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Information Geometry

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DUALLY FLAT MANIFOLDS AND GLOBAL INFORMATION GEOMETRY

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ABSTRACT. Dually flat manifolds constitute fundamental mathematical objects of information geometry. This note establishes some facts on the global properties and topology of dually flat manifolds which, in particular, provide answers to questions and problems in global information geometry posed by Amari and Amari-Nagaoka.

INTRODUCTION

Having emerged from the study of geometrical properties of manifolds of probability distributions, information geometry is nowadays applied in a broad variety of different fields and contexts which include, for instance, information theory, stochastic processes, dynamical systems and time series, statistical physics, quantum systems, and the mathematical theory of neural networks (compare [A1], [A2], [AN], and the further references given there).

It is well known that dual flatness constitutes a fundamental mathematical concept of information geometry. However, due to the fact that the global theory of dually flat manifolds is still far from being complete, its range of applications still suffers certain limitations since often only matters of mainly a local nature can be successfully pursued. Consequently, there is a strong need and desire for a further understanding of the global characteristics of dually flat manifolds.

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In particular, in [AN], the recent comprehensive treatise of the subject, in this regard the following basic problems and questions have been posed (see [AN], p. 180; cf. [A2]):

1. *Let (M, g) be a Riemannian manifold. Does there always exist a dually flat structure on M , i.e., a pair of affine connections ∇ and ∇^* such that (M, g, ∇, ∇^*) is dually flat?*
2. *If the answer to Question 1 is negative, find conditions and invariants which characterize the spaces for which this is possible.*
3. *Analyze the global structure of dually flat spaces.*

The present note sets out to investigate and clarify several aspects of the general global structure and topology of affinely flat and dually flat manifolds which are of importance to global information geometry. The facts established here provide in particular answers to the questions and problems from [AN] mentioned above. The remaining parts of the paper are organized as follows: Section 1 is devoted to the statements of the principal global and topological results, whose complete proofs are given in section 3. Their implications, especially in view of the above-mentioned questions, are discussed in section 2. We will freely use standard notions and facts from differential geometry and topology which can be found in, e.g., [Sa], [Spi], and [Spa]. Besides being positive dimensional, smooth, and without boundary, all manifolds in question will be assumed to be connected, since all of our statements carry over verbatim to connected components.

1. THE GLOBAL STRUCTURE AND TOPOLOGY OF DUALY FLAT MANIFOLDS

Following Amari (cf. [A1]), a dually flat manifold is defined as a smooth Riemannian manifold (M, g) equipped with a pair of flat torsion-free affine connections ∇ and ∇^* which are dual to each other in the sense that for all vector fields X, Y, Z on M ,

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z).$$

Our first result shows that there exist in fact general topological obstructions to the existence of dually flat structures:

1.1 Theorem *Let (M, g, ∇, ∇^*) be a dually flat manifold. If M is compact, then the fundamental group $\pi_1(M)$ of M must be infinite.*

1.2 Corollary *Compact Riemannian manifolds with trivial or finite fundamental group do never admit dually flat structures.*

We proceed by analyzing the global structure of dually flat manifolds under a completeness condition on one of the connections.

Notice that in many interesting situations and examples where dually flat manifolds (M, g, ∇, ∇^*) arise in information geometry, (at least) one of the two given connections on M is in fact *complete* in the sense that all of its geodesics are defined on the whole real line. This holds, for example, in the following important cases (cf. [AN]):

- (M, g) is an exponential family of probability distributions equipped with the Fisher metric, $\nabla = \nabla^{(e)}$ is the exponential connection which is Fisher dual to the canonical mixture connection $\nabla^* = \nabla^{(m)}$ on M . Here the exponential connection $\nabla^{(e)}$ is complete (whereas $\nabla^{(m)}$ is not complete).
- (M, g) is the set of positive density operators on a finite dimensional Hilbert space equipped with the Bogoliubov inner product, $\nabla = \nabla^{(e)}$ is the exponential connection and $\nabla^* = \nabla^{(m)}$ the mixture connection on M . Here again it holds that the exponential connection $\nabla^{(e)}$ is complete (though $\nabla^{(m)}$ is not).

For dually flat manifolds for which one of the given connections is complete in the above sense, now the following structure result holds:

1.3 Theorem *Let (M, g, ∇, ∇^*) be a dually flat manifold of dimension m . If one of the two connections on M , say, ∇ , is complete, then there exists a connection-preserving diffeomorphism $\Phi : (M, \nabla) \rightarrow (\mathbb{R}^m / \Gamma, \nabla^\Gamma)$, where $\Gamma \cong \pi_1(M)$ is a subgroup of the group $\mathbb{R}^m \ltimes GL(m, \mathbb{R})$ of affine motions of \mathbb{R}^m which acts freely and properly discontinuously on \mathbb{R}^m , and where ∇^Γ denotes the connection on \mathbb{R}^m / Γ which is induced by the canonical flat affine connection on \mathbb{R}^m .*

1.4 Corollary *Let (M, g, ∇, ∇^*) be a dually flat manifold of dimension m . If one of the two connections on M is complete, then the universal covering of M is diffeomorphic to Euclidean space \mathbb{R}^m and the fundamental group $\pi_1(M)$ of M is isomorphic to a subgroup of the group of affine motions of \mathbb{R}^m which acts freely and properly discontinuously on \mathbb{R}^m .*

1.5 Corollary *Let (M, g, ∇, ∇^*) be a dually flat manifold. If one of the two connections on M is complete, then the higher homotopy groups $\pi_k(M)$ of M must vanish for all $2 \leq k \in \mathbb{N}$.*

Notice that for a complete affine connection ∇ on a manifold M it is in general false that any two points in M could be joined by a geodesic of ∇ . This property, which is of special importance in applications, does in general not even hold if the manifold M is compact. It is therefore worth noting that dually flat manifolds actually do enjoy it:

1.6 Theorem *Let (M, g, ∇, ∇^*) be a dually flat manifold of dimension m . If one of the two connections on M , say, ∇ , is complete, then any two points in M can be joined by a ∇ -geodesic.*

2. FIRST CONSEQUENCES AND APPLICATIONS

Theorem 1.1 and Corollary 1.2 show that for compact manifolds there are general topological obstructions for the existence of dually flat structures which are even independent of the metric of the manifold. In particular, Corollary 1.2 yields a negative answer to Question 1 and a partial answer to Problem 2.

Theorem 1.3 gives a complete topological classification of dually flat manifolds under the completeness assumption on one of the connections and provides therefore in this case a complete answer to Problems 2 and 3. Moreover, Corollary 1.4 and 1.5 provide further and strong topological obstructions the existence of such dually flat structures by showing that any such manifold must be aspherical and possess a fundamental group of a very restricted type.

Theorem 1.6 illustrates in view of Problem 3 another special feature of the global structure of dually flat manifolds and justifies, in particular, projection constructions along geodesics which are a basic and widely-used tool of information geometry.

Applications of the above structure results to questions of quantum information geometry will be given in a sequel to this paper.

3. PROOFS

The results of section 1 will follow from more general statements about manifolds with complete flat connections and parallel torsion. We begin by introducing some notation as well as relevant facts whose details can be found in [Hi] and the references cited in the introduction.

Let G be a connected Lie group and $Aut(G)$ denote the group of continuous automorphisms of G . Then $G \rtimes Aut(G)$ is a group with group multiplication $(\lambda, k) \cdot (\mu, l) = (\lambda \cdot k(\mu), k \circ l)$, and an effective action of $G \rtimes Aut(G)$ on G is given as follows: If $(\lambda, k) \in G \rtimes Aut(G)$ and $g \in G$, then $(\lambda, k) \cdot g = \lambda \cdot k(g)$.

On the Lie group G there exists a canonical complete flat affine connection ∇ which is characterized by the fact that it is the one for which all left invariant vector fields on G are parallel. Moreover, $G \rtimes \text{Aut}(G)$ is isomorphic to the group of affine transformations of ∇ , i.e., isomorphic to the group of connection-preserving diffeomorphisms $G \rightarrow G$. This can be seen as follows: If $\phi : G \rightarrow G$ is a diffeomorphism, then ϕ is connection-preserving if and only if $\phi_*(\mathfrak{g}) = \mathfrak{g}$, where \mathfrak{g} denotes the Lie algebra of G . Thus the group of connection-preserving diffeomorphisms of G contains all left translations and all automorphisms of G . On the other hand, if ϕ is a connection-preserving diffeomorphism of G , set $\lambda := \phi(id)$, so that $\lambda^{-1} \circ \phi$ leaves \mathfrak{g} invariant. Thus $\lambda^{-1} \circ \phi$ is an automorphism of G and $\phi = \lambda \cdot (\lambda^{-1} \circ \phi)$. Therefore the group of connection-preserving diffeomorphisms of G is isomorphic to $G \rtimes \text{Aut}(G)$. It is also easily seen that the torsion tensor of the complete flat connection ∇ is invariant under parallel translation, and if $\Gamma < G \rtimes \text{Aut}(G)$ is a subgroup which acts properly discontinuously and freely on G , then the quotient space G/Γ is a manifold with fundamental group isomorphic to Γ and carries an induced flat and complete connection ∇^Γ whose torsion tensor is invariant under parallel transport as well.

Theorem 1.3 is now a consequence of the following more general structure statement. It is a slight extension of results of Hicks (cf. [Hi]) on affinely flat manifolds whose torsion tensor does not necessarily have to vanish but is covariantly constant.

Theorem 3.1 *Let (M, ∇) be a manifold with a flat connection whose torsion tensor is invariant under parallel translation. If ∇ is complete, then there exists a connection-preserving diffeomorphism $\Phi : (M, \nabla) \rightarrow (G/\Gamma, \nabla^\Gamma)$, where G is a connected and simply connected Lie group, $\Gamma \cong \pi_1(M)$ is a subgroup of the affine group $G \rtimes \text{Aut}(G)$ acting freely and properly discontinuously on G , and ∇^Γ denotes the connection induced from the canonical connection on G for which all left invariant vector fields are parallel.*

Proof of Theorem 3.1 Let M satisfy the assumptions of the theorem. Consider the universal covering \tilde{M} of M , and let Γ denote the group of deck transformations of this covering which acts freely and properly discontinuously on \tilde{M} . Since the projection $\tilde{M} \rightarrow M = \tilde{M}/\Gamma$ is a local diffeomorphism, \tilde{M} carries a complete flat connection whose torsion tensor is invariant under parallel translation and which induces the given connection on M . Since \tilde{M} is simply connected, by ([Hi], Thm. 5) \tilde{M} is connection-preserving diffeomorphic to a simply connected Lie group G equipped with its canonical connection.

The Lie group G is determined as follows (compare [Hi]): The flatness of the connection on \tilde{M} and the invariance of its torsion tensor under parallel transport imply that the vector fields on \tilde{M} , which are invariant under the parallel transport defined by this connection, form a finite-dimensional Lie algebra which in turn uniquely determines a simply connected and connected Lie group G .

Since each deck transformation is connection-preserving, Γ is isomorphic to a subgroup of $G \ltimes \text{Aut}(G)$ which acts freely and properly discontinuously on G , so that Theorem 3.1 is proved. \square

Proof of Theorem 1.3 If M satisfies the assumptions of Theorem 1.3, the torsion-freeness of ∇ implies (see the formula given in [Hi]) that the structural constants of the Lie algebra which appears in the proof of Theorem 3.1 are all trivial so that the Lie group in question here is simply flat Euclidean space \mathbb{R}^m . \square

Proof of Corollary 1.4 and Corollary 1.5 Corollary 1.4 follows directly from Theorem 1.3. Corollary 1.5 is a consequence of Corollary 1.4 and the fact that for a topological space whose universal covering is contractible all higher homotopy groups vanish. \square

Proof of Theorem 1.6 Suppose that M has dimension m so that by Theorem 1.3 there exists a connection-preserving diffeomorphism $\Phi : (M, \nabla) \rightarrow (\mathbb{R}^m/\Gamma, \nabla^\Gamma)$, where ∇^Γ denotes the connection on \mathbb{R}^m/Γ which is induced by the canonical flat affine connection on \mathbb{R}^m .

Given two points $p, q \in M = \mathbb{R}^m/\Gamma$, choose corresponding points $P, Q \in \mathbb{R}^m$ which project down to p and q , respectively. Now P and Q can be joined by a geodesic in \mathbb{R}^m whose projection is a $\nabla = \nabla^\Gamma$ -geodesic in M . \square

Proof of Theorem 1.1 Let M be a compact manifold with a flat and torsion-free affine connection ∇ . Let \tilde{M} be the universal covering of M . Using the fact that the holonomy group of the induced connection on \tilde{M} is trivial since \tilde{M} is simply connected, one easily sees that \tilde{M} admits a flat Riemannian metric \tilde{g} whose geodesics project onto the ∇ -geodesics of M . Suppose that the fundamental group of M is finite. Then \tilde{M} is compact as well, and therefore the metric \tilde{g} is complete. However, a complete and simply connected flat Riemannian manifold is isometric to some Euclidean space. This is a contradiction. \square

Proof of Corollary 1.2 This is obvious from Theorem 1.1. \square

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