v} \lim_{\eta \rightarrow 0} \int_{C^\eta(t)} [(U + k) n_j v_j - \sigma_{ij} u_{i,k} v_k n_j] \\ &= \lim_{\eta \rightarrow 0} \int_{C^\eta(t)} [(U + k) n_1 - \sigma_{ij} u_{i,1} n_j] \end{aligned} \quad (5.1)$$

where we used the asymptotic relationship: $\dot{u}_i \sim -u_{i,j}$. This is the well known expresion for the energy release rate \mathcal{G} , and the value in terms of the stress intensity factors can be found in Freund (1990), page 234:

$$\omega_1 \equiv \mathcal{G} = \frac{1 - \nu^2}{Y} [A_I(v) K_I^2 + A_{II}(v) K_{II}^2] + \frac{1}{2\mu} A_{III}(v) K_{III}^2, \quad (5.2)$$

where:

$$A_I(v) := \frac{v^2 \alpha_d}{(1 - \nu) c_s^2 D}, \quad A_{II}(v) := \frac{v^2 \alpha_s}{(1 - \nu) c_s^2 D}, \quad A_{III}(v) := \frac{1}{\alpha_s}, \quad (5.3)$$

Y is Young modulus, ν is Poisson ratio and the functions α_d , α_s , D are defined as follows:

$$\alpha_d(v) := \sqrt{1 - v^2/c_d^2}, \quad \alpha_s(v) := \sqrt{1 - v^2/c_s^2}, \quad (5.4)$$

$$D(v) := 4\alpha_d(v)\alpha_s(v) - (1 + \alpha_s^2(v))^2. \quad (5.5)$$

Finally, c_d and c_s are the dilatational and shear wave speeds of the material, defined by:

$$c_d := \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_s := \sqrt{\frac{\mu}{\rho}}.$$

Notice that $D(v)$ has a real positive root that is called the Rayleigh wave speed c_R . The functions A_I , A_{II} and A_{III} are universal functions, *ie* they do not depend on the boundary data nor on the geometric configuration of the body. They do depend on the properties of the elastic material as well as on fracture velocity. Each of them satisfy: $A \rightarrow 1$ when $v \rightarrow 0^+$, $dA/dv \rightarrow 0$ when $v \rightarrow 0^+$ and $A = O\{(c_R - v)^{-1}\}$ when $v \rightarrow c_R$.

In order to compute ω_2 we consider a path $\mathcal{C}_{\delta_1\delta_2}$ like the one in Figure 4 below.

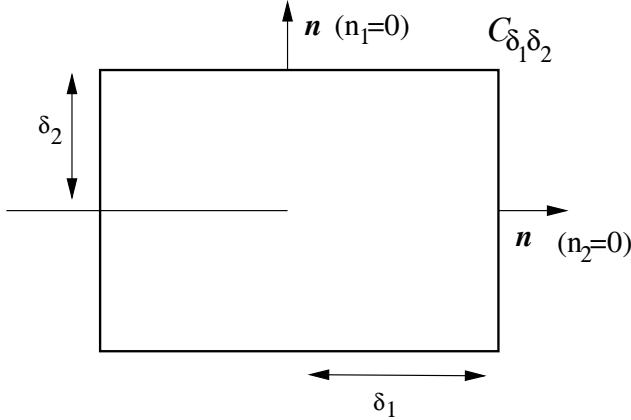


Fig. 4. The path $\mathcal{C}_{\delta_1\delta_2}$

We let first $\delta_1 \rightarrow 0$ and then $\delta_2 \rightarrow 0$ in (4.14). The computation is then simplified:

$$\begin{aligned} \omega_2 = \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} & \left(- \int_{-\delta_2, x_1=\delta_1}^{\delta_2} u_{i,2} (\sigma_{i1} + \rho v \dot{u}_i) dx_2 + \right. \\ & \left. \int_{-\delta_2, x_1=-\delta_1}^{\delta_2} u_{i,2} (\sigma_{i1} + \rho v \dot{u}_i) dx_2 \right). \quad (5.6) \end{aligned}$$

The singular fields obtained for modes *I* and *II* are uncoupled to the mode *III* field, see for example Freund (1990). The quantity corresponding to the in plane deformation is computed in detail in Oleaga (2001). We have then

the following identity:

$$\omega_2(t) = \frac{1-\nu^2}{Y} H(v) K_I K_{II} + \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} \left(- \int_{-\delta_2, x_1=\delta_1}^{\delta_2} u_{3,2} (\sigma_{31} + \rho v \dot{u}_3) dx_2 + \int_{-\delta_2, x_1=-\delta_1}^{\delta_2} u_{3,2} (\sigma_{31} + \rho v \dot{u}_3) \right) dx_2 , \quad (5.7)$$

where $H(v)$ is defined by:

$$H(v) := \frac{2}{1-\nu} \frac{\alpha_d(v)\alpha_s(v)(\alpha_s(v)-\alpha_d(v))(\alpha_s^4(v)-1)}{D^2(v)} .$$

Moreover, $H(0) = 1$, $H(v) \rightarrow +\infty$ for $v \rightarrow c_R$ and $H'(v) > 0$ for $0 \leq v < c_R$.

We write now:

$$u_{3,i} \sim \frac{K_{III}}{\mu \sqrt{2\pi r}} \Upsilon_{3i}^{III}(\theta, v) , \quad (5.8)$$

$$\sigma_{3i} \sim \frac{K_{III}}{\mu \sqrt{2\pi r}} \Sigma_{3i}^{III}(\theta, v) , \quad (5.9)$$

$$\dot{u}_3 \sim \frac{v K_{III}}{\sqrt{2\pi r}} V^{III}(\theta, v) . \quad (5.10)$$

Then we have that:

$$u_{3,2} (\sigma_{31} + \rho v \dot{u}_3) = \frac{K_{III}}{\mu 2\pi r} \Upsilon_{32}^{III} \left(\Sigma_{31}^{III} + \rho v^2 V^{III} \right) . \quad (5.11)$$

For this computation we need the quantities:

$$\Upsilon_{32}^{III}(\theta, v) = - \frac{\cos(\frac{\theta_s}{2})}{\gamma_s^{1/2}(v)} , \quad (5.12)$$

$$\Sigma_{31}^{III}(\theta, v) = - \frac{\sin(\frac{\theta_s}{2})}{\alpha_s(v) \gamma_s^{1/2}(v)} , \quad (5.13)$$

$$V^{III}(\theta, v) = \frac{\sin(\frac{\theta_s}{2})}{\alpha_s(v) \gamma_s^{1/2}(v)} , \quad (5.14)$$

where

$$\begin{aligned} \gamma_d(\theta, v) &= \sqrt{1 - (v \sin \theta / c_d)^2}, & \tan \theta_d &= \alpha_d \tan \theta, \\ \gamma_s(\theta, v) &= \sqrt{1 - (v \sin \theta / c_s)^2}, & \tan \theta_s &= \alpha_d \tan \theta . \end{aligned} \quad (5.15)$$

We observe that the quantity $u_{3,2} (\sigma_{31} + \rho v \dot{u}_3)$ is odd in θ , so we have that the integrals in (5.7) are equal to zero. We obtain then the same expression for ω_2 as in two dimensional propagation.

For the computation of ω_3 we must consider that the asymptotic fields involved are independent of x_3 . In the formula analogous to (5.6) we have that $u_{i,3} \sim 0$, and we find that the expression is identically zero.

Gathering the expressions for ω in terms of the stress intensity factors, we obtain:

$$\omega_1(K_I, K_{II}, K_{III}) = \frac{1 - \nu^2}{Y} [A_I(v)K_I^2 + A_{II}(v)K_{II}^2] + \frac{1}{2\mu} A_{III}(v)K_{III}^2, \quad (5.16)$$

$$\omega_2(K_I, K_{II}, K_{III}) = \frac{1 - \nu^2}{Y} H(v)K_I K_{II}, \quad (5.17)$$

$$\omega_3(K_I, K_{II}, K_{III}) = 0. \quad (5.18)$$

These are the same as the components of the dynamic Eshelby force obtained by Adda-Bedia *et al* (1999).

Taking into account that $\omega = (2\gamma, 0, 0)$ in this local coordinate system, the crack propagation law is given by:

$$\frac{1 - \nu^2}{Y} [A_I(v)K_I^2 + A_{II}(v)K_{II}^2] + \frac{1}{2\mu} A_{III}(v)K_{III}^2 = 2\gamma, \quad (5.19)$$

$$\frac{1 - \nu^2}{Y} H(v)K_I K_{II} = 0. \quad (5.20)$$

These expresions, together with the field equations (4.8), provide the full set of equations for the free boundary problem of dynamic crack propagation on a smooth path.

6 Concluding remarks

The analysis done in this work was based on the simple variational principle given in (3.7). This approach is well suited for the problem at hand, because the scalar fields involved are well defined and the formulation is wide enough to include dissipative phenomena and kinetic energy terms.

Notice that even though we considered a particular form of the surface energy (cf. 3.6), this is not a serious constraint of the formulation, and it is easy to extend the law to different effective surface energy terms, possibly even velocity dependent.

Moreover, the approach is not based on a Lagrangian formulation of the field equations, as in the case of the derivation of the Eshelby's force (cf. Eshelby, 1970), and this fact suggests that the same strategy can in principle be applied to handle non elastic materials.

Another advantage of this setting is that the law is obtained by freezing the actual state of the system and then trying different configurations for a fixed

time. It becomes clear then, that it does not make sense in rotating the front of the crack locally around an axis perpendicular to its plane, because that would imply that the crack grows on one side and heals on the other side. As long as healing is not allowed, the “rotation” of the front can be induced only through differences of velocity in the curve’s points. This implies, among other things, the absence of terms containing curvature effects. Thus, this point of view allows us to select carefully the appropriate kinematical constraints of the physical model, and we do not need to impose *a priori* simplifying assumptions concerning the local two dimensional character of the propagation law.

The effects of the presence of a mode III appear only through the scalar velocity v of the crack front (cf. 5.19-5.20). This implies that a planar crack under mode I and mode III loading will keep running in the corresponding plane, unless some instability is introduced. The presence of a mode II would produce a trend to depart from a planar configuration and the subsequent twisting when the first or the third modes are not uniform along the crack front.

A Computation of δE and $\int_{t_1}^{t_2} \delta T$

Applying Lemma 1 to the function

$$E_\varepsilon(t) = \int_{\mathcal{B}_\varepsilon} U(\mathbf{e}_\varepsilon)$$

we obtain the following expression, for each time t and $\eta > 0$:

$$\delta E(t) = \delta \left(\int_{\mathcal{R}_\varepsilon^\eta} U_\varepsilon \right) - \int_{\partial R_\eta} U \mathbf{n} \cdot \delta \boldsymbol{\varphi} + \int_{\mathcal{B}^\eta(t)} \delta U$$

We integrate by parts the last term, taking into account the following identity:

$$\delta U = \frac{\partial U}{\partial \mathbf{e}} \cdot \delta \mathbf{e} = \sigma_{ij} \delta u_{i,j}$$

In this way, we have:

$$\int_{\mathcal{B}^\eta(t)} \delta U = - \int_{\mathcal{B}^\eta(t)} \operatorname{div}(\boldsymbol{\sigma}) \cdot \delta \mathbf{u} + \int_{\partial(\mathcal{B}^\eta(t))} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta \mathbf{u}$$

Where we also used that $(\boldsymbol{\sigma} \cdot \delta \mathbf{u}) \cdot \mathbf{n} = (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta \mathbf{u}$ due to the symmetry of $\boldsymbol{\sigma}$. The last boundary set consists of

$$\partial(\mathcal{B}^\eta(t)) = (\mathcal{S}^\pm \setminus \mathcal{R}^\eta) \cup \partial \mathcal{B}_t \cup \partial \mathcal{B}_d \cup \partial \mathcal{R}^\eta$$

The set $\partial\mathcal{B}_t$ is the part of $\partial\mathcal{B}$ where tractions \mathbf{t} are prescribed. On \mathcal{B}_d we have prescribed displacements, so $\delta\mathbf{u} = \mathbf{0}$ there. Summing up, we find that:

$$\begin{aligned}\delta E(t) &= \delta \left(\int_{\mathcal{R}_\varepsilon^\eta} U_\varepsilon \right) - \int_{\partial R_\eta} [(\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta\mathbf{u} + U \mathbf{n} \cdot \delta\varphi] + \int_{\partial\mathcal{B}_t} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta\mathbf{u} \\ &\quad - \int_{\mathcal{S}^\pm \setminus \mathcal{R}^\eta} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \delta\mathbf{u} - \int_{\mathcal{B}^\eta(t)} \operatorname{div} \boldsymbol{\sigma}\end{aligned}$$

Notice that over $\partial\mathcal{R}^\eta$ and \mathcal{S}^\pm the normal vector is the one that points *into* the body.

We turn now to the integration of the kinetic energy term. Applying Lemma 1 to the function (for fixed time):

$$T_\varepsilon := \int_{\mathcal{B}_\varepsilon} k(\dot{\mathbf{u}}_\varepsilon)$$

we obtain, for each $\eta > 0$ and time t :

$$\delta T = \delta \left(\int_{\mathcal{R}_\varepsilon^\eta} k_\varepsilon \right) - \int_{\partial\mathcal{R}_\eta} k \mathbf{n} \cdot \delta\varphi + \int_{\mathcal{B}^\eta(t)} \rho \dot{\mathbf{u}} \cdot \delta\dot{\mathbf{u}} \quad (\text{A.1})$$

We want to integrate by parts the following term, with respect to the time:

$$\int_{t_1}^{t_2} \int_{\mathcal{B}_\eta(t)} \rho \dot{\mathbf{u}} \cdot \delta\dot{\mathbf{u}},$$

where the set $\mathcal{B}^\eta(t)$ is given by $\mathcal{B}^\eta(t) := \mathcal{B}(t) \setminus \mathcal{R}^\eta(t)$ and we made explicit the dependence of the sets \mathcal{R}^η on time because they have to isolate, for each value of time, the front $\mathcal{F}(t)$.

Now we calculate:

$$\frac{d}{dt} \int_{\mathcal{B}^\eta(t)} \rho \dot{\mathbf{u}} \cdot \delta\mathbf{u} = - \int_{\partial\mathcal{R}^\eta(t)} \rho (\dot{\mathbf{u}} \cdot \delta\mathbf{u})(\mathbf{v} \cdot \mathbf{n}) + \int_{\mathcal{B}^\eta(t)} \rho (\ddot{\mathbf{u}} \cdot \delta\mathbf{u} + \dot{\mathbf{u}} \cdot \delta\dot{\mathbf{u}}),$$

where \mathbf{v} is the velocity of the crack front. Integrating between t_1 and t_2 , and using the fact that $\delta\mathbf{u}(t_1) = \delta\mathbf{u}(t_2) \equiv \mathbf{0}$ we obtain:

$$\int_{t_1}^{t_2} \int_{\mathcal{B}^\eta(t)} \rho \dot{\mathbf{u}} \cdot \delta\dot{\mathbf{u}} = \int_{t_1}^{t_2} \left[\int_{\partial\mathcal{R}^\eta(t)} \rho (\dot{\mathbf{u}} \cdot \delta\mathbf{u})(\mathbf{v} \cdot \mathbf{n}) - \int_{\mathcal{B}^\eta(t)} \rho \ddot{\mathbf{u}} \cdot \delta\mathbf{u} \right]$$

Introducing in (A.1) the last expression and integrating between t_1 and t_2 we obtain (4.6).

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