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by

Matthias Kurzke and Heiko von der Mosel

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MATTHIAS KURZKE*
AND
HEIKO VON DER MOSEL**
* *Mathematisches Institut der Universität Leipzig*
** *Mathematisches Institut der Universität Bonn*

Abstract

Let $\mathcal{F}$ be a parametric variational double integral and $\Gamma \subset \mathbb{R}^n$ be a system of several distinct Jordan curves. We prove the existence of multiply connected, conformally parametrized minimizers of $\mathcal{F}$ spanned in Γ by solving the Douglas problem for parametric functionals on multiply connected schlicht domains. As a by-product we obtain a simple isoperimetric inequality for multiply connected $\mathcal{F}$-minimizers, and we discuss regularity results up to the boundary which follow from corresponding results for the Plateau problem.

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1 Introduction and main results

We consider parametric variational functionals $\mathcal{F}$ of the form

$$\mathcal{F}_B(X) := \int_B F(X, X_u \wedge X_v)\, dudv, \tag{1.1}$$

where F is of class $C^0(\mathbb{R}^n \times \mathbb{R}^N)$, $n \geq 3$, $N = n(n-1)/2$, satisfying the homogeneity condition

$$F(x, tz) = tF(x, z) \text{ for all } t > 0. \tag{H}$$

We assume that F is *positive definite*, that is, there are constants m_1, m_2 with $0 < m_1 \leq m_2$ such that

$$m_1|z| \leq F(x, z) \leq m_2|z| \text{ for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}^N. \tag{D}$$

Then $\mathcal{F}_B(X)$ is well-defined by (1.1) for any domain $B \subset \mathbb{R}^2$ and any mapping $X \in H^{1,2}(B, \mathbb{R}^n)$, where

$$a \wedge b := (a^j b^l - a^l b^j)_{j<l} \in \mathbb{R}^N$$

denotes the bivector generated by the two vectors $a = (a^1, \ldots, a^n)$, $b = (b^1, \ldots, b^n) \in \mathbb{R}^n$. Moreover, the homogeneity condition (H) implies that $\mathcal{F}$ is parameter invariant, i.e., invariant with respect to reparametrizations by means of C^1-diffeomorphisms with positive Jacobian, and even under bi-Lipschitz transformations with positive Jacobian a.e.

The parametric integrand F is said to be *semi-elliptic* if and only if

$$\text{(C)} \qquad F(x, \cdot) \text{ is convex on } \mathbb{R}^N \text{ for any } x \in \mathbb{R}^n.$$

A special Lagrangian of this kind is the *area integrand* $A(z) := |z|$, which leads to the *area functional*

$$\text{(1.2)} \qquad \mathcal{A}_B(X) := \int_B A(X_u \wedge X_v)\, dudv \text{ for } X \in H^{1,2}(B, \mathbb{R}^n).$$

We call a system $\Gamma := \langle \Gamma^1, \Gamma^2, \ldots, \Gamma^k \rangle$ consisting of k disjoint rectifiable closed Jordan curves $\Gamma^j \subset \mathbb{R}^n$, $j = 1, \ldots, k$, a *Jordan system.*

Our aim is to find a *multiply connected* surface spanning the contour Γ which minimizes the parametric functional (1.1). For the area functional this problem is known as the *Douglas problem* or *general problem of Plateau,* which was first attacked by Douglas [5], and then solved by Courant [2] for the genus zero case, and by Shiffman in [16] prescribing an arbitrary genus. Modern proofs and extensions can be found in the work of Tomi and Tromba [17] and Jost [13], and we refer to Chapter 11 in [4] for a more complete discussion of the Douglas problem. For a solution of this problem in the context of surfaces of constant, or variable prescribed mean curvature, we refer to Werner [18], and Luckhaus [15], respectively.

We are going to focus on the genus zero case, and will combine Courant's classical approach for minimal surfaces as described in [3, Ch.IV] with the method of conformal approximation of parametric functionals, which was first introduced in [8] and improved in [9] for the Plateau problem, and recently used in [10] to treat partially free boundary value problems for $\mathcal{F}$. This approximation procedure allows us to avoid the use of any conformal mapping theorem in our arguments.

Certain conditions on the solvability of the Douglas problem need to be imposed as may be illustrated by the particular boundary configuration consisting of two coaxial circles in parallel planes. Depending on the distance of the two planes either the (doubly connected) catenoid or the pair of spanning disks may have less area $\mathcal{A}$, see, e.g., the discussion in [3, pp. 141-142]. This motivates the so-called *Douglas condition* to exclude degenerate solutions consisting of two or more surfaces bounded by complementary parts of Γ. Another technical problem arises from the fact that k-fold connected domains of the same topological type may be of different conformal type for $k > 1$, so that we are led to consider minimizing $\mathcal{F}$ over pairs of admissible surfaces *and* domains.

To be more precise, we will restrict our attention to domains of the type

$$B_{r_1,\dots,r_k}(p_1,\dots,p_k) := B_{r_1}(p_1)\setminus \bigcup_{j=2}^{k} \overline{B_{r_j}(p_j)},$$

where $B_r(p)$ denotes the open ball in $\mathbb{R}^2$ of radius $r > 0$ centred at the point $p \in \mathbb{R}^2$. If $\overline{B_{r_j}(p_j)} \subset B_{r_1}(p_1)$ and $\overline{B_{r_j}(p_j)} \cap \overline{B_{r_i}(p_i)} = \emptyset$ for all $i, j \in \{2,\dots,k\}$, $i \neq j$, we call $B_{r_1,\dots,r_k}(p_1,\dots,p_k)$ a *k-circle domain.* We denote the class of all such domains as $\mathcal{K}_k$. A member of $\mathcal{K}_k$ with $p_1 = 0$, $r_1 = 1$ and $p_j = 0$ for some $j \in \{2,\dots,k\}$, is called a *unit k-circle domain.* The class of unit k-circle domains will be denoted by $\mathcal{K}_k^1$. Note that $\mathcal{K}_1^1 = \{B_1(0)\}$, and that one can perform elementary Möbius transformations to see that every k-circle domain is conformally equivalent to a unit k-circle domain.

We define the class of domains and surfaces admissible for the given boundary contour $\Gamma = \langle \Gamma^1,\dots,\Gamma^k\rangle$ as

$$\begin{aligned}\mathcal{C}(\Gamma) \quad := \quad &\{(B,X) : B \in \mathcal{K}_k, X \in H^{1,2}(B,\mathbb{R}^n) \cap C^0(\partial B,\mathbb{R}^n)\\ &X|_{\partial B} : \partial B \overset{\text{onto}}{\to} \Gamma \text{ is weakly monotonic}^1\},\end{aligned}$$

in contrast to the class of degenerate surfaces given by

$$\begin{aligned}\mathcal{C}^\dagger(\Gamma) \quad := \quad &\{(B,X) : B = \bigcup_{j=1}^{s} B_j,\ B_j \in \mathcal{K}_{k_j} \ \text{ disjoint},\ s > 1,\\ &\sum_{j=1}^{s} k_j = k,\ X \in H^{1,2}(B,\mathbb{R}^n) \cap C^0(\partial B,\mathbb{R}^n),\\ &X|_{\partial B} : \partial B \overset{\text{onto}}{\to} \Gamma \text{ is weakly monotonic}\}.\end{aligned}$$

[1] See [4, Vol.I, p. 231] for the notion of weakly monotonic mappings on the boundary.

Setting

$$
\begin{aligned}
d(\Gamma) &:= \inf_{(B,X)\in\mathcal{C}(\Gamma)} \mathcal{F}_B(X), \\
d^\dagger(\Gamma) &:= \inf_{(B,X)\in\mathcal{C}^\dagger(\Gamma)} \mathcal{F}_B(X),
\end{aligned}
\tag{1.3}
$$

we say that Γ satisfies the *Douglas condition* for $\mathcal{F}$ if and only if

$$
d(\Gamma) < d^\dagger(\Gamma). \tag{1.4}
$$

Notice that $\mathcal{C}^\dagger(\Gamma) \neq \emptyset$, since $\Gamma = \langle \Gamma^1, \dots, \Gamma^k\rangle$ consists of rectifiable components Γ^j, $j = 1, \dots, k$, each of which can be spanned by the harmonic extension of the respective boundary representation, compare [4, Vol. I, pp. 233–234]. Hence the right-hand side of (1.4) is finite.

For technical reasons[2] we impose a chord-arc condition on Γ, i.e., there are numbers $\delta > 0$, $M \geq 1$ such that, for any two points $P, Q \in \Gamma$ with $|P - Q| < \delta$, the shorter arc on Γ connecting P and Q, denoted by $\triangle(P,Q)$, has length $L(\triangle(P,Q)) \leq M|P-Q|$.

Then we will prove the existence of conformally parametrized minimizers of $\mathcal{F}$ in the class $\mathcal{C}(\Gamma)$:

Theorem 1.1. (Existence) *Suppose that F satisfies properties* (H),(D) *and* (C) *and the Douglas condition* (1.4) *holds. Assume also that Γ satisfies a chord-arc condition. Then there is a domain $B \in \mathcal{K}_k^1$ and a pair $(B,X) \in \mathcal{C}(\Gamma)$ with*

$$
\mathcal{F}_B(X) = \inf_{\mathcal{C}(\Gamma)} \mathcal{F}. \tag{1.5}
$$

Moreover, X is conformally parametrized, that is, X satisfies

$$
|X_u|^2 = |X_v|^2, \ \ \textit{and} \ \ X_u \cdot X_v = 0 \ \ \textit{a.e. on} \ \ B. \tag{1.6}
$$

In principle, condition (1.4) can be verified for concrete boundary configurations. If, for example, $\Gamma = \langle \Gamma^1, \Gamma^2\rangle$ consists of two coaxial circles of radius R in parallel planes which are a distance $\lambda > 0$ apart, then one can compare the total area $2\pi R^2$ of the two spanning disks with the area $2\pi R\lambda$ of the cylinder surface bounded by Γ. Using (D) it is easy to check that $\lambda < Rm_1/m_2$ is sufficient for (1.4) to hold.

As a by-product of the existence proof we obtain the following simple isoperimetric inequality for $\mathcal{F}$-minimizing surfaces.

[2]The chord-arc condition is not necessary for $k = 1$ in our arguments in agreement with the existence result proved in [8] and [9]. For $k > 1$ one could avoid this condition if one had an a priori L^∞-estimate for $\mathcal{F}$-minimizers.

THEOREM 1.2. (ISOPERIMETRIC INEQUALITY) *Let F satisfy* (D). *Then for any $(B,X) \in \mathcal{C}(\Gamma)$ satisfying* (1.5) *one has*

$$\mathcal{A}_B(X) \le \frac{m_2}{4\pi m_1} \sum_{l=1}^{k} L^2(\Gamma_l). \tag{1.7}$$

Since regularity considerations are of local nature we can apply the regularity results of [8] and [9] to obtain

THEOREM 1.3. (REGULARITY) *Let F satisfy* (H),(D) *and assume that* Γ *satisfies a chord-arc condition. Then every conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ is of class $C^{0,\alpha}(\overline{B},\mathbb{R}^n)\cap H^{1,q}_{\mathrm{loc}}(B,\mathbb{R}^n)$ for some $\alpha \in (0,1/2]$ and some $q > 2$.*

In addition, one can prove $C^{1,\alpha}$-smoothness up to the boundary of any conformally parametrized $\mathcal{F}$-minimizer in the presence of a *perfect dominance function*, if $\Gamma \in C^3$, and if $F \in C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$ is elliptic, we refer the reader to [9] and [11] for the details.

The present paper is organized as follows.

In Section 2 we present suitable versions of the Courant-Lebesgue Lemma for sequences of multiply connected domains with boundary contours approximating a given contour in the Fréchet-sense. In addition, we prove Courant's compactness result for domains in $\mathcal{K}^1_k$. Guided by Courant's ideas we work out different methods to manipulate multiply connected surfaces in Lemmas 3.1–3.3 of Section 3. In Section 4 we describe how to approximate the minimization problem for $\mathcal{F}$ on $\mathcal{C}(\Gamma)$ by introducing additional L^∞-bounds, and by adding a small multiple of the Dirichlet energy to the parametric functional $\mathcal{F}$. The technical result Theorem 4.7 contains the key ingredient, which, in combination with the Douglas condition (1.4), is used later to exclude degenerate minimal sequences, since it implies Courant's *condition of cohesion.* The existence proof in Section 5 proceeds in three steps: first we establish the existence of conformal minimizers for the perturbed functional under an additional L^∞-bound in Lemma 5.1, then we show how to remove the L^∞-bound in Lemmas 5.2 and 5.3, before we prove Theorem 1.1 approximating $\mathcal{F}$ by the perturbed functionals. The appendix contains two technical lemmas about Jordan systems.

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2 Definitions and preliminaries for the Douglas problem

A sequence of k-circle domains $\{B_m\} \subset \mathcal{K}_k$ with

$$B_m = B_{r_{1,m},\dots,r_{k,m}}(p_{1,m},\dots,p_{k,m})$$

is said to *converge* to

$$B = B_{r_1,\dots,r_k}(p_1,\dots,p_k),$$

i.e., $B_m \to B$ as $m \to \infty$, if and only if there are permutations σ_m of the set $\{2,\dots,k\}$ such that for $j = 2,\dots,k$,

$$\lim_{m\to\infty} r_{1,m} = r_1, \quad \lim_{m\to\infty} p_{1,m} = p_1, \quad \lim_{m\to\infty} r_{\sigma_m(j)} = r_j, \quad \lim_{m\to\infty} p_{\sigma_m(j)} = p_j.$$

A sequence $\{(B_m, X_m)\}$ of domains $B_m \in \mathcal{K}_k$ and mappings X_m of class $H^{1,2}(B_m,\mathbb{R}^n)$ is called *separating* if for all $\epsilon > 0$ there is $m_0 = m_0(\epsilon) \in \mathbb{N}$, such that for each $m > m_0$ there is a domain $D_m \in \mathcal{K}_k$, a bi-Lipschitz continuous mapping $T_m : \overline{B_m} \to \overline{D_m}$, a closed Jordan curve $c_m \subset \overline{D_m}$ not homotopic to zero in $\overline{D_m}$ and bounding a Lipschitz domain in $\mathbb{R}^2$, a representative Z_m of X_m, such that $Z_m \circ T_m^{-1}|_{c_m}$ is continuous and coincides with the boundary trace[3] $X_m \circ T_m^{-1}|_{c_m}$ on c_m and satisfies $\operatorname{diam}(Z_m \circ T_m^{-1}(c_m)) < \epsilon$. If a sequence $\{(B_m, X_m)\}$, with $B_m \in \mathcal{K}_k$, $X \in H^{1,2}(B_m,\mathbb{R}^n)$, possesses no separating subsequence, it is said to be *cohesive*.

For $X \in H^{1,2}(B,\mathbb{R}^n)$ the *Dirichlet energy* $\mathcal{D}_B(X)$ is given by

$$\mathcal{D}_B(X) := \frac{1}{2}\int_B |\nabla X|^2 \, dudv. \tag{2.1}$$

We will repeatedly use the following version of the Courant-Lebesgue Lemma for sequences of multiply connected surfaces with uniformly bounded Dirichlet energy, a proof of which can be modelled after the proof for $k = 1$ in [4, Ch. 4.4].

Lemma 2.1. (Courant-Lebesgue) *Let $B_m \in \mathcal{K}_k^1$ and $X_m \in H^{1,2}(B_m,\mathbb{R}^n)$, $m \in \mathbb{N}$, be a sequence of k-times connected surfaces, such that there is a constant M with*

$$\mathcal{D}_{B_m}(X_m) \le M < \infty \ \textit{ for all } \ m \in \mathbb{N}.$$

Let $x_0 \in \mathbb{R}^2$ and $\delta \in (0,1)$ be fixed. Then for all $m \in \mathbb{N}$ there exists a set $I_m \subset (\delta, \sqrt{\delta})$ with $\mathcal{L}^1(I_m) > 0$, and a representative Z_m of X_m such that for

[3]For the definition of the boundary trace operator see, e.g., [19, Ch. 4.4].

all $\rho \in I_m$ *the mapping* Z_m *is absolutely continuous on* $\partial B_\rho(x_0) \cap B_m$ *and coincides*[4] *with the boundary trace* $X_m|_{\partial B_\rho(x_0)\cap B_m}$*, and satisfies*

$$|Z_m(x_0 + \rho e^{i\theta_1}) - Z_m(x_0 + \rho e^{i\theta_2})| \le \sqrt{\frac{4M}{\log \frac{1}{\delta}}} |\theta_1 - \theta_2|^{1/2} \tag{2.2}$$

for all $\theta_1, \theta_2 \in \{\theta \in \mathbb{R} : x_0 + \rho e^{i\theta} \in B_m\}$*. In particular, for all* $\epsilon > 0$ *there is some number* $\delta_1 = \delta_1(\epsilon) > 0$*, such that for all* $\delta \in (0, \delta_1)$ *there exists a set* $I_m \subset (\delta, \sqrt{\delta})$ *with* $\mathcal{L}^1(I_m) > 0$ *such that*

$$\operatorname{diam}(Z_m(\partial B_\rho(x_0) \cap B_m)) < \epsilon \text{ for all } \rho \in I_m,\ m \in \mathbb{N}. \tag{2.3}$$

Denote $w_1 := (1, 0)$, $w_2 := (0, i)$, $w_3 := (-1, 0) \in B_1(0) \subset \mathbb{R}^2$, and fix three distinct points $P_i \in \Gamma$, $i = 1, 2, 3$, where $\Gamma \subset \mathbb{R}^n$ is a single (closed) Jordan curve. For a sequence of Jordan curves Γ_m converging to Γ in the Fréchet-sense as $m \to \infty$ we say that a sequence of mappings $X_m : B_1(0) \to \mathbb{R}^n$ with $\{(B_1(0), X_m)\} \subset \mathcal{C}(\Gamma_m)$ satisfies a *three-point condition* for $P_i \in \Gamma$, $i = 1, 2, 3$, if and only if

$$\lim_{m\to\infty} X_m(w_i) = P_i \text{ for } i = 1, 2, 3. \tag{2.4}$$

As a consequence of Lemma 2.1 important for controlling the boundary values of sequences of surfaces we have

Corollary 2.2. *Let* $\{(B, X_m)\} \subset \mathcal{C}(\Gamma_m)$ *with* $\Gamma_m \to \Gamma$ *in the Fréchet-sense. Suppose that there is a constant* M *such that* $\mathcal{D}_B(X_m) \le M$ *for all* $m \in \mathbb{N}$*. Then, if*

(i) $\{X_m\}$ *satisfies a three-point condition in the case* $k = 1$*, where* $B = B_1(0)$*, or if*

(ii) $\{(B, X_m)\}$ *is cohesive in the case* $k > 1$*,*

the sequence $X_m|_{\partial B}$ *is uniformly equicontinuous and contains a subsequence converging uniformly to some vector-valued function* $X^* \in C^0(\partial B, \mathbb{R}^n)$ *with* $X^* : \partial B \to \Gamma$ *weakly monotonic.*

Proof:

Once we have shown the uniform convergence of a subsequence of the X_m to X^* the continuity and the weak monotonicity follows easily. In the

[4] For this statement we refer to [6, Vol. II, p. 232].

case $k = 1$, $B = B_1(0)$, let $\epsilon_0 := \min_{j \neq l} |P_j - P_l|$, and take $m_0 \in \mathbb{N}$ so large that, by the three-point condition,

$$\min_{j \neq l} |X_m(w_j) - X_m(w_l)| \geq \epsilon_0/2 \text{ for all } \ m \geq m_0. \tag{2.5}$$

By Lemma A.1 of the appendix we find for every $\epsilon > 0$ a constant $\lambda = \lambda(\epsilon) > 0$ and $m_1 = m_1(\epsilon) \in \mathbb{N}$ such that for $Q_1^m, Q_2^m \in \Gamma_m$ with $0 < |Q_1^m - Q_2^m| < \lambda$ the shorter arc $\triangle(Q_1^m, Q_2^m) \subset \Gamma_m$ connecting Q_1^m and Q_2^m satisfies

$$\operatorname{diam}(\triangle(Q_1^m, Q_2^m)) < \epsilon \text{ for all } \ m \geq m_1. \tag{2.6}$$

Hence, if $0 < \epsilon < \epsilon_0/2$, then $\triangle(Q_1^m, Q_2^m)$ contains at most one of the points $X_m(w_j)$, $j = 1, 2, 3$, for all $m \geq m_2 := \max\{m_0, m_1\}$.

Let $\delta_0 \in (0, 1)$ be a fixed number with

$$2\sqrt{\delta_0} < \min_{j \neq l} |w_j - w_l|. \tag{2.7}$$

For arbitrary $\epsilon \in (0, \epsilon_0/2)$ choose $\delta = \delta(\epsilon) > 0$ such that

$$\sqrt{\frac{4\pi M}{\log(1/\delta)}} < \lambda(\epsilon) \text{ and } \ \delta < \delta_0. \tag{2.8}$$

For $z_0 \in \partial B$ and $m \geq m_2$ let $\rho_m \in I_m \subset (\delta, \sqrt{\delta})$ be such that for $\{z_m, z_m'\} = \partial B \cap \partial B_{\rho_m}(z_0)$

$$|X_m(z_m) - X_m(z_m')| \leq \sqrt{\frac{4\pi M}{\log(1/\delta)}}, \tag{2.9}$$

which is possible by Lemma 2.1. Then (2.8) implies that $|X_m(z_m) - X(z_m')| < \lambda(\epsilon)$, whence by (2.6) $\operatorname{diam}(\triangle(X_m(z_m), X_m(z_m'))) < \epsilon$ for all $m \geq m_2$. Since $\epsilon < \epsilon_0/2$ we know that $\triangle(X_m(z_m), X_m(z_m'))$ contains at most one of the points $X_m(w_j)$, $j = 1, 2, 3$, for all $m \geq m_2$. On the other hand by the three-point condition, the second part of (2.8), by (2.7), and the weak monotonicity of $X_m|_{\partial B}$, we know that $X_m(\partial B \cap \overline{B_{\rho_m}(z_0)})$ contains at most one of the points $X_m(w_j)$. Therefore $X_m(\partial B \cap \overline{B_{\rho_m}(z_0)}) = \triangle(X_m(z_m), X_m(z_m'))$ and

$$\operatorname{diam}(X_m(\partial B \cap \overline{B_\delta(z_0)})) < \operatorname{diam}(X_m(\partial B \cap \overline{B_{\rho_m}(z_0)})) < \epsilon.$$

Since $z_0 \in \partial B$ was arbitrary, we get the equicontinuity on ∂B :

$$|X_m(w) - X_m(w')| < \epsilon \text{ for all } \ w, w' \in \partial B \text{ with } \ |w - w'| < \delta.$$

From $X_m(\partial B) = \Gamma_m \to \Gamma$ as $m \to \infty$ we infer also that the mappings $X_m|_{\partial B}$ are uniformly bounded, and the Theorem of Arzela and Ascoli can be applied to obtain a subsequence uniformly convergent to some continuous vector-valued function $X^* : \partial B \to \Gamma$.

For $k > 1$ it suffices to show that the X_m are uniformly equicontinuous on any one of the components $\beta^1, \dots, \beta^k$ of ∂B. Relabeling and taking a subsequence we may assume that $X_m(\beta^j) = \Gamma_m^j$ for all m, and that $\Gamma_m^j \to \Gamma^j$. Take an arbitrary $x \in \beta^j$ and some $\epsilon > 0$, and let r_j be the radius of β^j. Then according to Lemma A.1 of the appendix there is some $\lambda = \lambda(\epsilon) > 0$, and $m_1 = m_1(\epsilon)$, such that for all $Q_1^m, Q_2^m \in \Gamma_m^j$ with $0 < |Q_1^m - Q_2^m| < \lambda$:

$$\operatorname{diam}(\triangle(Q_1^m, Q_2^m)) < \epsilon/2 \text{ for all } m \geq m_1. \tag{2.10}$$

Applying Lemma 2.1 we find $\delta_1 > 0$ such that for all

$$\delta < \min\left\{\delta_1, 1, 4r_j^2, \tfrac{1}{2}\min_{i\neq l} \operatorname{dist}^2(\beta^i, \beta^l)\right\}, \tag{2.11}$$

there is a set $I_m \subset (\delta, \sqrt{\delta})$ with $\mathcal{L}^1(I_m) > 0$ such that for all $\rho \in I_m$

$$\operatorname{diam}(X_m(\partial B_\rho(x) \cap B)) < \min\{\lambda, \epsilon/2\} \text{ for all } m \in \mathbb{N}, \tag{2.12}$$

and

$$\overline{B_\rho(x)} \cap \beta^\nu = \emptyset \text{ for all } \rho \in I_m, \nu \neq j,$$

where we tacitly assumed that X_m is already the representative which is continuous and coincides with the boundary trace on $\partial B_\rho(x) \cap B$. Define $\beta_\rho^j := \beta^j \cap \overline{B_\rho(x)}$, and $\{Q_{1,\rho}^m, Q_{2,\rho}^m\} := X_m(\partial B_\rho(x) \cap \beta^j)$ for $\rho \in I_m$. Then by (2.12) and (2.10) we have for $\triangle(Q_{1,\rho}^m, Q_{2,\rho}^m) \subset \Gamma_m^j$,

$$\operatorname{diam}(\triangle(Q_{1,\rho}^m, Q_{2,\rho}^m)) < \epsilon/2 \text{ for all } m \geq m_1, \rho \in I_m. \tag{2.13}$$

If we had $X_m(\beta^j \backslash \beta_\rho^j) = \triangle(Q_{1,\rho}^m, Q_{2,\rho}^m)$ for infinitely many $m \in \mathbb{N}$, then the curves

$$c_{\rho,m} := (\beta^j \backslash \beta_\rho^j) \cup (\partial B_\rho(x) \cap B), \text{ for } \rho \in I_m$$

would not be homotopic to zero in $\overline{B}$, and by (2.12),(2.13),

$$\begin{aligned} \operatorname{diam}(X_m(c_{\rho,m})) &\leq \operatorname{diam}(\triangle(Q_{1,\rho}^m, Q_{2,\rho}^m)) + \operatorname{diam}(X_m(\partial B_\rho(x) \cap B)) \\ &< \epsilon, \end{aligned}$$

contradicting our assumption that $\{(B, X_m)\}$ is cohesive[5]. Hence $X_m(\beta^j_\rho) = \triangle(Q^m_{1,\rho}, Q^m_{2,\rho})$ for all (but finitely many) $m \in \mathbb{N}$, which leads to

$$\mathrm{diam}\,(X_m(\beta^j \cap B_\delta(x))) \le \mathrm{diam}\,(X_m(\beta^j_\rho)) \underset{(2.13)}{<} \epsilon/2,$$

because $\rho \in I_m \subset (\delta, \sqrt{\delta})$. Since $x \in \beta_j$ was arbitrary, this concludes the proof. □

In order to control the domains for minimal sequences we will prove a slightly generalized version of Courant's compactness result [3, pp. 146–148] for normalized unit k-circle domains.

Lemma 2.3. *Let $\Gamma_m \to \Gamma$ in the Fréchet-sense, where $\Gamma_m, \Gamma \subset \mathbb{R}^n$ are Jordan systems. Suppose that $\{(B_m, X_m)\} \subset \mathcal{C}(\Gamma_m)$ with $B_m \in \mathcal{K}^1_k$, $X_m \in H^{1,2}(B_m, \mathbb{R}^n)$, is a cohesive sequence with $\mathcal{D}_{B_m}(X_m) \le M$ for all $m \in \mathbb{N}$, where $0 \le M < \infty$ is a constant independent of $m \in \mathbb{N}$. Then there is a subsequence $\{B_{m_\nu}\}$ and a domain $B \in \mathcal{K}^1_k$ such that $B_{m_\nu} \to B$ as $\nu \to \infty$.*

Proof: Since $\Gamma_m \to \Gamma$ in the Fréchet-sense, we may assume that $\Gamma_m = \langle \Gamma^1_m \dots, \Gamma^k_m \rangle$ with disjoint (closed) Jordan curves Γ^j_m converging to $\Gamma^j \in \Gamma = \langle \Gamma^1, \dots, \Gamma^k \rangle$ in the Fréchet-sense as $m \to \infty$ for $j = 1, \dots, k$. Let

$$B_m = B_{1,r_{2,m},r_{3,m},\dots,r_{k,m}}(0, 0, p_{3,m}, \dots, p_{k,m})$$

and by relabeling we can assume

$$X_m(\partial B_{r_{j,m}}(p_{j,m})) = \Gamma^j_m. \tag{2.14}$$

For m sufficiently large we have

$$\mathrm{dist}\,(\Gamma^j_m, \Gamma^l_m) \ge c/2 \ \text{ for } \ j \ne l, \tag{2.15}$$

where $c := \min_{i \ne j}\{\mathrm{dist}\,(\Gamma^i, \Gamma^j)\}$. Since $|p_{j,m}| < 1$ and $0 < r_{j,m} < 1$ for all $j = 2, \dots, k$ and all $m \in \mathbb{N}$, we can take a subsequence (again labeled by m) such that

(2.16)
$$p_{j,m} \to p_j \in \overline{B_1(0)}, \quad r_{j,m} \to r_j \in [0,1], \ \text{ for } \ j = 2, \dots, k \ \text{ as } \ m \to \infty.$$

We need to show that $\overline{B_{r_j}(p_j)} \subset B_1(0)$ and $\overline{B_{r_j}(p_j)} \cap \overline{B_{r_l}(p_l)} = \emptyset$ for all $j, l \in \{2, \dots k\}$, $j \ne l$. In other words we claim

[5] For the transformation T_m required in the definition of a separating sequence we can simply take the identity map for all m. The condition (2.11) ensures that $c_{\rho,m}$ bounds a Lipschitz domain in $\mathbb{R}^2$.

(i) $|p_j - p_l| > r_j + r_l$,

(ii) $|p_j| + r_j < 1$,

(iii) $0 < r_j$

for all $j, l \in \{2, \dots, k\}, j \neq l$. Since $B_m \in \mathcal{K}_k^1$ it is clear that $0 \leq r_j \leq 1$, $|p_j| + r_j \leq 1$ and $|p_j - p_l| \geq r_j + r_l$, because the corresponding strict inequalities hold for $p_{j,m}$, $r_{j,m}$ for all m. To show (i)–(iii) we just need to exclude equality, and for that we apply an indirect reasoning. Lemma 2.1 implies that for all $\epsilon \in (0, c)$ there exists $\delta_1 > 0$ such that for all $\delta < \min\{\delta_1, 1\}$, $x \in \mathbb{R}^2$ there is a set $I_m \subset (\delta, \sqrt{\delta})$ with $\mathcal{L}^1(I_m) > 0$ such that for all $r \in I_m$, X_m is continuous[6] on $(\partial B_r(x) \cap \overline{B}_m) \cup \partial B_m$, and

$$\operatorname{diam}\left(X_m(\partial B_r(x) \cap \overline{B}_m)\right) < \epsilon/2. \tag{2.17}$$

Case A. There is at least one $j \in \{2, \dots, k\}$ such that $r_j = 0$. Let $\mathcal{L}_j := \{l \in \{1, \dots, k\} : r_l > 0\}$.

A1. If $|p_l - p_j| > r_l$ for all $l \in \mathcal{L}_j$, $l \neq 1$, and if $|p_j| < 1$, then we have for

$$\delta := \tfrac{1}{2} \min \left\{ \delta_1, 1, \big(\min_{\substack{l \neq j \\ l \in \mathcal{L}_j}} \{ \big| |p_l - p_j| - r_l \big| \}\big)^2 \right\}, \tag{2.18}$$

for $\rho \in I_m \subset (\delta, \sqrt{\delta})$, and for m sufficiently large,

$$\overline{B_\rho(p_j)} \cap \overline{B_{r_{l,m}}(p_{l,m})} = \emptyset \ \text{ for all } \ l \in \mathcal{L}_j, l \neq 1, \ \text{ and} \tag{2.19}$$

$$\overline{B_\rho(p_j)} \subset B_1(0). \tag{2.20}$$

Moreover, for all $i \in \{1, \dots, k\}$ with $r_i = 0$ and $p_i \neq p_j$ we have

$$\overline{B_\rho(p_j)} \cap \overline{B_{r_{i,m}}(p_{i,m})} = \emptyset \ \text{ for all } \ m \text{ sufficiently large.} \tag{2.21}$$

For all $\nu \in \{1, \dots, k\}$ with $r_\nu = 0$ and $p_\nu = p_j$ we have

$$\overline{B_{r_{\nu,m}}(p_{\nu,m})} \subset B_\rho(p_j) \ \text{ for all } \ m \text{ sufficiently large.} \tag{2.22}$$

By (2.19)–(2.22), and (2.17) for $x := p_j$ we obtain that the curve $\partial B_\rho(p_j)$ is not homotopic to zero in B_m for m sufficiently large, but with

$$\operatorname{diam}\left(X_m(\partial B_\rho(p_j))\right) < \epsilon/2,$$

[6] As before we have identified X_m with the particular representative being continuous and coinciding with the boundary trace on $(\partial B_r(x) \cap \overline{B}_m) \cup \partial B_m$ for each $r \in I_m$.

which is not possible since $\{(B_m, X_m)\}$ is cohesive[7].

A2. If $|p_l - p_j| = r_l$ for exactly one index $l \in \mathcal{L}_j$, then take

$$\delta := \tfrac{1}{2}\min\left\{\delta_1, 1, e^{-4\pi M\lambda^{-2}(\epsilon)}, (\min_{\substack{s\neq l\\ s\in\mathcal{L}_j}}\{\big||p_s-p_j|-r_s\big|\})^2\right\}, \tag{2.23}$$

where $\lambda(\epsilon)$ is the constant appearing in Lemma A.1 in the appendix. Consequently, applying this lemma we obtain for all m sufficiently large

$$\operatorname{diam}\left(\triangle(Q_1^m, Q_2^m)\right) < \epsilon/2 \tag{2.24}$$

for all $Q_1^m, Q_2^m \in \Gamma_m^l$ with $|Q_1^m - Q_2^m| < \sqrt{\frac{4\pi M}{\log(1/\delta)}}$. Applying Lemma 2.1 we infer for all $\rho \in I_m \subset (\delta, \sqrt{\delta})$ and for m sufficiently large

$$\overline{B_\rho(p_j)} \cap \overline{B_{r_{s,m}}(p_{s,m})} = \emptyset \ \text{ for all } \ s \in \mathcal{L}_j \backslash \{l\}, \tag{2.25}$$

$$\overline{B_\rho(p_j)} \cap \overline{B_{r_{i,m}}(p_{i,m})} = \emptyset \tag{2.26}$$

for all $i \in \{1, \dots, k\}$ with $r_i = 0, p_i \neq p_j$. For all $\nu \in \{1, \dots, k\}$ with $r_\nu = 0$ and $p_\nu = p_j$ we have as before

$$\overline{B_{r_{\nu,m}}(p_{\nu,m})} \subset B_\rho(p_j) \ \text{ for all } \ m \,\text{sufficiently large.} \tag{2.27}$$

In addition,

$$|X_m(w_{1,m}) - X_m(w_{2,m})| < \sqrt{\frac{4\pi M}{\log(1/\delta)}} \tag{2.28}$$

for $\{w_{1,m}, w_{2,m}\} = \partial B_\rho(p_j) \cap \partial B_{r_{l,m}}(p_{l,m})$. Defining the arcs $c_{\rho,m} := \partial B_m \cap \overline{B_\rho(p_j)}$ we obtain curves $\gamma_{\rho,m} := c_{\rho,m} \cup \partial B_\rho(p_j)$ not homotopic to zero, with $\operatorname{diam}(X_m(\gamma_{\rho,m})) < \epsilon$ by (2.28), (2.24) and (2.17) for $x := p_j$, contradicting the fact that $\{(B_m, X_m)\}$ is a cohesive sequence.

[7]As before, to obtain a separating sequence here, and in the following cases, satisfying all requirements in the definition one can take $T_m := \mathrm{Id}_{\mathbb{R}^2}$ for all m. Choosing δ sufficiently small as done in (2.18) or in (2.23) we can ensure that the respective nonzero homotopic curves bound a Lipschitz domain in $\mathbb{R}^2$, since they consist of circular arcs that meet transversally.

A3. If $|p_l - p_j| = r_l$ for at least two indices $l = l_1, l_2 \in \mathcal{L}_j$, then we distinguish the two cases $l_1, l_2 \neq 1$ and, say $l_1 = 1$. In the former case we obtain

$$\begin{aligned} r_{l_1} + r_{l_2} &\leq |p_{l_1} - p_{l_2}| \leq |p_{l_1} - p_j| + |p_j - p_{l_2}| \\ &= r_{l_1} + r_{l_2}, \end{aligned}$$

i.e., $|p_{l_1} - p_{l_2}| = r_{l_1} + r_{l_2}$. In the second case, we calculate using the facts $r_j = 0$, $|p_j| = 1$, $|p_{l_2} - p_j| = r_{l_2}$, and $|p_{l_2}| + r_{l_2} \leq 1$,

$$\begin{aligned} |p_{l_2}| + r_{l_2} &\geq |p_j| - |p_{l_2} - p_j| + r_{l_2} \\ &= 1 - r_{l_2} + r_{l_2} = 1, \end{aligned}$$

that is, $|p_{l_2}| + r_{l_2} = 1$. In both cases we obtain

$$\partial B_{r_{l_1,m}}(p_{l_1,m}) \cap \partial B_\rho(p_j) \neq \emptyset, \quad \partial B_{r_{l_2,m}}(p_{l_2,m}) \cap \partial B_\rho(p_j) \neq \emptyset, \tag{2.29}$$

for all $\rho \in I_m \subset (\delta, \sqrt{\delta})$, δ sufficiently small, and m sufficiently large. Thus, we find by (2.17) for $x := p_j$

$$\begin{aligned} &\operatorname{dist}(\Gamma_m^{l_1}, \Gamma_m^{l_2}) \\ \leq\ &\operatorname{dist}(X_m(\partial B_{r_{l_1,m}}(p_{l_1,m})) \cap \partial B_\rho(p_j)), X_m(\partial B_{r_{l_2,m}}(p_{l_2,m})) \cap \partial B_\rho(p_j))) \\ <\ &\epsilon/2, \end{aligned}$$

contradicting (2.15), since $\epsilon < c$.

Case B. If $r_j > 0$ for all $j \in \{1, \dots, k\}$, then we distinguish two situations:

B1. If $|p_i - p_j| = r_i + r_j$ for some $i \neq j$, $i, j \in \{2, \dots, k\}$, then we argue as in the first case of item A3 above using (2.17) for $x := \partial B_{r_i}(p_i) \cap \partial B_{r_j}(p_j)$ and m sufficiently large.

B2. If $|p_i| + r_i = 1$ for some $i \in \{2, \dots, k\}$ then we argue as in the second case of item A3 above, again by (2.17) this time for $x := \partial B_1(0) \cap \partial B_{r_i}(p_i)$ and m sufficiently large.

$\square$

3 Manipulating multiply connected surfaces

For comparison arguments it is important to work on a fixed domain $B \in \mathcal{K}_k^1$, for which we will prove the following Lemma guided by the ideas of Courant.

LEMMA 3.1. *Let $B_m \in \mathcal{K}_k^1$ and $X_m \in H^{1,2}(B_m, \mathbb{R}^n)$, $m \in \mathbb{N}$, with $B_m \to B \in \mathcal{K}_k^1$ and $\mathcal{D}_{B_m}(X_m) \to d$ as $m \to \infty$. Then there is $m_0 \in \mathbb{N}$ such that for $m > m_0$ there exist bi-Lipschitz continuous mappings $\zeta_m : \overline{B} \to \overline{B}_m$ with the properties*

(i) $Z_m := X_m \circ \zeta_m \in H^{1,2}(B, \mathbb{R}^n)$ *for all* $m \in \mathbb{N}$,

(ii) $\mathcal{D}_B(Z_m) \to d$ *as* $m \to \infty$.

(iii) $\{(B, Z_m)\}$ *is cohesive iff* $\{B_m, X_m)\}$ *is cohesive.*

PROOF: Let

$$B_m := B_{1,r_{2,m},\dots,r_{k,m}}(0, p_{2,m}, \dots, p_{k,m}),$$
$$B := B_{1,r_2,\dots,r_k}(0, p_2, \dots, p_k),$$

with $p_{j,m} \to p_j$, and $r_{j,m} \to r_j$ for all $j = 2, \dots, k$. For

$$0 < s < \frac{1}{2} \min_{i \neq j} \operatorname{dist}\left(\partial B_{r_i}(p_i), \partial B_{r_j}(p_j)\right)$$

one has

$$B' := B_{1,r_2+s,\dots,r_k+s}(0, p_2, \dots, p_k) \subset B,$$

and for $i, j = 2, \dots, k$, $i \neq j$

$$\overline{B_{r_j+s}(p_j)} \subset B_1(0) \text{ and } \overline{B_{r_j+s}(p_j)} \cap \overline{B_{r_i+s}(p_i)} = \emptyset.$$

On the other hand, we find $m_0 \in \mathbb{N}$ such that

$$\overline{B_{r_{j,m}}(p_{j,m})} \subset B_{r_j+s}(p_j) \text{ for all } m > m_0, \; j = 2, \dots, k.$$

For $m > m_0$ define bi-Lipschitz continuous mappings $\zeta_m : \overline{B} \to \overline{B_m}$ by

(3.1)
$$\zeta_m(w) := \begin{cases} w & \text{for } w \in \overline{B'}, \\ \tau_{p_{j,m}-p_j, r_{j,m}, r_j, r_j+s}(w - p_j) + p_j & \text{for } w \in \overline{B} \cap B_{r_j+s}(p_j), \end{cases}$$

where $\tau_{p,r,a,R}$ for $r, a \in (0, R)$, $p \in \mathbb{R}^2$, $|p| + R < 1$, has the properties

(a) $\tau_{p,r,a,R} : \overline{B_R(0)}\backslash B_r(0) \to \overline{B_R(0)}\backslash B_a(p)$ is a diffeomorphism,

(b) For $p_n \to 0$ and $a_n \to 0$ one has $\tau_{p_n,a_n,r,R} \to \mathrm{Id}_{\mathbb{R}^2}$ in $C^1(\overline{B_R(0)}\backslash B_r(0))$,

(c) $\tau_{p,r,a,R}(w) = w$ for $w \in \partial B_R(0)$.

One can easily check that such a mapping can be constructed in polar coordinates $w = \rho e^{i\theta} \cong (\rho, \theta)$ as

$$(\rho,\theta) \mapsto \tau_{p,r,a,R}(\rho,\theta) := p + \big[a + \frac{\rho - r}{R - r}(b(\theta) - a)\big]e^{i\theta},$$

where $b(\theta)$ is the positive solution of the quadratic equation $R^2 = |p + b(\theta)e^{i\theta}|^2$, for details see [14, pp. 13,14].

Statement (i) follows from the Lipschitz continuity of ζ_m and [19, Thm. 2.2.2], assertion (ii) from an explicit calculation of $\mathcal{D}_B(X_m \circ \zeta_m)$, see [14, Lemmas 2.8 and 2.13] for the details.

(iii) Any curve $\beta \subset \overline{B}$ not homotopic to zero corresponds to a curve $\gamma_m := \zeta_m \circ \beta \subset \overline{B_m}$ not homotopic to zero and vice versa. If X_m is not cohesive, then it must possess a separating subsequence X_{m_i}. This implies that the subsequence $Z_{m_i} := X_{m_i} \circ \zeta_{m_i}$ is separating, so Z_m is not cohesive. We can argue in the same way starting with the assumption that Z_m is not cohesive. □

For technical reasons we introduce the sets

$$\mathcal{C}_K(\Gamma) := \{(B,X) \in \mathcal{C}(\Gamma) : |X| \le K \text{ a.e. on } B\} \text{ for } K \in (0,\infty].$$

Notice that $\mathcal{C}_\infty(\Gamma) = \mathcal{C}(\Gamma)$.

We are going to show that we can replace small parts of a surface by the constant map $Z \equiv 0$ without gaining too much energy. The Lemma is formulated for general functionals with quadratic growth.

Lemma 3.2. *Let $\mathcal{G}_\Omega(Y) := \int_\Omega G(Y, \nabla Y)\, dudv$ be a functional with a Lagrangian $G \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n})$ satisfying $G(x,p) \le (\mu/2)|p|^2$ for some constant $\mu > 0$, and let $(B,X) \in C_K(\Gamma)$ for some $K \in (0,\infty]$. Then for any $\delta > 0$ and $p \in B$ there exists $r_0 \in (0, \operatorname{dist}(p, \partial B))$ depending on δ and on K such that for any $r \in (0, r_0)$ there is $(B, Z_r) \in C_K(\Gamma)$ with*

(i) $\mathcal{G}_B(Z_r) < \mathcal{G}_B(X) + \delta$,

(ii) $Z_r|_{B_r(p)} \equiv 0$.

Proof: For $p \in B$ there is $R \in (0, \operatorname{dist}(p, \partial B))$ such that

$$\mathcal{D}_{B_\rho(p)}(X) < \frac{\delta}{6\mu} =: \delta_0 \text{ for all } \rho \in (0, R). \tag{3.2}$$

Take $\rho \in (0, R)$ such that a representative of X (again denoted by X) is absolutely continuous and satisfies $|X| \le K$ on $\partial B_\rho(p)$.

Then choose $H \in H^{1,2}(B_\rho(p), \mathbb{R}^n)$ such that

(i) $H - X \in \overset{\circ}{H}{}^{1,2}(B_\rho(p), \mathbb{R}^n)$,

(ii) $\Delta H = 0$ in $B_\rho(p)$.

By the maximum principle for harmonic functions we obtain

$$\sup_{\overline{B_\rho(p)}} |H| = \sup_{\partial B_\rho(p)} |H| = \sup_{\partial B_\rho(p)} |X| =: M, \tag{3.3}$$

(hence $M \le K$,) and, using the Dirichlet principle,

$$\mathcal{D}_{B_\rho(p)}(H) \le \mathcal{D}_{B_\rho(p)}(X) < \delta_0. \tag{3.4}$$

For some constant $\eta < \rho$ to be chosen later set

$$h_{\eta^2}(s) := \begin{cases} 1 & \text{if } s > \eta, \\ 1 + \frac{\log \eta - \log s}{\log \eta} & \text{if } \eta^2 \le s \le \eta, \\ 0 & \text{if } s < \eta^2. \end{cases}$$

Then the mapping

$$Z_{\eta^2}(w) := \begin{cases} X(w) & \text{if } |w - p| \ge \rho, \\ h_{\eta^2}(|w - p|) H(w) & \text{if } |w - p| < \rho \end{cases}$$

is of class $H^{1,2}(B, \mathbb{R}^n)$ and $(B, Z_{\eta^2}) \in \mathcal{C}(\Gamma)$, since we can estimate for the Dirichlet energy on $B_\rho(p)$ (writing h_u for $(h_{\eta^2}(|w-p|))_u$) using Hölder's inequality and the Dirichlet principle

$$\begin{aligned}
\mathcal{D}_{B_\rho(p)}(Z_{\eta^2}) &= \frac{1}{2}\int_{B_\rho(p)} |h_u H + hH_u|^2 + |h_v H + hH_v|^2 \, dudv \\
&= \frac{1}{2}\int_{B_\rho(p)} \big[(h_u^2 + h_v^2)|H|^2 + |h|^2(|H_u|^2 + |H_v|^2)\big] \, dudv \\
&\qquad + \int_{B_\rho(p)} 2hH\cdot(h_u H_u + h_v H_v)\big] \, dudv \\
&\le M^2\mathcal{D}_{B_\rho(p)}(h_{\eta^2}) + \mathcal{D}_{B_\rho(p)}(H) \\
&\quad +2M\left[\int_{B_\eta(p)\setminus B_{\eta^2}(p)} |\nabla h|^2 \, dudv\right]^{1/2}\left[\int_{B_\eta(p)\setminus B_{\eta^2}(p)} |\nabla H|^2 \, dudv\right]^{1/2} \\
&\le M^2\mathcal{D}_{B_\rho(p)}(h_{\eta^2}) + \mathcal{D}_{B_\rho(p)}(H) + 2M\sqrt{2\mathcal{D}_{B_\rho(p)}(h_{\eta^2})}\sqrt{2\mathcal{D}_{B_\rho(p)}(H)}\,.
\end{aligned}$$

An easy calculation shows that $2\mathcal{D}_{B_\rho(p)}(h_{\eta^2}) = -2\pi/(\log\eta) =: \delta_1(\eta)$, hence by (3.4),

$$\mathcal{D}_{B_\rho(p)}(Z_{\eta^2}) \le M^2\delta_1/2 + \delta_0 + 2M\sqrt{\delta_1}\sqrt{2\delta_0}.$$

Now we choose η_0 so small that $M^2\delta_1(\eta) < 2\delta_0$ for all $\eta \in (0,\eta_0)$, whence

$$\mathcal{D}_{B_\rho(p)}(Z_{\eta^2}) \le 6\delta_0 \text{ for all } \eta \in (0,\eta_0). \tag{3.5}$$

Setting $r_0 := \eta_0^2$ we obtain for any $r = \eta^2 < \eta_0^2 = r_0$

$$\begin{aligned}
\mathcal{G}_B(Z_r) &= \mathcal{G}_{B\setminus B_\rho(p)}(X) + \mathcal{G}_{B_\rho(p)}(Z_r) \\
&\le \mathcal{G}_B(X) + \mu\mathcal{D}_{B_\rho(p)}(Z_r) \\
&\le \mathcal{G}_B(X) + 6\mu\delta_0 = \mathcal{G}_B(X) + \delta,
\end{aligned}$$

and $Z_r|_{B_r(p)} = Z_{\eta^2}|_{B_{\eta^2}(p)} \equiv 0$ by construction. Moreover, $Z_r(w) = X(w)$ for $|w-p| \ge \rho$, which implies $|Z_r(w)| \le K$ for a.e. $w \in B\setminus B_\rho(p)$. On the other hand, $|Z_r(w)| \le |H(w)|$ for $|w-p| < \rho$, hence by (3.3) $|Z_r(w)| \le \sup_{\partial B_\rho(p)}|X| \le K$. Thus $(B, Z_r) \in \mathcal{C}_K(\Gamma)$. □

Essential for our existence proof in the following section is a “pinching”-Lemma (modeled after Courant's corresponding result for the Dirichlet energy) that allows us to contract a part of a surface contained in a small ball to the center of that ball without gaining too much energy.

LEMMA 3.3. *Let $(B,X) \in \mathcal{C}_K(\Gamma)$ and $P \in \overline{B_K(0)} \subset \mathbb{R}^n$, $\delta > 0$, $\epsilon_0 > 0$ be given. Then there is $\eta_0 \in (0, \min\{K,1\})$ depending on the modulus of continuity of $F|_{\overline{B_K(0)}\times S^{n-1}}$, on Γ, δ, and on the value $\mathcal{F}_B^{\epsilon_0}(X)$, such that for every $\eta \in (0,\eta_0)$ there is a mapping $\Phi_\eta : \mathbb{R}^n \to \mathbb{R}^n$, such that*

(i) $\Phi_\eta(\Gamma)$ *is a Jordan curve with* $\|\Phi_\eta(\Gamma)-\Gamma\|_{C^0}<\delta$,

(ii) $\Phi_\eta\circ X\in\mathcal{C}_K(\Phi_\eta(\Gamma))$,

(iii) $\Phi_\eta=\mathrm{Id}_{\mathbb{R}^n}$ *on* $\mathbb{R}^n\backslash B_\eta(P)$,

(iv) $\Phi_\eta\equiv P$ *on* $\overline{B_{\eta^2}(P)}$,

(v) $\mathcal{F}^\epsilon_B(\Phi_\eta\circ X)<\mathcal{F}^\epsilon_B(X)+\delta$ *for all* $\epsilon\in[0,\epsilon_0]$.

Proof: Define

(3.6)
$$\omega_F(\sigma):=\max\left\{|F(x,z)-F(y,z)|:x,y\in\overline{B_K(0)},z\in S^{n-1},|x-y|\le\sigma\right\},$$

and choose $\eta_0\in(0,K)$ so small that

$$\max\left\{\frac{1}{m_1}[\omega_F(\eta_0)+\frac{m_2}{|\log\eta_0|}],\frac{3}{|\log\eta_0|}\right\}<\frac{\delta}{\mathcal{F}^{\epsilon_0}_B(X)}. \tag{3.7}$$

Then, for $\eta\in(0,\eta_0)$ and $x\in\mathbb{R}^n$, set $\Phi_\eta(x):=P+h_\eta(|x-P|)(x-P)$ with

$$h_\eta(r):=\begin{cases}1 & \text{if } r>\eta,\\ 1+\frac{\log(\eta/r)}{\log\eta} & \text{if } \eta^2\le r\le\eta,\\ 0 & \text{if } r<\eta^2.\end{cases} \tag{3.8}$$

Parts (iii) and (iv) follow immediately from this definition, and Part (i) is proven in the appendix, Lemma A.2, as well as the weak monotonicity of $\Phi_\eta\circ X$ on ∂B. By [19, Thm. 2.1.11] $\Phi_\eta\circ X\in H^{1,2}(B,\mathbb{R}^n)$ since Φ_η is Lipschitz continuous, and $\Phi_\eta(X(w))\in[P,X(w)]$ for any $w\in\overline{B}$. This implies $|\Phi_\eta(X(w))|\le K$ for a.e. $w\in\overline{B}$, since both P and $X(w)$ are contained in $\overline{B_K(0)}\subset\mathbb{R}^n$. This together with (i) proves Part (ii). It remains to show Part (v). First notice that

$$|\Phi_\eta\circ X-X|\le\eta \text{ on } B. \tag{3.9}$$

Then we calculate on $B^*:=\{w\in B:\eta^2<|X(w)-P|<\eta\}$

$$\begin{aligned}[h_\eta(|X-P|)]_u &= -\frac{1}{|X-P|\log\eta}\left[\frac{X-P}{|X-P|}\cdot X_u\right],\\ [h_\eta(|X-P|)]_v &= -\frac{1}{|X-P|\log\eta}\left[\frac{X-P}{|X-P|}\cdot X_v\right],\end{aligned}$$

which implies

$$
(3.10)\qquad \begin{aligned}
(\Phi_\eta \circ X)_u &= h_\eta(|X-P|)X_u - \frac{1}{\log \eta}\left[\frac{X-P}{|X-P|}\cdot X_u\right]\frac{X-P}{|X-P|},\\
(\Phi_\eta \circ X)_v &= h_\eta(|X-P|)X_v - \frac{1}{\log \eta}\left[\frac{X-P}{|X-P|}\cdot X_v\right]\frac{X-P}{|X-P|}
\end{aligned}
$$

on B^*. Hence

$$
\begin{aligned}
(\Phi_\eta \circ X)_u^2 &= h_\eta^2(|X-P|)X_u^2 + \frac{1}{(\log\eta)^2}[1-2h_\eta \log\eta]\left[\frac{X-P}{|X-P|}\cdot X_u\right]^2\\
&\leq h_\eta^2(|X-P|)X_u^2 + \frac{3}{|\log\eta|}X_u^2 \quad \text{on} \quad B^*, \quad \text{and}\\
(\Phi_\eta \circ X)_v^2 &\leq h_\eta^2(|X-P|)X_v^2 + \frac{3}{|\log\eta|}X_v^2 \quad \text{on} \quad B^*.
\end{aligned}
$$

Thus, we can estimate the Dirichlet energy as

$$
(3.11)\qquad \begin{aligned}
\mathcal{D}_{B^*}(\Phi_\eta \circ X) &\leq \mathcal{D}_{B^*}(X) + \frac{3}{|\log\eta|}\mathcal{D}_{B^*}(X)\\
&\underset{(3.7)}{\leq} (1+\frac{\delta}{\mathcal{F}_B^{\epsilon_0}(X)})\mathcal{D}_{B^*}(X).
\end{aligned}
$$

On the other hand by (3.9),(3.10),

$$
\begin{aligned}
(\Phi_\eta \circ X)_u \wedge (\Phi_\eta \circ X)_v &= h_\eta^2(|X-P|)X_u \wedge X_v\\
&\quad -\frac{h_\eta(|X-P|)}{\log\eta}\left[\frac{X-P}{|X-P|}\cdot X_v\right]X_u \wedge \frac{X-P}{|X-P|}\\
&\quad -\frac{h_\eta(|X-P|)}{\log\eta}\left[\frac{X-P}{|X-P|}\cdot X_u\right]\frac{X-P}{|X-P|}\wedge X_v.
\end{aligned}
$$

For $A, B, E \in \mathbb{R}^n$ with $|E| = 1$, we have the estimate

$$
|(B\cdot E)A\wedge E + (A\cdot E)E\wedge B| \leq |A\wedge B|,
$$

hence

$$
(3.12)\qquad (\Phi_\eta \circ X)_u \wedge (\Phi_\eta \circ X)_v = h_\eta^2(|X-P|)X_u \wedge X_v + R
$$

with $|R| \leq |\log\eta|^{-1}|X_u \wedge X_v|$.

Using the assumptions (C),(D), and (H) on the parametric integrand F we estimate on B^*

$$
\begin{aligned}
&F(\Phi_\eta \circ X, (\Phi_\eta \circ X)_u \wedge (\Phi_\eta \circ X)_v) \\
\underset{(3.12)}{=} \quad & F(\Phi_\eta \circ X, \frac{1}{2}(2h_\eta^2(|X-P|)X_u \wedge X_v) + \frac{1}{2}2R) \\
\underset{(C)}{\leq} \quad & \frac{1}{2}F(\Phi_\eta \circ X, 2h_\eta^2(|X-P|)X_u \wedge X_v) + \frac{1}{2}F(\Phi_\eta \circ X, 2R) \\
\underset{(H)}{=} \quad & h_\eta^2(|X-P|)F(\Phi_\eta \circ X, X_u \wedge X_v) + F(\Phi_\eta \circ X, R) \\
\underset{(H),(D)}{\leq} \quad & h_\eta^2(|X-P|)F(\Phi_\eta \circ X, \frac{X_u \wedge X_v}{|X_u \wedge X_v|})|X_u \wedge X_v| + \frac{m_2}{|\log \eta|}|X_u \wedge X_v| \\
\underset{(D)}{\leq} \quad & h_\eta^2(|X-P|)\left[F(\Phi_\eta \circ X, \frac{X_u \wedge X_v}{|X_u \wedge X_v|}) - F(X, \frac{X_u \wedge X_v}{|X_u \wedge X_v|})\right]|X_u \wedge X_v| \\
& \qquad + \left[h_\eta^2(|X-P|) + \frac{m_2}{m_1|\log \eta|}\right] F(X, X_u \wedge X_v) \\
\underset{h_\eta^2 \leq 1,(D)}{\leq} \quad & \left(1 + m_1^{-1}\left[\omega_F(\eta) + \frac{m_2}{|\log \eta|}\right]\right) F(X, X_u \wedge X_v) \\
\underset{(3.7)}{\leq} \quad & F(X, X_u \wedge X_v)(1 + \frac{\delta}{\mathcal{F}_B^{\epsilon_0}(X)}).
\end{aligned}
$$

Integrating over B^* we arrive at

$$
\mathcal{F}_{B^*}(\Phi_\eta \circ X) \leq \mathcal{F}_{B^*}(X) + \delta \frac{\mathcal{F}_{B^*}(X)}{\mathcal{F}_B^{\epsilon_0}(X)},
$$

and together with (3.7) and (3.11) this leads to

$$
\mathcal{F}_{B^*}^{\epsilon}(\Phi_\eta \circ X) \underset{(3.7),(3.11)}{\leq} \mathcal{F}_{B^*}^{\epsilon}(X) + \delta \frac{\mathcal{F}_{B^*}^{\epsilon}(X)}{\mathcal{F}_B^{\epsilon_0}(X)} \leq \mathcal{F}_{B^*}^{\epsilon}(X) + \delta
$$

for all $\epsilon \in [0, \epsilon_0]$, since $\mathcal{F}_{B^*}^{\epsilon}(X) \leq \mathcal{F}_B^{\epsilon}(X) \leq \mathcal{F}_B^{\epsilon_0}(X)$ for all $\epsilon \in [0, \epsilon_0]$. □

4 Conformal approximation of parametric functionals

We consider the conformally invariant functionals

$$
\mathcal{F}_B^{\epsilon}(X) := \mathcal{F}_B(X) + \epsilon \mathcal{D}_B(X) \quad \text{for} \quad \epsilon \geq 0 \quad \text{and} \quad X \in H^{1,2}(B, \mathbb{R}^n).
$$

Recall from the previous section the definition of the class $\mathcal{C}_K(\Gamma)$ and define analogously

$$\mathcal{C}_K^\dagger(\Gamma) := \{(B,X) \in \mathcal{C}^\dagger(\Gamma) : |X| \le K \ \text{ a.e. } \ \text{on} \ \ B\}.$$

Set for $\epsilon \ge 0$ and $K \in (0,\infty]$

$$d_K(\Gamma,\epsilon) := \inf_{(B,X)\in\mathcal{C}_K(\Gamma)} \mathcal{F}_B^\epsilon(X) \quad \text{and} \quad d_K^\dagger(\Gamma,\epsilon) := \inf_{(B,X)\in\mathcal{C}_K^\dagger(\Gamma)} \mathcal{F}_B^\epsilon(X).$$

Notice that $d_\infty(\Gamma,0) = d(\Gamma)$ and $d_\infty^\dagger(\Gamma,0) = d^\dagger(\Gamma)$ by definition.

Considering separating sequences we define the infimum of $\mathcal{F}^\epsilon$ over all such sequences as

$$d_K^*(\Gamma,\epsilon) := \inf_{\substack{(B_m,X_m)\in C_K(\Gamma)\\ (B_m,X_m)\,\text{sep.}}} \liminf_{m\to\infty} \mathcal{F}_{B_m}^\epsilon(X_m) \tag{4.1}$$

for $\epsilon \in [0,\infty)$, $K \in (0,\infty]$.

Lemma 4.1. *The mappings $\epsilon \mapsto d_K(\Gamma,\epsilon)$, $\epsilon \mapsto d_K^\dagger(\Gamma,\epsilon)$, and $\epsilon \mapsto d_K^*(\Gamma,\epsilon)$ are nondecreasing, and*

$$d_K(\Gamma,0) = \lim_{\epsilon\to+0} d_K(\Gamma,\epsilon) \quad \text{and} \quad d_K^\dagger(\Gamma,0) = \lim_{\epsilon\to+0} d_K^\dagger(\Gamma,\epsilon) \tag{4.2}$$

for all $K \in (0,\infty]$.

Proof: The functions $\epsilon \mapsto d_K(\Gamma,\epsilon)$, $\epsilon \mapsto d_K^\dagger(\Gamma,\epsilon)$, and $\epsilon \mapsto d_K^*(\Gamma,\epsilon)$, are nondecreasing on $(0,\infty)$, since for $0 < \epsilon_1 \le \epsilon_2$ one has $\mathcal{F}_B^{\epsilon_1}(X) \le \mathcal{F}_B^{\epsilon_2}(X)$ for any $(B,X) \in C_K(\Gamma)$, $0 < K \le \infty$. Hence the respective limits on the right-hand side of (4.2) exist unless $C_K(\Gamma) = \emptyset$, or $C_K^\dagger(\Gamma) = \emptyset$. In that case both sides of the respective equation in (4.2) become infinite and we are done. Since $\mathcal{F}_B(X) \le \mathcal{F}_B^\epsilon(X)$ for all $\epsilon \ge 0$, $(X,B) \in \mathcal{C}_K(\Gamma)$, or $(X,B) \in \mathcal{C}_K^\dagger(\Gamma)$, respectively, we have

$$d_K(\Gamma,0) \le \lim_{\epsilon\to+0} d_K(\Gamma,\epsilon) \ \text{ and } \ d_K^\dagger(\Gamma,0) \le \lim_{\epsilon\to+0} d_K^\dagger(\Gamma,\epsilon)$$

If we had $d_K(\Gamma,0) < \lim_{\epsilon\to+0} d_K(\Gamma,\epsilon)$ then we could find $(B,X) \in C_K(\Gamma)$ such that $\mathcal{F}_B(X) < \lim_{\epsilon\to+0} d_K(\Gamma,\epsilon)$. Consequently we could choose $\epsilon_1 \ll 1$ such that also

$$\mathcal{F}_B^{\epsilon_1}(X) < \lim_{\epsilon\to+0} d_K(\Gamma,\epsilon) \le d(\Gamma,\epsilon_1) \le \mathcal{F}_B^{\epsilon_1}(X),$$

which is absurd, i.e., $d_K(\Gamma,0) = \lim_{\epsilon\to+0} d_K(\Gamma,\epsilon)$. The same argument works for $d_K^\dagger$. $\square$

Lemma 4.2. *Let $K \in (0,\infty]$ be a given number. Then the following holds: If $d_K(\Gamma,0) < d_K^\dagger(\Gamma,0)$, then for ϵ_1 sufficiently small we have $d_K(\Gamma,\epsilon) < d_K^\dagger(\Gamma,\epsilon)$ for all $0<\epsilon\le\epsilon_1$.*

Proof: Assuming the contrary we find a sequence $\epsilon_i \to 0$ with $d_K(\Gamma,\epsilon_i) \ge d_K^\dagger(\Gamma,\epsilon_i)$ for all $i \in \mathbb{N}$. Going to the limit $i\to\infty$ we can use (4.2) to obtain

$$d_K(\Gamma,0) \ge \lim_{\epsilon_i\to 0} d_K^\dagger(\Gamma,\epsilon_i) = d_K^\dagger(\Gamma,0),$$

contradicting our assumption. □

Lemma 4.3. *The mappings $K \mapsto d_K(\Gamma,\epsilon)$, $K\mapsto d_K^\dagger(\Gamma,\epsilon)$ and $K\mapsto d_K^*(\Gamma,\epsilon)$ are monotonically decreasing, have respective limits as $K\to\infty$, and we have*

$$\lim_{K\to\infty} d_K(\Gamma,\epsilon) = d(\Gamma,\epsilon) \quad \textit{and} \quad \lim_{K\to\infty} d_K^\dagger(\Gamma,\epsilon) = d^\dagger(\Gamma,\epsilon) \tag{4.3}$$

for all $\epsilon \ge 0$.

Proof: It suffices to show the claim for $d_K(\Gamma,\epsilon)$, the result for $d_K^\dagger(\Gamma,\epsilon)$ follows by induction on the components. The function $K\mapsto d_K(\Gamma,\epsilon)$ is decreasing and bounded from below for each $\epsilon\ge 0$. Moreover,

$$0 \le d(\Gamma) \le d(\Gamma,\epsilon) \le d_K(\Gamma,\epsilon) \text{ for all } K\in(0,\infty], \epsilon\ge 0.$$

Hence the limits

$$d_\infty(\epsilon) := \lim_{K\to\infty} d_K(\Gamma,\epsilon)$$

exist and

$$d(\Gamma,\epsilon) \le d_\infty(\epsilon) \le d_K(\Gamma,\epsilon) \text{ for all } \epsilon\ge 0,\ K\in(0,\infty]. \tag{4.4}$$

To prove the claim we assume to the contrary that $d_\infty(\epsilon) > d(\Gamma,\epsilon)$ for some $\epsilon \ge 0$. Then we find $(B,X)\in\mathcal{C}(\Gamma)$ such that

$$d_\infty(\epsilon) > \mathcal{F}_B^\epsilon(X) \ge d(\Gamma,\epsilon). \tag{4.5}$$

For $\eta \in (0, d_\infty(\epsilon) - \mathcal{F}_B^\epsilon(X))$ we find $\delta>0$ such that

$$(m_2+\epsilon)\mathcal{D}_E(X) \le \eta \text{ for all } E\subset B \text{ with } \mathcal{L}^2(E)<\delta, \tag{4.6}$$

since $X \in H^{1,2}(B, \mathbb{R}^n)$. Moreover, we have for $K \geq 0$

(4.7)
$$K^2\mathcal{L}^2(\{w \in B : |X^i(w)|) \geq K\}) \leq \int_{\{|X^i(w)| \geq K\}} |X(w)|^2 \, dudv \leq \|X\|^2_{L^2}.$$

Thus we may choose $K_0 > \|\Gamma\|_\infty$ so large that the set

$$\kappa := \bigcup_{i=1}^n \{w \in B : |X^i(w)| \geq K_0\}$$

has measure

$$\mathcal{L}^2(\kappa) < \delta. \tag{4.8}$$

Then the truncated surface Y with the components

$$Y^i(w) := \begin{cases} \operatorname{sign}(X^i(w))K_0 & \text{if } \ |X^i(w)| \geq K_0, \\ X^i(w) & \text{if } \ |X^i(w)| < K_0, \end{cases}$$

is in the class $\mathcal{C}_{\sqrt{n}K_0}(\Gamma)$, and satisfies

$$\begin{aligned} \mathcal{F}^\epsilon_B(Y) &= \mathcal{F}^\epsilon_{B\setminus\kappa}(Y) + \mathcal{F}^\epsilon_\kappa(Y) \\ &\leq \mathcal{F}^\epsilon_{B\setminus\kappa}(X) + (m_2 + \epsilon)\mathcal{D}_\kappa(Y) \\ &\leq \mathcal{F}^\epsilon_B(X) + (m_2 + \epsilon)\mathcal{D}_\kappa(X). \end{aligned}$$

Thus we infer from (4.6) and (4.8) that

$$d_{\sqrt{n}K_0}(\Gamma, \epsilon) \leq \mathcal{F}^\epsilon_B(Y) \underset{(4.6)}{\leq} \mathcal{F}^\epsilon_B(X) + \eta < d_\infty(\epsilon)$$

by our choice of η, contradicting (4.4) for $K := \sqrt{n}K_0$. $\square$

A similar argument as in the proof of Lemma 4.2 leads to

Corollary 4.4. *Let $\epsilon \geq 0$ be arbitary. If $d(\Gamma, \epsilon) < d^\dagger(\Gamma, \epsilon)$, then there exists $K_1 \in (0, \infty)$ such that $d_K(\Gamma, \epsilon) < d^\dagger_K(\Gamma, \epsilon)$ for all $K \in [K_1, \infty]$.*

Lemma 4.5. *Let $\epsilon \geq 0$ and $K \in (0, \infty]$ be arbitrary, and suppose that $\{(B_m, X_m)\} \subset \mathcal{C}_K(\Gamma_m)$ is cohesive with*

$$\mathcal{D}_{B_m}(X_m) \leq C \ \text{ for all } \ m \in \mathbb{N}, \tag{4.9}$$

where C does not depend on m. Then there exist $B \in \mathcal{K}^1_k$, $X \in H^{1,2}(B, \mathbb{R}^n)$ such that

$$d_K(\Gamma, \epsilon) \leq \mathcal{F}^\epsilon_B(X) \leq \liminf_{m\to\infty} \mathcal{F}^\epsilon_{B_m}(X_m). \tag{4.10}$$

Recall that for $k=1$ one has $\mathcal{K}^1_k=\{B_1(0)\}$, hence all sequences $\{(B_m,X_m)\}=\{(B_1(0),X_m)\}\subset\mathcal{C}_K(\Gamma_m)$ are cohesive for $k=1$.

Proof: For $k=1$ we may assume that the X_m satisfy a three-point condition for preassigned distinct points $P_i\in\Gamma$, $i=1,2,3$. Thus by (4.9) and a suitable version of Poincaré's inequality and Corollary 2.2 we find a surface $X\in H^{1,2}(B_1(0),\mathbb{R}^n)\cap C^0(\partial B_1(0),\mathbb{R}^n)$ with a weakly monotonic boundary mapping $X|_{\partial B_1(0)}:\partial B_1(0)\to\Gamma$ and with

$$X_{m_\nu}\rightharpoonup X\ \text{ in }\ H^{1,2}(B_1(0),\mathbb{R}^n),$$
$$X_{m_\nu}|_{\partial B_1(0)}\longrightarrow X|_{\partial B_1(0)}\ \text{ in }\ C^0(\partial B_1(0),\mathbb{R}^n)$$

for a subsequence $m_\nu\to\infty$. Hence $|X(w)|\le K$ for a.e. $w\in B_1(0)$, and thus, $(B_1(0),X)\in\mathcal{C}_K(\Gamma)$. This and the lower-semicontinuity result of Acerbi and Fusco in [1] applied to the functional $\mathcal{F}^\epsilon_B$ imply (4.10).

For $k>1$ we may assume that $B_m\in\mathcal{K}^1_k$ for all $m\in\mathbb{N}$ by elementary Möbius transformations.

We deduce from (4.9) that

$$\mathcal{D}_{B_{m_\nu}}(X_{m_\nu})\longrightarrow D\in[0,C]\ \text{ as }\ \nu\to\infty \tag{4.11}$$

for a subsequence, and taking a further subsequence we may assume that

$$\lim_{\nu\to\infty}\mathcal{F}^\epsilon_{B_{m_\nu}}(X_{m_\nu})=\liminf_{m\to\infty}\mathcal{F}^\epsilon_{B_m}(X_m). \tag{4.12}$$

Thus by Lemma 2.3 we can find a unit k-circle domain $B\in\mathcal{K}^1_k$ and a subsequence $\{B_l\}\subset\{B_{m_\nu}\}$, such that $B_l\to B$ as $l\to\infty$. Using the reparametrized surfaces $Z_l\in H^{1,2}(B,\mathbb{R}^n)$ for l sufficiently large, constructed according to Lemma 3.1, we find

$$\mathcal{D}_B(Z_l)\to D\ \text{ as }\ l\to\infty. \tag{4.13}$$

A suitable version of Poincaré's inequality yields a uniform bound on the $H^{1,2}$-norm of the Z_l independent of l so that we find a surface $Z\in H^{1,2}(B,\mathbb{R}^n)$ such that (for a relabeled subsequence)

$$Z_l\rightharpoonup Z\ \text{ in }\ H^{1,2}(B,\mathbb{R}^n)\ \text{ as }\ l\to\infty.$$

In addition, we can apply Corollary 2.2 to get

$$Z_l|_{\partial B}\to Z|_{\partial B}\ \text{ in }\ C^0(\partial B,\mathbb{R}^n)\ \text{ as }\ l\to\infty,$$

where $Z|_{\partial B}$ is weakly monotonic, since the reparametrized sequence $\{(B,Z_l)\}$ remains cohesive according to Part (iii) of Lemma 3.1. In addition, we have

$|Z| \leq K$ a.e. on B, since $|Z_l| \leq K$ for all $l \in \mathbb{N}$. Consequently, $Z \in \mathcal{C}_K(\Gamma)$, and we may apply the lower-semicontinuity result of [1] valid for the functional $\mathcal{F}_B^\epsilon$ to conclude

$$d_K(\Gamma, \epsilon) \leq \mathcal{F}_B^\epsilon(Z) \leq \liminf_{l\to\infty} \mathcal{F}_B^\epsilon(Z_l). \tag{4.14}$$

The parametric invariance of $\mathcal{F}$ implies

$$\mathcal{F}_B^\epsilon(Z_l) = \mathcal{F}_{B_{m_l}}^\epsilon(X_{m_l}) + \epsilon(\mathcal{D}_B(Z_l) - \mathcal{D}_{B_{m_l}}(X_{m_l})) \text{ for all } l \in \mathbb{N},$$

hence by (4.11)–(4.13),

$$\liminf_{l\to\infty} \mathcal{F}_B^\epsilon(Z_l) = \liminf_{l\to\infty} \mathcal{F}_{B_{m_l}}^\epsilon(X_{m_l}) = \liminf_{m\to\infty} \mathcal{F}_{B_m}^\epsilon(X_m),$$

which proves (4.10) according to (4.14). □

Lemma 4.6. *For all $\epsilon \geq 0$, $0 < K \leq \infty$ one has*

$$d_K(\Gamma, \epsilon) \leq d_K^*(\Gamma, \epsilon) \leq d_K^\dagger(\Gamma, \epsilon).$$

Proof: The first inequality is obvious, since $d_K(\Gamma, \epsilon) \leq \mathcal{F}_{B_m}^\epsilon(X_m)$ for any member (B_m, X_m) of a separating sequence in $\mathcal{C}_K(\Gamma)$.

For the second inequality it suffices to consider the case $k > 1$, since for $k = 1$ we have $\mathcal{C}_K^\dagger(\Gamma) = \emptyset$, whence $d_K^\dagger(\Gamma, \epsilon) = \infty$ for any $K \in (0, \infty]$ and $\epsilon \geq 0$.

Moreover, it is enough to prove the second inequality for $s = 2$ components (see the definition of $\mathcal{C}_K^\dagger(\Gamma)$), the rest follows from a simple inductive argument over the number of components. Hence we may assume that $\Gamma = \langle \Gamma^1, \Gamma^2 \rangle$, and for $\delta > 0$ we choose $(B_1, X_1) \in \mathcal{C}_K(\Gamma^1)$ and $(B_2, X_2) \in \mathcal{C}_K(\Gamma^2)$, with $B_1 \in \mathcal{K}_{k_1}$ and $B_2 \in \mathcal{K}_{k_2}$, where $k_1 + k_2 = k$, and such that

$$\mathcal{F}_{B_1}^\epsilon(X_1) \leq d_K(\Gamma^1, \epsilon) + \delta \text{ and } \mathcal{F}_{B_1}^\epsilon(X_1) \leq d_K(\Gamma^1, \epsilon) + \delta. \tag{4.15}$$

Applying Lemma 3.2 for $\mathcal{G} := \mathcal{F}^\epsilon$ we obtain new surfaces $(B_1, Z_1) \in \mathcal{C}_K(\Gamma^1)$, $(B_2, Z_2) \in \mathcal{C}_K(\Gamma^2)$, such that there are balls $B_{r_1}(p_1) \subset\subset B_1$, $B_{r_2}(p_2) \subset\subset B_2$, with $Z_1|_{B_{r_1}(p_1)} \equiv 0$, $Z_2|_{B_{r_2}(p_2)} \equiv 0$, and such that

$$\mathcal{F}_{B_1}^\epsilon(Z_1) \leq \mathcal{F}_{B_1}^\epsilon(X_1) + \delta \text{ and } \mathcal{F}_{B_2}^\epsilon(Z_2) \leq \mathcal{F}_{B_2}^\epsilon(X_2) + \delta. \tag{4.16}$$

Let B_2^* be the reflection of B_2 at $\partial B_{r_2/2}(p_2)$, and Z_2^* be the reflected surface defined on B_2^*, i.e., $Z_2^*(w) := Z_2(\tau^{-1}(w))$ for $w \in \tau(B_2)$, where τ denotes the reflection mapping at $\partial B_{r_2/2}(p_2)$. By dilation and translation which are conformal mappings we can assume that $\partial B_{r_2/2}(p_2) = \partial B_{r_1/2}(p_1)$, and we introduce the k-circle domain $B \in \mathcal{K}_k$ by

$$B := (B_1 \backslash B_{r_1/2}(p_1)) \cup B_2^*.$$

Now define a mapping $Y \in H^{1,2}(B, \mathbb{R}^n)$ by

$$Y := \begin{cases} Z_1 & \text{on } \; B_1 \backslash B_{r_1}(p_1), \\ 0 & \text{on } \; B_{r_1}(p_1) \backslash B_2^*, \\ Z_2^* & \text{on } \; B_2^*, \end{cases}$$

to obtain $(B, Y) \in \mathcal{C}_K(\Gamma)$. The conformal invariance of the functional $\mathcal{F}^\epsilon$ and (4.16) imply

$$\begin{aligned} \mathcal{F}_B^\epsilon(Y) &= \mathcal{F}^\epsilon_{B_1 \backslash B_{r_1}(p_1)}(Y) + \mathcal{F}^\epsilon_{B_{r_1}(p_1) \backslash B_2^*}(Y) + \mathcal{F}^\epsilon_{B_2^*}(Y) \\ &= \mathcal{F}^\epsilon_{B_1 \backslash B_{r_1}(p_1)}(Z_1) + \mathcal{F}^\epsilon_{B_{r_1}(p_1) \backslash B_2^*}(0) + \mathcal{F}^\epsilon_{B_2^*}(Z_2^*) \\ &\underset{(4.16)}{\leq} \mathcal{F}^\epsilon_{B_1}(X_1) + \delta + 0 + \mathcal{F}^\epsilon_{B_2}(X_2) + \delta \\ &\leq d(\Gamma^1, \epsilon) + d(\Gamma^2, \epsilon) + 4\delta. \end{aligned}$$

Since Y maps the reflected image $\tau(\partial B_\rho(p_1))$ onto zero for all $\rho \in (r_1/2, r_1)$, the constant sequence $(B_m, Y_m) \equiv (B, Y) \in \mathcal{C}_K(\Gamma)$ is separating[8] by definition. Therefore

$$d_K^*(\Gamma, \epsilon) \leq \mathcal{F}_B^\epsilon(Y) \leq d_K(\Gamma^1, \epsilon) + d_K(\Gamma^2, \epsilon) + 4\delta.$$

Since the partition $\Gamma = \langle \Gamma^1, \Gamma^2 \rangle$ was arbitrary we obtain $d_K^*(\Gamma, \epsilon) \leq d_K^\dagger(\Gamma, \epsilon)$ for all $\epsilon \geq 0$. $\square$

The central result for the existence proof in the next section is

Theorem 4.7. *Let Γ_m, Γ be Jordan systems with $\Gamma_m \to \Gamma$ in the Fréchet-sense as $m \to \infty$. Then*

(i)
$$d_K(\Gamma, \epsilon) \leq \liminf_{\substack{m \to \infty \\ (X_m, B_m) \in C_K(\Gamma_m)}} \mathcal{F}^\epsilon_{B_m}(X_m).$$

[8] As before we can take the required bi-Lipschitz transformation T_m in the definition of separating sequences to be the identity in the plane.

(ii)
$$d_K^\dagger(\Gamma, \epsilon) = d_K^*(\Gamma, \epsilon),$$

for all $\epsilon > 0$, $0 < K < \infty$, *and for all sequences* $\{(B_m, X_m)\} \subset \mathcal{C}_K(\Gamma_m)$.

PROOF: (i) We prove the claim by induction over k. In the case $k = 1$ we may take a sequence $\{(B_1(0), X_m)\} \subset \mathcal{C}_K(\Gamma_m)$ with

$$\lim_{m\to\infty} \mathcal{F}_B^\epsilon(X_m) = \liminf_{m\to\infty} \mathcal{F}_B^\epsilon(X_m) < \infty,$$

(if the right-hand side is infinite there is nothing to prove). This implies for $\epsilon > 0$

$$\mathcal{D}_B(X_m) \le c(\epsilon) \text{ for all } m \in \mathbb{N}, \tag{4.17}$$

where $c(\epsilon)$ is a constant depending on ϵ. Now (i) follows from Lemma 4.5.

For the induction step $k-1 \mapsto k$ we consider a sequence $\{(B_m, X_m)\} \subset \mathcal{C}_K(\Gamma_m)$ with
$$\lim_{m\to\infty} \mathcal{F}_{B_m}^\epsilon(X_m) = \liminf_{m\to\infty} \mathcal{F}_{B_m}^\epsilon(X_m)$$
and we may apply elementary Möbius transformations to get $B_m \in \mathcal{K}_k^1$ for all $m \in \mathbb{N}$.

We distinguish two cases:

Case A. Assume $\{(B_m, X_m)\} \subset \mathcal{C}_K(\Gamma_m)$ is separating. Then (since $|X_m| \le K$ a.e. on B_m) there are points $Q_m \in \overline{B_K(0)} \subset \mathbb{R}^n$, domains $D_m \in \mathcal{K}_k$ containing closed Jordan curves $c_m \subset \overline{D_m}$ not homotopic to zero in $\overline{D_m}$ and bounding a Lipschitz domain in $\mathbb{R}^2$, bi-Lipschitz mappings $T_m : \overline{B_m} \to \overline{D_m}$, and constants $\eta_m \to 0$ as $m \to \infty$, such that a representative Z_m of $X_m \circ T_m^{-1}$ restricted to c_m is absolutely continuous and coincides with the boundary trace on c_m, and satisfies

$$\|Z_m - Q_m\|_{\infty, c_m} < \eta_m^2.$$

Taking subsequences we may assume that $Q_m \to Q \in \overline{B_K(0)}$ and adjusting the constants $\eta_m > 0$ we get

$$\|Z_m - Q\|_{\infty, c_m} < \eta_m^2. \tag{4.18}$$

Let $\delta_i \to 0$ be a given sequence of numbers. Then, according to Lemma 3.3 we find for fixed $\epsilon > 0$ and for each $i \in \mathbb{N}$ some number $\eta_{0,i} > 0$, and (taking a subsequence of the η_m above and relabeling) $\eta_i < \eta_{0,i}$, mappings $\Phi_{\eta_i} : \mathbb{R}^n \to \mathbb{R}^n$, such that

(i) $\Phi_{\eta_i}(\Gamma_i)$ is a Jordan curve with

$$\|\Phi_{\eta_i}(\Gamma_i) - \Gamma_i\|_{C^0} < \delta_i,$$

(ii) $\Phi_{\eta_i} \circ Z_i \in \mathcal{C}_K(\Phi_{\eta_i}(\Gamma_i))$,

(iii) $\Phi_{\eta_i} = \mathrm{Id}_{\mathbb{R}^n}$ on $\mathbb{R}^n \backslash B_{\eta_i}(Q)$,

(iv) $\Phi_{\eta_i} \equiv Q$ on $\overline{B_{\eta_i^2}(Q)}$,

(v) $\mathcal{F}^\epsilon_{D_i}(\Phi_{\eta_i} \circ Z_i) < \mathcal{F}^\epsilon_{B_i}(X_i) + \delta_i$.

In particular, we have $\Phi_{\eta_i} \circ Z_i|_{c_i} \equiv Q$ for all $i \in \mathbb{N}$. We define new surfaces[9] $Y_i^1 \in H^{1,2}(E_i^1, \mathbb{R}^n)$, $Y_i^2 \in H^{1,2}(E_i^2, \mathbb{R}^n)$, by

$$Y_i^1 := \begin{cases} \Phi_{\eta_i} \circ Z_i \circ \tau^{-1} & \text{on } \ \tau(D_i^1) \subset E_i^1, \\ Q & \text{elsewhere on } E_i^1, \end{cases}$$

and

$$Y_i^2 := \begin{cases} \Phi_{\eta_i} \circ Z_i & \text{on } \ \tau(D_i^2) \subset E_i^2, \\ Q & \text{elsewhere on } E_i^2, \end{cases}$$

where

$$D_i^1 := \mathrm{int}\,(c_i) \cap D_i, \quad D_i^2 := D_i \backslash D_i^1,$$
$$E_i^1 := \mathrm{int}\,(\tau(c_i)) \cup \tau(D_i^1), \quad E_i^2 := \mathrm{int}\,(c_i) \cup D_i^2,$$

where $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the reflection at an arbitrary circle contained in the set $\overline{\mathrm{int}\,(c_i)} \cap \overline{D_i}$.

One can check that E_i^1 and E_i^2 are of lower topological type, and that $(E_i^1, Y_i^1) \in \mathcal{C}_K(\Phi_i(\Gamma_i^1))$, and $(E_i^2, Y_i^2) \in \mathcal{C}_K(\Phi_i(\Gamma_i^2))$, with $\Gamma_i^1 \dot\cup \Gamma_i^2 = \Gamma_i$ for all $i \in \mathbb{N}$, by property (i),(ii) of the Φ_{η_i}. In addition, by property (i), we get $\Phi_{\eta_i}(\Gamma_i^1) \to \Gamma_i^1$, and $\Phi_{\eta_i}(\Gamma_i^2) \to \Gamma_i^2$, as $i \to \infty$, where $\Gamma^1 \dot\cup \Gamma^2 = \Gamma$. Moreover, we have by construction

$$\mathcal{F}^\epsilon_{E_i^1}(Y_i^1) + \mathcal{F}^\epsilon_{E_i^2}(Y_i^2) = \mathcal{F}^\epsilon_{D_i}(\Phi_{\eta_i} \circ Z_i). \tag{4.19}$$

Using the induction hypothesis

$$d_K(\Gamma^1, \epsilon) \le \liminf_{i\to\infty} \mathcal{F}^\epsilon_{E_i^1}(Y_i^1) \ \text{ and } \ d_K(\Gamma^2, \epsilon) \le \liminf_{i\to\infty} \mathcal{F}^\epsilon_{E_i^2}(Y_i^2),$$

[9] Y_i^1 and Y_i^2 are in fact in the right Sobolev class since they are constructed from $H^{1,2}$-surfaces with matching boundary traces by the assumptions on the values on the curves c_i.

we obtain by Lemma 4.6, (4.19) and the definition of $d_K^\dagger(\Gamma,\epsilon)$

$$
\begin{aligned}
d_K(\Gamma,\epsilon) &\underset{\text{L.4.6}}{\leq} d_K^\dagger(\Gamma,\epsilon)\\
&\leq d_K(\Gamma^1,\epsilon)+d_K(\Gamma^2,\epsilon)\\
&\underset{(4.19)}{\leq} \liminf_{i\to\infty} \mathcal{F}^\epsilon_{D_i}(\Phi_{\eta_i}\circ Z_i)\\
&\underset{(\text{v})}{\leq} \liminf_{m\to\infty} \mathcal{F}^\epsilon_{B_m}(X_m),
\end{aligned}
\tag{4.20}
$$

which proves (i) in this case.

Case B. Now we assume that $\{(X_m,B_m)\}\subset\mathcal{C}_K(\Gamma_m)$ is cohesive. If $\liminf_{m\to\infty}\mathcal{F}^\epsilon_{B_m}(X_m)$ is infinite, there is nothing to prove for Part (i). If, on the other hand, there is a constant c such that

$$\liminf_{m\to\infty}\mathcal{F}^\epsilon_{B_m}(X_m)\leq C,$$

then we find (for fixed ϵ) another constant $c(\epsilon)$ such that

$$\mathcal{D}_{B_m}(X_m)\leq c(\epsilon)\ \text{ for all }\ m\in\mathbb{N},$$

which allows us to apply Lemma 4.5 to prove (i) in Case B as well.

(ii) If $d_K^*(\Gamma,\epsilon)$ is infinite, (which is the case, e.g., for $k=1$) then so is $d_K^\dagger(\Gamma,\epsilon)$ by Lemma 4.6, and there is nothing to prove. If $d_K^*(\Gamma,\epsilon)$ is finite, then we choose for arbitrary $\delta>0$ a separating sequence $\{(B_m,X_m)\}\subset\mathcal{C}_K(\Gamma_m)$ with

$$\lim_{m\to\infty}\mathcal{F}^\epsilon_{B_m}(X_m)\leq d_K^*(\Gamma,\epsilon)+\delta.$$

Now we apply the construction of Case A above to this sequence to arrive at (4.20), i.e.,

$$d_K(\Gamma,\epsilon)\leq d_K^\dagger(\Gamma,\epsilon)\leq\lim_{m\to\infty}\mathcal{F}^\epsilon_{B_m}(X_m)\leq d^*(\Gamma,\epsilon)+\delta,$$

and let $\delta\to 0$. This together with Lemma 4.6 finishes the proof of (ii). □

5 Existence proof

Lemma 5.1. *There exist $\epsilon_1>0$ and $K_1>0$ such that for each $\epsilon\in(0,\epsilon_1]$, $K\in[K_1,\infty)$ there is a domain $B_K^\epsilon\in\mathcal{K}_k^1$, and $(B_K^\epsilon,X_K^\epsilon)\in\mathcal{C}_K(\Gamma)$ with*

$$\mathcal{F}^\epsilon_{B_K^\epsilon}(X_K^\epsilon)=d_K(\Gamma,\epsilon). \tag{5.1}$$

Moreover,

$$|(X_K^\epsilon)_u|^2 = |(X_K^\epsilon)_v|^2 \ \ \text{and} \ \ (X_K^\epsilon)_u \cdot (X_K^\epsilon)_v = 0 \ \ \text{a.e.} \ \ \text{on} \ \ B_K. \tag{5.2}$$

PROOF: For arbitrary $\epsilon > 0$ we take a minimal sequence $\{(B_m, X_m)\} \subset \mathcal{C}_K(\Gamma)$ with

$$\lim_{m\to\infty} \mathcal{F}^\epsilon_{B_m}(X_m) = d_K(\Gamma, \epsilon). \tag{5.3}$$

Since $\mathcal{C}_K^\dagger(\Gamma) \neq \emptyset$ for K sufficiently large, and since $d_K(\Gamma,\epsilon) \le d_K^\dagger(\Gamma,\epsilon)$ by Lemma 4.6, the right-hand side is finite, and we find a constant $c(\epsilon)$ independent of m such that

$$\mathcal{D}_{B_m}(X_m) \le c(\epsilon) \ \text{ for all } \ m \in \mathbb{N}.$$

If $\{(B_m, X_m)\}$ is cohesive (which is automatically the case for $k = 1$) then we can apply Lemma 4.5 for fixed $\epsilon > 0$ to obtain a pair $(B_K^\epsilon, X_K^\epsilon) \in \mathcal{C}_K(\Gamma)$ satisfying (4.10) and hence (5.1) by (5.3).

We claim that, for ϵ sufficiently small and K sufficiently large, there is no separating minimal sequence for $\mathcal{F}^\epsilon$ in $\mathcal{C}_K(\Gamma)$. Indeed, the Douglas condition (1.4) implies by Lemma 4.2 for $K := \infty$, that there exists $\epsilon_1 > 0$ such that

$$d(\Gamma,\epsilon) < d^\dagger(\Gamma,\epsilon) \ \text{ for all } \ \epsilon \in (0, \epsilon_1]. \tag{5.4}$$

This in turn yields by Corollary 4.4 that there is $K_1 \in (0,\infty)$ such that

$$d_K(\Gamma,\epsilon) < d_K^\dagger(\Gamma,\epsilon) \ \text{ for all } \ K \ge K_1, \, \epsilon \in (0,\epsilon_1]. \tag{5.5}$$

Consequently, any minimal sequence $\{(B_m, X_m)\} \subset \mathcal{C}_K(\Gamma)$ with

$$\lim_{m\to\infty} \mathcal{F}^\epsilon_{B_m}(X_m) = d_K(\Gamma, \epsilon) \tag{5.6}$$

must be cohesive for $\epsilon \in (0, \epsilon_1]$, $K \in [K_1, \infty)$, since otherwise by (5.5) and Theorem 4.7, Part (ii) we would have

$$\begin{aligned} d_K(\Gamma,\epsilon) \ \underset{(5.5)}{<} \ & d_K^\dagger(\Gamma,\epsilon) \underset{\text{Thm.4.7(ii)}}{=} d_K^*(\Gamma,\epsilon) \\ \le \ & \liminf_{m\to\infty} \mathcal{F}^\epsilon_{B_m}(X_m) = \lim_{m\to\infty} \mathcal{F}^\epsilon_{B_m}(X_m) \underset{(5.6)}{=} d_K(\Gamma,\epsilon), \end{aligned}$$

which is absurd.

Finally we have to show the conformality relations (5.2). The minimality (5.1) implies in particular,

$$\mathcal{F}^{\epsilon}_{B^{\epsilon}_K}(X^{\epsilon}_K) \leq \mathcal{F}^{\epsilon}_{B^{\epsilon}_K}(Y) \tag{5.7}$$

for all $Y \in H^{1,2}(B^{\epsilon}_K, \mathbb{R}^n)$ with $(B^{\epsilon}_K, Y) \in \mathcal{C}_K(\Gamma)$. As in [8, p.254] we obtain $\partial\mathcal{F}^{\epsilon}(X^{\epsilon}_K, \eta) = 0$ for the inner variation of $\mathcal{F}^{\epsilon}_{B^{\epsilon}_K}$ at X^{ϵ}_K in the direction of an arbitrary vectorfield $\eta \in C^1(\overline{B^{\epsilon}_K}, \mathbb{R}^2)$. Since $\mathcal{F}$ is parameter invariant, we have $\partial\mathcal{F}(X^{\epsilon}_K, \eta) = 0$, and thus $\partial\mathcal{D}(X^{\epsilon}_K, \eta) = 0$ for any[10] $\eta \in C^1(\overline{B^{\epsilon}_K}, \mathbb{R}^2)$, which implies (5.2), see [4, Vol. I, p. 246]. □

Lemma 5.2. *Assume Γ satisfies a chord-arc condition. Then there is a constant $\chi = \chi(\Gamma)$ depending on Γ but independent of ϵ and K, such that $X^{\epsilon}_K \in C^{0,\alpha}(\overline{B^{\epsilon}_K}, \mathbb{R}^n)$ for some $\alpha \in (0, 1/2]$ independent of $\epsilon \in (0, \epsilon_1]$ and $K \in [K_1, \infty)$, with*

$$\|X^{\epsilon}_K\|_{C^{0,\alpha}(\overline{B^{\epsilon}_K}, \mathbb{R}^n)} \leq \chi \ \textit{for all} \ \ \epsilon \in (0, \epsilon_1],\ K \in [K_1, \infty). \tag{5.8}$$

In particular, $X^{\epsilon}_K \in \mathcal{C}_{\chi}(\Gamma)$ for all $\epsilon \in (0, \epsilon_1]$, $K \in [K_1, \infty)$.

Proof: By assumption (D) on the parametric integrand F and by (5.1) and (5.2) we obtain

$$\begin{aligned}
(m_1 + \epsilon)\mathcal{D}_{B^{\epsilon}_K}(X^{\epsilon}_K) &\underset{(5.2)}{=} m_1\mathcal{A}_{B^{\epsilon}_K}(X^{\epsilon}_K) + \epsilon\mathcal{D}_{B^{\epsilon}_K}(X^{\epsilon}_K) \\
&\underset{(D)}{\leq} \mathcal{F}^{\epsilon}_{B^{\epsilon}_K}(X^{\epsilon}_K) \underset{(5.1)}{\leq} \mathcal{F}^{\epsilon}_{B^{\epsilon}_K}(Z) \\
&\underset{(D)}{\leq} m_2\mathcal{A}_{B^{\epsilon}_K}(Z) + \epsilon\mathcal{D}_{B^{\epsilon}_K}(Z) \\
&\leq (m_2 + \epsilon)\mathcal{D}_{B^{\epsilon}_K}(Z)
\end{aligned}$$

for any $(B^{\epsilon}_K, Z) \in \mathcal{C}_K(\Gamma)$ and for every $\epsilon \in (0, \epsilon_1]$, $K \in [K_1, \infty)$. Since $(m_2 + \epsilon)(m_1 + \epsilon)^{-1} \leq m_2/m_1$ for any $\epsilon > 0$, we arrive at

$$\mathcal{D}_{B^{\epsilon}_K}(X^{\epsilon}_K) \leq \frac{m_2}{m_1}\mathcal{D}_{B^{\epsilon}_K}(Z) \ \text{for all} \ \ (B^{\epsilon}_K, Z) \in \mathcal{C}_K(\Gamma). \tag{5.9}$$

We can now use comparison arguments as in [8, pp. 261–263] by replacing X^{ϵ}_K locally by harmonic vectors (without leaving the class $\mathcal{C}_K(\Gamma)$ due to the maximum principle) to finish the proof. □

---

[10] To avoid Riemann's mapping theorem for multiply connected domains one can restrict the inner variations to a special class of vector fields (but still sufficiently large to obtain conformality), as done in [3, pp. 169–178], [14, Ch. 3]. These vector fields generate diffeomorphisms mapping $\overline{B^{\epsilon}_K}$ onto a k-circle domain.

Lemma 5.3. *For all* $\epsilon \in (0,\epsilon_1]$ *there is a domain* $B^\epsilon \in \mathcal{K}^1_k$ *and a pair* $(B^\epsilon, X^\epsilon) \in \mathcal{C}(\Gamma)$ *with*

$$\mathcal{F}^\epsilon_{B^\epsilon}(X^\epsilon) = d(\Gamma,\epsilon) \tag{5.10}$$

Moreover,

$$|X^\epsilon_u|^2 = |X^\epsilon_v|^2 \ \textit{and} \ X^\epsilon_u \cdot X^\epsilon_v = 0 \ \textit{a.e.} \quad \textit{on} \ B^\epsilon. \tag{5.11}$$

Proof: By Lemma 4.3 we know that

$$\mathcal{F}^\epsilon_{B^\epsilon_K}(X^\epsilon_K) = d_K(\Gamma,\epsilon) \longrightarrow d(\Gamma,\epsilon) \ \text{ as } \ K \to \infty. \tag{5.12}$$

The right-hand side is bounded by Lemma 4.6 for $K := \infty$ and the fact that $\mathcal{C}^\dagger(\Gamma) \neq \emptyset$. Thus there is a constant C independent of $K \in [K_1,\infty)$ such that for fixed $\epsilon \in (0,\epsilon_1]$

$$\mathcal{F}^\epsilon_{B^\epsilon_K}(X^\epsilon_K) \le C \ \text{ for all } \ K \in [K_1,\infty).$$

This yields a constant $C(\epsilon)$ depending on ϵ but not on K, such that

$$\mathcal{D}_{B^\epsilon_K}(X^\epsilon_K) \le C(\epsilon) \ \text{ for all } \ K \in [K_1,\infty). \tag{5.13}$$

If $\{(B^\epsilon_K, X^\epsilon_K)\} \subset \mathcal{C}(\Gamma)$ is cohesive (which is always true for $k=1$) then we may use Lemma 4.5 (with $K := \infty$ there) to obtain a pair $(B^\epsilon, X^\epsilon) \in \mathcal{C}(\Gamma)$ with (4.10) for $K = \infty$, and thus by (5.12) the desired identity (5.10).

For $k > 1$ we claim that the sequence $\{(B^\epsilon_K, X^\epsilon_K)\}$ is cohesive. In fact, we infer from (5.8) in Lemma 5.2 that

$$(B^\epsilon_K, X^\epsilon_K) \in \mathcal{C}_\chi(\Gamma) \ \text{ for all } \ K \in [K_1,\infty), \epsilon \in (0,\epsilon_1]. \tag{5.14}$$

If $\{(B^\epsilon_K, X^\epsilon_K)\}$ were separating we would obtain from (5.12)

$$d^*_\chi(\Gamma,\epsilon) \le \liminf_{K\to\infty} \mathcal{F}^\epsilon_{B^\epsilon_K}(X^\epsilon_K) \underset{(5.12)}{=} d(\Gamma,\epsilon). \tag{5.15}$$

This together with (5.4), Lemma 4.3, and Part (i) of Theorem 4.7 leads to

$$d^*_\chi(\Gamma,\epsilon) \underset{(5.15)}{\le} d(\Gamma,\epsilon) \underset{(5.4)}{<} d^\dagger(\Gamma,\epsilon) \underset{\text{L.4.3}}{\le} d^\dagger_\chi(\Gamma,\epsilon) \underset{\text{Thm.4.7}}{=} d^*_\chi(\Gamma,\epsilon).$$

This is a contradiction and proves that $\{(B^\epsilon_K, X^\epsilon_K)\}$ is cohesive.

The conformality relations (5.11) follow by taking inner variations of $\mathcal{F}^\epsilon_{B^\epsilon}$ at X^ϵ as in the proof of (5.2) in Lemma 5.1. □

PROOF OF THEOREM 1.1: By assumption (D), the conformality relations (5.11), and the minimality (5.10) we deduce for the $(B^\epsilon, X^\epsilon) \in \mathcal{C}(\Gamma)$ obtained in Lemma 5.3

$$
\begin{aligned}
(m_1+\epsilon)\mathcal{D}_{B^\epsilon}(X^\epsilon) &\underset{(5.11)}{=} m_1\mathcal{A}_{B^\epsilon}(X^\epsilon) + \epsilon\mathcal{D}_{B^\epsilon}(X^\epsilon) \\
&\underset{(D)}{\leq} \mathcal{F}^\epsilon_{B^\epsilon}(X^\epsilon) \underset{(5.10)}{\leq} \mathcal{F}^\epsilon_{B^\epsilon}(Z) \\
&\leq (m_2+\epsilon)\mathcal{D}_{B^\epsilon}(Z)
\end{aligned}
\tag{5.16}
$$

for any $(B^\epsilon, Z) \in \mathcal{C}(\Gamma)$ and every $\epsilon \in (0, \epsilon_1]$. Since $(m_2+\epsilon)(m_1+\epsilon)^{-1} \leq m_2/m_1$ for any $\epsilon > 0$, we arrive at

$$
\mathcal{D}_{B^\epsilon}(X^\epsilon) \leq \frac{m_2}{m_1}\mathcal{D}_{B^\epsilon}(Z) \ \text{ for all } \ (B^\epsilon, Z) \in \mathcal{C}(\Gamma). \tag{5.17}
$$

Using the fact $\mathcal{F}^\epsilon_{B^\epsilon}(X^\epsilon) = d(\Gamma, \epsilon)$ and Lemma 4.6, we can replace (5.16) by

$$
\begin{aligned}
(m_1+\epsilon)\mathcal{D}_{B^\epsilon}(X^\epsilon) &\leq d(\Gamma,\epsilon) \underset{\text{L.4.6}}{\leq} d^\dagger(\Gamma,\epsilon) \\
&\leq \mathcal{F}^\epsilon_\Omega(Y) \leq (m_2+\epsilon)\mathcal{D}_\Omega(Y)
\end{aligned}
$$

for any $(\Omega, Y) \in \mathcal{C}^\dagger(\Gamma)$, and thus,

$$
\mathcal{D}_{B^\epsilon}(X^\epsilon) \leq \frac{m_2}{m_1}\mathcal{D}_\Omega(Y) \ \text{ for all } \ (\Omega, Y) \in \mathcal{C}^\dagger(\Gamma). \tag{5.18}
$$

In particular, if $\Gamma = \langle \Gamma^1 \dots, \Gamma^k \rangle$, we may take $(\Omega, Y) \in \mathcal{C}^\dagger(\Gamma)$ to be the collection of k disk-type minimal surfaces Y_j spanned in Γ^j, $j = 1, \dots, k$, defined on

$$
\Omega := \bigcup_{j=1}^k B_j,
$$

where the disks $B_j \in \mathcal{K}_1$, $l = 1, \dots, k$, are disjoint. More precisely,

$$
Y := \begin{cases} Y_1 & \text{on } \ B_1, \\ \vdots & \vdots \\ Y_k & \text{on } \ B_k, \end{cases}
$$

where $Y_j \in H^{1,2}(B_j, \mathbb{R}^n)$ is a minimal surface such that $(B_j, Y_j) \in \mathcal{C}(\Gamma^j)$, $j = 1, \dots, k$. By the classical isoperimetric inequality for minimal surfaces (see e.g., [4, Vol. I, Ch. 6.3]) we infer from (5.18)

$$
\mathcal{D}_{B^\epsilon}(X^\epsilon) \leq \frac{m_2}{4\pi m_1}\sum_{j=1}^k L^2(\Gamma^j) =: c(\Gamma, m_1, m_2) \ \text{ for all } \ \epsilon \in (0, \epsilon_1], \tag{5.19}
$$

where $L(\Gamma^j)$ denotes the length of the j-th component Γ^j of Γ. The right-hand side does not depend on $\epsilon \in (0, \epsilon_1]$, so there is a sequence $\epsilon_j \to 0$ such that

$$\mathcal{D}_{B^{\epsilon_j}}(X^{\epsilon_j}) \longrightarrow D \in [0, c(\Gamma, m_1, m_2)]. \tag{5.20}$$

As in the proofs of Lemma 5.1 and 5.3 we have to show that (for $k > 1$) the sequence $\{(B^{\epsilon_j}, X^{\epsilon_j})\} \subset \mathcal{C}(\Gamma)$ is cohesive, since then we may apply Lemma 4.5 (for $K := \infty$ and $\epsilon := 0$) to obtain $(B, X) \in \mathcal{C}(\Gamma)$ such that

$$d(\Gamma) = d(\Gamma, 0) \leq \mathcal{F}_B(X) \leq \liminf_{j \to \infty} \mathcal{F}^{\epsilon_j}(X^{\epsilon_j}) = d(\Gamma),$$

which proves (1.5).

We claim that $\{(B^{\epsilon_j}, X^{\epsilon_j})\}$ is cohesive. Indeed, otherwise we could infer from the Douglas condition, the estimate (5.8), (5.20), (5.10), Part (ii) of Theorem 4.7, and from the Lemmas 4.1 and 4.3 the following:

$$\begin{aligned}
d(\Gamma) \quad & \underset{(1.4)}{<} \quad d^\dagger(\Gamma) = d^\dagger(\Gamma, 0) \underset{\text{L.4.3}}{=} \lim_{K \to \infty} d^\dagger_K(\Gamma, 0) \\
& \underset{\text{L.4.1}}{=} \quad \lim_{K \to \infty} \lim_{j \to \infty} d^\dagger_K(\Gamma, \epsilon_j) \underset{\text{L.4.3}}{\leq} \lim_{j \to \infty} d^\dagger_\chi(\Gamma, \epsilon_j) \\
& \underset{\text{Thm.4.7(ii)}}{=} \quad \lim_{j \to \infty} d^*_\chi(\Gamma, \epsilon_j) \\
& \underset{(5.8)}{\leq} \quad \lim_{j \to \infty} \liminf_{m \to \infty} \mathcal{F}^{\epsilon_j}_{B^{\epsilon_m}}(X^{\epsilon_m}) \\
& = \quad \lim_{j \to \infty} \Big[\liminf_{m \to \infty} \mathcal{F}_{B^{\epsilon_m}}(X^{\epsilon_m}) + \epsilon_j \lim_{m \to \infty} \mathcal{D}_{B^{\epsilon_m}}(X^{\epsilon_m}) \Big] \\
& \underset{(5.20)}{=} \quad \lim_{j \to \infty} \Big[\liminf_{m \to \infty} \mathcal{F}_{B^{\epsilon_m}}(X^{\epsilon_m}) + \epsilon_j D \Big] \\
& \leq \quad \lim_{j \to \infty} \Big[\liminf_{m \to \infty} \mathcal{F}^{\epsilon_m}_{B^{\epsilon_m}}(X^{\epsilon_m}) + \epsilon_j D \Big] \\
& \underset{(5.10)}{=} \quad \lim_{j \to \infty} \Big[\liminf_{m \to \infty} d(\Gamma, \epsilon_m) + \epsilon_j D \Big] \\
& \underset{\text{L.4.1}}{=} \quad \lim_{j \to \infty} \Big[d(\Gamma) + \epsilon_j D \Big] = d(\Gamma),
\end{aligned}$$

which is absurd.

To prove (1.6) we argue as follows. By (5.10) and (1.5) and the fact that both $(B^{\epsilon_j}, X^{\epsilon_j})$ and (B, X) are contained in the class $\mathcal{C}(\Gamma)$ (where

$B = B^{\epsilon_j} = B_1(0)$ for all $j \in \mathbb{N}$ in the case $k = 1$), we obtain for all $\epsilon_j > 0$

$$\begin{aligned}\mathcal{F}_{B^{\epsilon_j}}(X^{\epsilon_j}) + \epsilon_j \mathcal{D}_{B^{\epsilon_j}}(X^{\epsilon_j}) \quad &= \quad \mathcal{F}^{\epsilon_j}_{B^{\epsilon_j}}(X^{\epsilon_j}) \\ &\underset{(5.10)}{\leq} \quad \mathcal{F}^{\epsilon_j}_{B}(X) = \mathcal{F}_B(X) + \epsilon_j \mathcal{D}_B(X) \\ &\underset{(1.5)}{\leq} \quad \mathcal{F}^{\epsilon_j}_{B^{\epsilon_j}}(X^{\epsilon_j}) + \epsilon_j \mathcal{D}_B(X).\end{aligned}$$

This implies

$$\mathcal{D}_{B^{\epsilon_j}}(X^{\epsilon_j}) \leq \mathcal{D}_B(X) \text{ for all } \epsilon_j > 0. \tag{5.21}$$

Recall from the proof of Lemma 4.5 (applied to $\{(B^{\epsilon_j}, X^{\epsilon_j})\}$ for $K := \infty$, and $\epsilon := 0$) that there are reparametrizations $Z_j \in H^{1,2}(B, \mathbb{R}^n)$ of X^{ϵ_j} such that by (5.20)

$$\mathcal{D}_B(Z_j) \longrightarrow D, \tag{5.22}$$

$$Z_j \rightharpoonup X \text{ in } H^{1,2}(B, \mathbb{R}^n), \quad \text{as } j \to \infty \tag{5.23}$$

This together with the weak lower semicontinuity of $\mathcal{D}_B(\cdot)$ in $H^{1,2}(B, \mathbb{R}^n)$ leads to

$$\limsup_{j\to\infty} \mathcal{D}_B(Z_j) \underset{(5.22)}{=} D \underset{(5.20)}{=} \lim_{j\to\infty} \mathcal{D}_{B^{\epsilon_j}}(X^{\epsilon_j}) \underset{(5.21)}{\leq} \mathcal{D}_B(X) \leq \liminf_{j\to\infty} \mathcal{D}_B(Z_j).$$

Thus,

$$\lim_{j\to\infty} \mathcal{D}_B(Z_j) = \mathcal{D}_B(X) = D, \tag{5.24}$$

and (5.23) then implies the strong convergence

$$Z_j \to X \text{ in } H^{1,2}(B, \mathbb{R}^n) \text{ as } j \to \infty. \tag{5.25}$$

Since the area functional $\mathcal{A}$ is parameter invariant and by (5.11) we have

$$\mathcal{A}_B(Z_j) = \mathcal{A}_{B^{\epsilon_j}}(X^{\epsilon_j}) \underset{(5.11)}{=} \mathcal{D}_{B^{\epsilon_j}}(X^{\epsilon_j}),$$

which implies by (5.24),(5.20) and (5.25),

$$\mathcal{D}_B(X) \underset{(5.24)}{=} D \underset{(5.20)}{=} \lim_{j\to\infty} \mathcal{D}_{B^{\epsilon_j}}(X^{\epsilon_j}) = \lim_{j\to\infty} \mathcal{A}_B(Z_j) \underset{(5.25)}{=} \mathcal{A}_B(X). \tag{5.26}$$

This proves (1.6) and finishes the proof of Theorem 1.1. □

The Proof of Theorem 1.2 follows directly from (5.19) and (5.26).

A Auxiliary facts about Jordan systems

Let Γ_m, and Γ be Jordan systems in $\mathbb{R}^n$.

LEMMA A.1. *If $\Gamma_m \to \Gamma$ in the Fréchet-sense, then for any $\epsilon > 0$ there is $\lambda = \lambda(\epsilon)$ and $m_1 = m_1(\epsilon)$ such that for all Q_1^m, $Q_2^m \in \Gamma_m$, and for all $Q_1, Q_2 \in \Gamma$ with $0 < |Q_1^m - Q_2^m| < \lambda$, $0 < |Q_1 - Q_2| < \lambda$,*

$$\operatorname{diam}(\triangle(Q_1,Q_2)) < \epsilon/2 \ \textit{and} \ \operatorname{diam}(\triangle(Q_1^m,Q_2^m)) < \epsilon/2 \ \textit{for all} \ m \geq m_1,$$

where $\triangle(Q_1^m,Q_2^m)$ and $\triangle(Q_1,Q_2)$ denote the shorter arcs on Γ_m and Γ, connecting Q_1^m and Q_2^m, and Q_1 and Q_2, respectively.

PROOF: We prove the statement for Γ first. Assume for contradiction that there is a sequence of points $Q_1^\nu \neq Q_2^\nu$ on Γ with $|Q_1^\nu - Q_2^\nu| \to 0$ such that $\operatorname{diam}(\triangle(Q_1^\nu,Q_2^\nu)) \geq \epsilon$ for some $\epsilon > 0$. Then

$$\begin{aligned} &\triangle(Q_1^\nu,Q_2^\nu) \cap \partial B_\epsilon(Q_1^\nu) \neq \emptyset \ \text{and} \\ &(\Gamma \backslash \triangle(Q_1^\nu,Q_2^\nu)) \cap \partial B_\epsilon(Q_1^\nu) \neq \emptyset, \end{aligned} \tag{A.27}$$

and we find points $R_\nu \in \triangle(Q_1^\nu,Q_2^\nu) \cap \partial B_\epsilon(Q_1^\nu)$, and $S_\nu \in (\Gamma\backslash\triangle(Q_1^\nu,Q_2^\nu)) \cap \partial B_\epsilon(Q_1^\nu)$. Since Γ is a Jordan system, we can assume (taking ν sufficiently large) that R_ν and S_ν belong to the same component Γ^j of Γ, and we can pick an injective continuous arc length parametrization $\gamma : [0,L] \to \Gamma^j \subset \mathbb{R}^n$ with $Q_1^\nu = \gamma(s_1^\nu)$, $Q_2^\nu = \gamma(s_2^\nu)$, $R_\mu = \gamma(\sigma_\nu)$, and $S_\nu = \gamma(\tau_\nu)$, where $\sigma_\nu \in (s_1^\nu, s_2^\nu)$ and $\tau_\nu \in [0,L]\backslash(s_1^\nu,s_2^\nu)$. Assume without loss of generality that

$$\gamma((s_1^\nu,s_2^\nu)) = \triangle(Q_1^\nu,Q_2^\nu), \ \text{ i.e. } \ |s_1^\nu - s_2^\nu| \geq \epsilon/2. \tag{A.28}$$

Taking subsequences we get $s_1^\nu \to s_1$, $s_2^\nu \to s_2$, $\sigma_\nu \to \sigma \in [s_1,s_2]$, and $\tau_\nu \to \tau \in [0,L] - (s_1,s_2)$ as $\nu \to \infty$. Note that $|s_1 - s_2| < L$, since otherwise

$$\operatorname{diam}(\gamma(s_1^\nu,s_2^\nu)) \to \operatorname{diam}(\gamma(s_1,s_2)) = \operatorname{diam}(\gamma(0,L)) = \operatorname{diam}(\Gamma^j)$$

contradicting (A.28) for ν sufficiently large. Consequently, by

$$\gamma(s_1) = \lim_{\nu\to\infty} Q_1^\nu = \lim_{\nu\to\infty} Q_2^\nu = \gamma(s_2),$$

we have $s_1 = s_2$ since γ is injective, contradicting (A.28) again.

For the corresponding result on Γ_ν let $\epsilon > 0$ be given and take $\eta = \eta(\epsilon)$ be such that

$$\operatorname{diam}(\triangle(Q_1,Q_2)) < \epsilon/4 \ \text{ for all } \ |Q_1 - Q_2| < \eta.$$

Choose ν_0 so large that for all $\nu \geq \nu_0$

$$\|\Gamma_\nu - \Gamma\|_{C^0} < \min\{\eta/4, \epsilon/8\}.$$

Then $|Q_1^\nu - Q_2^\nu| < \eta/2$ implies $|Q_1 - Q_2| < \eta$ for $Q_i := \lim_{\nu\to\infty} Q_i^\nu$, $i = 1, 2$. Hence we obtain $\operatorname{diam}(\triangle(Q_1, Q_2)) < \epsilon/4$, and this implies

$$\operatorname{diam}(\triangle(Q_1^\nu, Q_2^\nu)) < \epsilon/4 + 2\|\Gamma_\nu - \Gamma\|_{C^0} < \epsilon/2.$$

$\square$

Lemma A.2. *Let $\Gamma \subset \mathbb{R}^n$ be a Jordan system, and $P \in \mathbb{R}^n$. Then for every $\delta > 0$ there is $\eta_0 = \eta_0(\Gamma, \delta)$ such that for all $\eta \in (0, \eta_0)$, $\Phi_\eta(\Gamma)$ is a Jordan curve with $\|\Phi_\eta(\Gamma) - \Gamma\|_{C^0} < \delta$, where $\Phi_\eta : \mathbb{R}^n \to \mathbb{R}^n$ is defined as in Lemma 3.3. In addition, $\Phi_\eta \circ X : \partial B \to \mathbb{R}^n$ is continuous and weakly monotonic for any weakly monotonic mapping $X \in C^0(\partial B, \mathbb{R}^n)$.*

Proof: Choose $\eta_0 \in (0, \delta)$ so small that at most one component Γ^j of the Jordan system $\Gamma = \langle\Gamma^1 \dots, \Gamma^k\rangle$ has nonempty intersection with the ball $\overline{B_{\eta_0}(P)}$, and such that the set $\Gamma^j \cap \overline{B_{\eta_0^2}(P)}$ is connected. One can check that Φ_η restricted to the set $\mathbb{R}^n \backslash \overline{B_{\eta^2}(P)}$ is a homeomorphism onto $\mathbb{R}^n \backslash \{P\}$.

Hence $\Phi_\eta(\Gamma \backslash \Gamma^j)$ is a Jordan system consisting of $k-1$ components and the part $\Phi \circ (\Gamma^j \cap (\mathbb{R}^n \backslash \overline{B_{\eta^2}(P)})$ is a Jordan arc. Let $\gamma_l : S^1 \to \mathbb{R}^n$ be the injective arc length parametrization of Γ^j. Since Γ is a Jordan system we know that the set

$$J := \{s \in S^1 : \gamma_l(s) \in \overline{B_{\eta^2}(P)}\}$$

is a closed arc on S^1, with $\Phi_\eta \circ \Gamma(J) = P$. We can cut out the open interior $\mathring{J}$ and identify the endpoints of J to obtain a new circle denoted by S^* with perimeter $2\pi - |J|$. Thus we obtain a homeomorphism $\Phi \circ \gamma_l : S^* \backslash \mathring{J} \to \mathbb{R}^n$, which can be rescaled by $2\pi/(2\pi - |J|)$ in the domain, to yield a closed Jordan curve Γ_*^j which can be identified with $\Phi_\eta(\Gamma^j)$. This proves the first assertion.

For the second claim we only need to consider the component Γ^j of Γ, since Φ_η is the identity map on $\mathbb{R}^n \backslash \overline{B_\eta(P)}$ by definition. For $x \in \Gamma^j$ we obtain by definition of Φ_η

$$|\Phi_\eta(x) - x| = \begin{cases} 0 & \text{for } x \in \overline{B_{\eta^2}(P)}, \\ \leq \eta & \text{for } x \in \mathbb{R}^n \backslash \overline{B_{\eta^2}(P)}, \end{cases}$$

see (3.8).

The last claim follows from the fact that Φ_η is a homeomorphism of $\mathbb{R}^n \setminus \overline{B_{\eta^2}(P)}$ onto $\mathbb{R}^n \setminus \{P\}$. The composition of a weakly monotonic mapping with a homeomorphism remains weakly monotonic, and the composition of a weakly monotonic mapping with the constant map is also weakly monotonic by definition. □

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Matthias Kurzke
Mathematisches Institut
Universität Leipzig
Augustusplatz 10/11
D-04109 Leipzig
GERMANY
E-mail: `kurzke@mathematik.uni-leipzig.de`

Heiko von der Mosel
Mathematisches Institut
Universität Bonn
Beringstraße 1
D-53115 Bonn
GERMANY
E-mail: `heiko@math.uni-bonn.de`