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Heat asymptotics with spectral boundary conditions II

by

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ABSTRACT: Let P be an operator of Dirac type on a compact Riemannian manifold with smooth boundary. We impose spectral boundary conditions and study the asymptotics of the heat trace of the associated operator of Laplace type.
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We recall the notational conventions established in [10]. Let M be a compact m-dimensional Riemannian manifold with smooth boundary ∂M. We suppose given unitary vector bundles E_i over M and an elliptic complex

$$P : C^\infty(E_1) \to C^\infty(E_2). \tag{1}$$

We assume that (1) defines an elliptic complex of Dirac type. We impose spectral boundary conditions $\mathcal{B}$; Atiyah, Patodi, and Singer [2] showed that an elliptic complex of Dirac type need not admit local boundary conditions.

Apart from the mathematical interest, spectral boundary conditions are of relevance in one-loop quantum cosmology and supergravity (see e.g. [13,14]). Furthermore, they are consistent with a non-zero index and have been intensively discussed in the context of fermion number fractionization [15,23].

Let $P_\mathcal{B}$ and $D_\mathcal{B} := (P_\mathcal{B})^* P_\mathcal{B}$ be the associated realizations. Let $F \in C^\infty(E_1)$ be an auxiliary function used for localization. Results of Grubb and Seeley [19–21] show that there is an asymptotic series as $t \downarrow 0$ of the form:

$$\mathrm{Tr}_{L^2}\{Fe^{-tD_\mathcal{B}}\} \sim \sum_{0\le k\le m-1} a_k(F,D,\mathcal{B})t^{(k-m)/2} + O(t^{-1/8}). \tag{2}$$

(There is in fact a complete asymptotic series with log terms, but we shall only be interested in the first few terms in the series). The coefficients a_k in equation (2) are locally computable. We determined the coefficients a_0, a_1, and a_2 previously [10]; these results are summarized in Theorem 1 below. In this paper, we determine the coefficient a_3. We shall assume henceforth that $m \ge 4$ so that the series in equation (2) gives this term.

We shall express the coefficients a_k invariantly in terms of the following data. Let γ be the leading symbol of the operator P. Since the elliptic complex is of Dirac type, $\gamma + \gamma^*$ defines a unitary Clifford module structure on $E_1 \oplus E_2$. Let $\nabla = \nabla_1 \oplus \nabla_2$ be a compatible unitary connection; this means that

$$\nabla(\gamma + \gamma^*) = 0 \text{ and } (\nabla s, \tilde{s}) + (s, \nabla\tilde{s}) = d(s,\tilde{s}). \tag{3}$$

Such connections always exist [7] but are not unique. If $y = (y^1, ..., y^{m-1})$ are local coordinates on ∂M, let $x = (y, x^m)$ be local coordinates on the collar where x^m is the geodesic distance to the boundary; the curves $y \to (y,t)$ are unit speed geodesics perpendicular to ∂M. Let $\partial_\mu := \frac{\partial}{\partial x^\mu}$; ∂_m is the **inward** geodesic normal vector field on the collar. Let ∇_μ be covariant differentiation with respect to ∂_μ. Decompose

$$P = \gamma^\mu \nabla_\mu + \psi$$

where we adopt the Einstein convention and sum over repeated indices. Here ψ is a 0^{th} order operator; the structures γ, ∇, and ψ can depend on the normal variable. Since P is of Dirac type, we have the Clifford commutation relations:

$$(\gamma^\mu)^*\gamma^\nu + (\gamma^\nu)^*\gamma^\mu = 2g^{\mu\nu}. \tag{4}$$

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Near the boundary and relative to a local frame which is parallel along the normal geodesic rays, we have $\nabla_m = \partial_m$. We freeze the coefficients and set $x^m = 0$ to define a tangential operator

$$B(y) := \gamma^m(y,0)^{-1}\{\textstyle\sum_{\alpha<m}\gamma^\alpha(y,0)\nabla_\alpha + \psi(y,0)\} \text{ on } C^\infty(E_1|_{\partial M}).$$

Let Θ be an auxiliary self-adjoint endomorphism of $E_1|_{\partial M}$. We take the adjoint of B with respect to the structures on the boundary to define a self-adjoint tangential operator of Dirac type on $C^\infty(E_1|_{\partial M})$:

$$A := \frac{B+B^*}{2} + \Theta.$$

The boundary operator $\mathcal{B}$ whose vanishing defines spectral boundary conditions is orthogonal projection on the span of the eigenspaces for the non-negative spectrum of A. Replacing the words "non-negative" by "positive" would not change the local invariants a_n.

We shall let Roman indices i and j range from 1 to m and index a local orthonormal frame for the tangent bundle of M; Greek indices will index a local coordinate frame. Near the boundary, we choose the frame so that e_m is the inward unit geodesic normal vector; we let indices a and b range from 1 through $m-1$ and index the corresponding frame for the tangent bundle of the boundary. We adopt the Einstein convention and sum over repeated indices. We let ';' denote multiple covariant differentiation of the tensors involved. Let Γ be the Christoffel symbols of the Levi-Civita connection on M. There is a canonical connection ${}^D\nabla$ on the bundle E_1 and there is a canonical endomorphism E of the bundle E_1 so that $D = -(\mathrm{Tr}\{{}^D\nabla^2\} + E)$; see [17] for details. Note that ${}^D\nabla$ is not in general a compatible connection. Let ω be the connection 1 form of ${}^D\nabla$. We have the following equations of structure:

$$\begin{aligned}
&D = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b) = -(\mathrm{Tr}\{{}^D\nabla^2\} + E),\\
&\omega_\delta := \tfrac12 g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu), \text{ and}\\
&E := b - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma).
\end{aligned} \tag{5}$$

Decompose $P = \gamma_i\nabla_i + \psi$. Let R_{ijkl} be the Riemann curvature tensor. Let

$$\hat\psi := \gamma_m^{-1}\psi,\ \tau := R_{ijji},\ \rho_{ij} = R_{ikkj}, \tag{6}$$
$$\beta(m) := \Gamma(\tfrac{m}{2})\Gamma(\tfrac12)^{-1}\Gamma(\tfrac{m+1}{2})^{-1}. \tag{7}$$

Let ';' and ':' denote multiple covariant differentiation with respect to the background connection ∇ and the Levi-Civita connections on M and ∂M respectively. Let $L_{ab} := \Gamma_{abm}$ be the second fundamental form. The following is the main result of this paper.

Theorem 1 *We have*

1. $a_0(F,D,\mathcal{B}) = (4\pi)^{-m/2}\int_M \mathrm{Tr}\{F\}$.
2. $a_1(F,D,\mathcal{B}) = (4\pi)^{-(m-1)/2}\frac14(\beta(m)-1)\int_{\partial M}\mathrm{Tr}\{F\}$.
3. $a_2(F,D,\mathcal{B}) = (4\pi)^{-m/2}\int_M \frac16\mathrm{Tr}\{F(\tau+6E)\} + (4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\{\frac12[\hat\psi+\hat\psi^*]F + \frac13(1-\frac34\pi\beta(m))L_{aa}F$
$-\frac{m-1}{2(m-2)}(1-\frac12\pi\beta(m))F_{;m}\}$.
4. $a_3(F,D,\mathcal{B}) = (4\pi)^{-(m-1)/2}\int_{\partial M} F\mathrm{Tr}\{\frac{1}{32}(1-\frac{\beta(m)}{m-2})(\hat\psi\hat\psi+\hat\psi^*\hat\psi^*) + \frac{1}{16}(5-2m+\frac{7-8m+2m^2}{m-2}\beta(m))\hat\psi\hat\psi^*$
$+\frac{1}{32(m-1)}(2m-3-\frac{2m^2-6m+5}{m-2}\beta(m))(\gamma_a^T\hat\psi\gamma_a^T\hat\psi+\gamma_a^T\hat\psi^*\gamma_a^T\hat\psi^*) + \frac{1}{16(m-1)}(1+\frac{3-2m}{m-2}\beta(m))\gamma_a^T\hat\psi\gamma_a^T\hat\psi^*$
$-\frac{1}{48}(\frac{m-1}{m-2}\beta(m)-1)\tau + \frac{1}{48}(1-\frac{4m-10}{m-2}\beta(m))\rho_{mm} + \frac{1}{48(m+1)}(\frac{17+5m}{4}+\frac{23-2m-4m^2}{m-2}\beta(m))L_{ab}L_{ab}$
$+\frac{1}{48(m^2-1)}(-\frac{17+7m^2}{8}+\frac{4m^3-11m^2+5m-1}{m-2}\beta(m))L_{aa}L_{bb} + \frac{1}{8(m-2)}\beta(m)(\Theta\Theta+\frac{1}{m-1}\gamma_a^T\Theta\gamma_a^T\Theta)\}$
$+\frac{1}{8(m-3)}(\frac{5m-7}{8}-\frac{5m-9}{3}\beta(m))L_{aa}F_{;m}\mathrm{Tr}\{I\} + \frac{m-1}{16(m-3)}(2\beta(m)-1)F_{;mm}\mathrm{Tr}\{I\}$.

We refer to [10] for the proof of assertions (1)-(3); the remainder of this article is devoted to the proof of assertion (4). We begin by giving a general recipe for the invariant a_3. The coefficients a_{2k} involve both interior and boundary integrals. The coefficients a_{2k+1} only involve boundary integrals. One can use dimensional analysis to see that the boundary integrand for a_3 can be expressed in terms of local invariants which are homogeneous of order 2 in the jets of the total symbols. Since $P = \gamma^\nu\nabla_\nu + \psi$, we must consider invariant expressions determined by the jets of γ, ∇, and ψ. Instead of using the curvature Ω_{ij} of the background connection ∇ as one of our basic invariants, we shall instead use the tensor

$$W_{ij} := \Omega_{ij} - \tfrac{1}{4}R_{ijkl}\gamma_k^*\gamma_l. \tag{8}$$

Since the background connection ∇ is assumed to be compatible, we have $[\gamma, W] = 0$; see [7] for details. Since $\nabla\gamma = 0$, the covariant derivatives of γ do not enter. We define:

$$\gamma_a^T := \gamma_m^{-1}\gamma_a \text{ for } 1 \leq a \leq m-1. \tag{9}$$

Equation (4) implies that γ^T is a unitary Clifford module structure, i.e. that we have the relations:

$$\gamma_a^T\gamma_b^T + \gamma_b^T\gamma_a^T = -2\delta_{ab} \text{ and } (\gamma_a^T)^* = -\gamma_a^T. \tag{10}$$

The Grubb-Seeley calculus controls the pure γ terms and after a bit of work using the Weyl calculus, one can show the following Lemma holds where we adopt the notation of equations (5-9)

Lemma 2 *There exist universal constants $d_i = d_i(m)$ for $0 \leq i \leq 20$ and $e_i = e_i(m)$ for $0 \leq i \leq 8$ so that $a_3(F, D, \mathcal{B}) = (4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\,\{a_3(F, D, \mathcal{B})(y)\}dy$ where:*

$$\begin{aligned}
a_3(F, D, \mathcal{B}, y) = {} & F(d_0[\hat\psi\hat\psi + \hat\psi^*\hat\psi^*] + d_1[\hat\psi\hat\psi - \hat\psi^*\hat\psi^*] + d_2\hat\psi^*\hat\psi + d_3[\gamma_a^T\hat\psi\gamma_a^T\hat\psi + \gamma_a^T\hat\psi^*\gamma_a^T\hat\psi^*] \\
& + d_4[\gamma_a^T\hat\psi\gamma_a^T\hat\psi - \gamma_a^T\hat\psi^*\gamma_a^T\hat\psi^*] + d_5\gamma_a^T\hat\psi^*\gamma_a^T\hat\psi + d_6[\hat\psi_{;m} + \hat\psi^*_{;m}] + d_7[\hat\psi_{;m} - \hat\psi^*_{;m}] \\
& + d_8[\gamma_a^T\hat\psi_{:a} + \gamma_a^T\hat\psi^*_{:a}] + d_9[\gamma_a^T\hat\psi_{:a} - \gamma_a^T\hat\psi^*_{:a}] + d_{10}L_{aa}[\hat\psi + \hat\psi^*] + d_{11}L_{aa}[\hat\psi - \hat\psi^*] \\
& + d_{12}\tau + d_{13}\rho_{mm} + d_{14}W_{ab}\gamma_a^T\gamma_b^T + d_{15}W_{am}\gamma_a^T + d_{16}L_{ab}L_{ab} + d_{17}L_{aa}L_{aa}) \\
& + F_{;m}(d_{18}[\hat\psi + \hat\psi^*] + d_{19}[\hat\psi - \hat\psi^*] + d_{20}L_{aa}) + d_{21}F_{;mm} \\
& + F(e_0\Theta\Theta + e_1\gamma_a^T\Theta\gamma_a^T\Theta + e_2\gamma_a^T\Theta_{:a} + e_3L_{aa}\Theta + e_4\Theta[\hat\psi + \hat\psi^*] + e_5\Theta[\hat\psi - \hat\psi^*] \\
& + e_6\gamma_a^T\Theta\gamma_a^T[\hat\psi + \hat\psi^*] + e_7\gamma_a^T\Theta\gamma_a^T[\hat\psi - \hat\psi^*]) + e_8F_{;m}\Theta.
\end{aligned}$$

We prove Theorem 1 by determining the unknown constants of Lemma 2. We first establish some technical results.

Lemma 3

1. *We have that $d_9\int_{\partial M}F\mathrm{Tr}\,\{\gamma_a^T(\hat\psi - \hat\psi^*)_{:a}\} = -d_9\int_{\partial M}F_{:a}\mathrm{Tr}\,\{\gamma_a^T(\hat\psi - \hat\psi^*)\}$.*

2. *The dual boundary condition for the formal adjoint P^* is projection on the non-negative spectrum of the operator $A_2 := -\gamma_m A_1\gamma_m^{-1}$. Furthermore A_2 is defined by $\Theta_2 = -\gamma_m\Theta_1\gamma_m^{-1} + L_{aa}$.*

Proof. We shall derive equation (2.18) of [16] showing $\gamma_{a:a}^T = 0$; assertion(1) then follows by integration by parts:

$$\begin{aligned}
\gamma_{a:a}^T &= \nabla_a\gamma_a^T - \gamma_a^T\nabla_a - \Gamma_{aab}\gamma_b^T = -\nabla_a\gamma_m\gamma_a + \gamma_m\gamma_a\nabla_a + \Gamma_{aab}\gamma_m\gamma_b \\
&= -(\nabla_a\gamma_m - \gamma_m\nabla_a)\gamma_a - \gamma_m(\nabla_a\gamma_a - \gamma_a\nabla_a) + \Gamma_{aab}\gamma_m\gamma_b \\
&= -\gamma_{m;a}\gamma_a - \gamma_m\gamma_{a;a} - \Gamma_{amb}\gamma_b\gamma_a - \Gamma_{aai}\gamma_m\gamma_i + \Gamma_{aab}\gamma_m\gamma_b \\
&= L_{ab}\gamma_b\gamma_a - L_{aa}\gamma_m\gamma_m = 0.
\end{aligned}$$

We compute the Green's formula to prove assertion (2). We have $\gamma^\nu : E_1 \to E_2$. The operator $\gamma^\nu : E_2 \to E_1$ was defined by $(\phi_1, \gamma^\nu\phi_2) = -(\gamma^\nu\phi_1, \phi_2)$. We compute:

$$(P\phi_1, \phi_2)_{L^2} - (\phi_1, P^*\phi_2)_{L^2} = \int_M(\gamma^\nu\nabla_\nu\phi_1, \phi_2) - (\phi_1, \nabla_\nu\gamma^\nu\phi_2) = \int_M\partial_\nu(\gamma^\nu\phi_1, \phi_2) = -\int_{\partial M}(\gamma_m\phi_1, \phi_2). \tag{11}$$

We introduce the following tangential partial differential operators:

$$B_1 := \gamma_m^{-1}\gamma_a\nabla_a + \gamma_m^{-1}\psi_1, \quad A_1 := \tfrac{1}{2}(B_1 + B_1^*) + \Theta_1, \text{ and } A_2 := -\gamma_m A_1\gamma_m^{-1}.$$

Let $E(\lambda, A_i) := \{\phi \in C^\infty(E_i|_{\partial M}) : A_i\phi = \lambda\phi\}$ be the eigenspaces of A_i. Let

$$\begin{aligned}
\mathcal{L}_1^> &:= \text{Closed span}\{E(\lambda, A_1) : \lambda > 0\}, \quad \mathcal{L}_1^\leq := \text{Closed span}\{E(\lambda, A_1) : \lambda \leq 0\}, \\
\mathcal{L}_2^\geq &:= \text{Closed span}\{E(\lambda, A_2) : \lambda \geq 0\}, \quad \mathcal{L}_2^< := \text{Closed span}\{E(\lambda, A_2) : \lambda < 0\}.
\end{aligned}$$

We then have orthogonal direct sum decompositions

$$L^2(E_1|_{\partial M}) = \mathcal{L}_1^{>} \oplus \mathcal{L}_1^{\leq} \text{ and } L^2(E_2|_{\partial M}) = \mathcal{L}_2^{\geq} \oplus \mathcal{L}_2^{<}.$$

Let $\mathcal{B}_i\phi_i$ be orthogonal projection of $\phi_i|_{\partial M}$ on $\mathcal{L}_1^{>}$ and $\mathcal{L}_2^{\geq}$ respectively. As $\gamma_m A_1 = -A_2\gamma_m$, $\gamma_m E(\lambda, A_1) = E(-\lambda, A_2)$. Consequently we have that

$$\gamma_m \mathcal{L}_1^{>} = \mathcal{L}_2^{<} \text{ and } \gamma_m \mathcal{L}_1^{\leq} = \mathcal{L}_2^{\geq}.$$

Let $\phi_i \in C^\infty(E_i)$. We have $\phi_1 \in \text{Domain}(P)$ if and only if $\mathcal{B}_1\phi_1 = 0$ or equivalently if $\phi_1|_{\partial M} \in \mathcal{L}_1^{\leq}$. We use equation (11) to see that the following assertions are equivalent:

(1) $\phi_2 \in \text{Domain}(P^*)$.

(2) $(\gamma_m\phi_1, \phi_2)_{L^2(\partial M)} = 0$ for every $\phi_1 \in \text{Domain}(P)$.

(3) $\phi_2|_{\partial M} \in \{\gamma_m \mathcal{L}_1^{\leq}\}^\perp = (\mathcal{L}_2^{\geq})^\perp = \mathcal{L}_2^{<}$ i.e. $\phi_2 \in \ker \mathcal{B}_2$.

Thus $\mathcal{B}_2$ defines the adjoint boundary condition. As $\nabla\gamma = 0$,

$$\begin{aligned}
\nabla_a\gamma_m &= \gamma_m\nabla_a + \Gamma_{amb}\gamma_b = \gamma_m\nabla_a - L_{ab}\gamma_b,\\
B_2 :\ &= -\gamma_m B_1\gamma_m^{-1} = -\gamma_m\gamma_m^{-1}\gamma_a\nabla_a\gamma_m^{-1} - \psi_1\gamma_m^{-1}\\
&= \gamma_m^{-1}\gamma_a\nabla_a - L_{ab}\gamma_a\gamma_b - \psi_1\gamma_m^{-1} = \gamma_m^{-1}\gamma_a\nabla_a + L_{aa} - \psi_1\gamma_m^{-1},\\
A_2 :\ &= -\tfrac{1}{2}\gamma_m(B_1 + B_1^*)\gamma_m^{-1} - \gamma_m\Theta_1\gamma_m^{-1}\\
&= \tfrac{1}{2}(\gamma_m^{-1}\gamma_a\nabla_a + (\gamma_m^{-1}\gamma_a\nabla_a)^* - \psi_1\gamma_m^{-1} - \gamma_m\psi_1^*) + L_{aa} - \gamma_m\Theta_1\gamma_m^{-1}.
\end{aligned}$$

On the other hand since $\psi_2 = \psi_1^*$ and $\gamma_m^* = \gamma_m^{-1} = -\gamma_m$, we have

$$\begin{aligned}
&A_2 = \tfrac{1}{2}(\gamma_m^{-1}\gamma_a\nabla_a + (\gamma_m^{-1}\gamma_a\nabla_a)^* - \gamma_m\psi_1^* - \psi_1\gamma_m^{-1}) + \Theta_2 \text{ so}\\
&\Theta_2 = -\gamma_m\Theta_1\gamma_m^{-1} + L_{aa}. \square
\end{aligned}$$

We use functorial properties of the invariants a_n to establish the following Lemma. Recall that we defined

$$\beta(m) := \Gamma(\tfrac{m}{2})\Gamma(\tfrac{1}{2})^{-1}\Gamma(\tfrac{m+1}{2})^{-1}.$$

Lemma 4

1. *We have* $0 = d_1 = d_4 = d_7 = d_8 = d_{11} = d_{19} = e_2 = e_5 = e_7.$

2. *We have 2a)* $0 = e_3 = e_8$,
   2b) $0 = e_0 - (m-1)e_1$, *and*
   2c) $0 = e_4 - (m-1)e_6$.

3. *We may take* $d_{14} = 0$ *and* $d_{15} = 0$.

4. *We have* $0 = d_6 = d_{10}$.

5. *We have 5a)* $0 = d_{18}$,
   5b) $0 = 2(m-1)d_{12} + d_{13} - 2d_{16} + 2(1-m)d_{17} + (3-m)d_{20}$, *and*
   5c) $0 = 2(1-m)d_{12} + (1-m)d_{13} + (3-m)d_{21}$.

6. *We have 6a)* $0 = 2d_0 + d_2 + (m-3)(2d_3 + d_5)$,
   6b) $0 = -2d_0 + d_2 + (m-1)(2d_3 - d_5)$,
   6c) $0 = e_4 + (m-3)e_6$, *and*
   6d) $0 = d_9$.

7. *We have* $0 = -2d_0 + d_2 - (m-1)(2d_3 - d_5) - \frac{m-2}{4}(\beta(m) - 1)$.

8. *We have* $0 = \frac{1}{4}(\beta(m) - 1) + 2d_0 + d_2 + 2(m-1)d_3 + (m-1)d_5 + e_0 + e_1(m-1) - 2e_4 - 2e_6(m-1)$.

9. We have 9a) $2d_0+d_2=\frac{m-3}{8}(\frac{m-1}{m-2}\beta(m)-1)$
9b) $2d_3+d_5=-\frac{1}{8}(\frac{m-1}{m-2}\beta(m)-1)$*,and*
9c) $d_{12}=-\frac{1}{48}(\frac{m-1}{m-2}\beta(m)-1)$.

10. We have 10a) $d_{16}+(m-1)d_{17}=\frac{17-7m}{384}+\frac{4m-11}{48}\beta(m)$,
10b) $d_{20}=\frac{1}{8(m-3)}(\frac{5m-7}{8}-\frac{5m-9}{3}\beta(m))$*, and*
10c) $d_{21}=\frac{m-1}{16(m-3)}(-1+2\beta(m))$.

11. We have $d_{16}+d_{17}=\frac{1}{16(m^2-1)}(\frac{m^2+8m-17}{8}-(3m-4)\beta(m))$.

Remark We use equations (2c) and (6c) to see $e_4=e_6=0$. Equation (6a) is not independent from (9a) and (9b). Using (9a) and (9b) in (8), an equation for e_0 and e_1 follows. Together with (2b) this determines e_0 and e_1. We solve equations (6b), (7), (9a), and (9b) to determine d_0, d_2, d_3, and d_5. Thus we complete the proof of Theorem 1 (4) by checking that the non-zero coefficients are given by:

$d_0=\frac{1}{32}(1-\frac{\beta(m)}{m-2})$	$d_2=\frac{1}{16}(5-2m+\frac{7-8m+2m^2}{m-2}\beta(m))$
$d_3=\frac{1}{32(m-1)}(2m-3-\frac{2m^2-6m+5}{m-2}\beta(m))$	$d_5=\frac{1}{16(m-1)}(1+\frac{3-2m}{m-2}\beta(m))$
$d_{12}=-\frac{1}{48}(\frac{m-1}{m-2}\beta(m)-1)$	$d_{13}=\frac{1}{48}(1-\frac{4m-10}{m-2}\beta(m))$
$d_{16}=\frac{17+5m}{192(m+1)}+\frac{23-2m-4m^2}{48(m-2)(m+1)}\beta(m)$	$d_{17}=-\frac{17+7m^2}{384(m^2-1)}+\frac{4m^3-11m^2+5m-1}{48(m^2-1)(m-2)}\beta(m)$
$d_{20}=\frac{1}{8(m-3)}(\frac{5m-7}{8}-\frac{5m-9}{3}\beta(m))$	$d_{21}=\frac{m-1}{16(m-3)}(-1+2\beta(m))$
$e_0=\frac{1}{8(m-2)}\beta(m)$	$e_1=\frac{1}{8(m-1)(m-2)}\beta(m)$

Proof of (1). We shall always choose a real localizing (or smearing) function F. If the bundles E_i and the data (γ,ψ) are real, then a_3 is real. Thus the coefficients d_i are all real. Furthermore, since $D_{\mathcal{B}}$ is a self-adjoint operator, the invariant a_3 is real in the general case. Thus anti-Hermitian invariants must appear with zero coefficient. By equation (10), γ_a^T is skew-Hermitian. We assumed Θ is Hermitian. Assertion (1) now follows as the following terms are skew-Hermitian:

$$d_1F[\hat\psi\hat\psi-\hat\psi^*\hat\psi^*],\quad d_4F[\gamma_a^T\hat\psi\gamma_a^T\hat\psi-\gamma_a^T\hat\psi^*\gamma_a^T\hat\psi^*],\quad d_7F[\hat\psi_{;m}-\hat\psi^*_{;m}],\quad d_8F[\gamma_a^T\hat\psi_{:a}+\gamma_a^T\hat\psi^*_{:a}],$$
$$d_{11}FL_{aa}[\hat\psi-\hat\psi^*],\quad d_{19}F_{;m}[\hat\psi-\hat\psi^*],\quad e_2F\gamma_a^T\Theta_{:a},\quad e_5F\Theta[\hat\psi-\hat\psi^*],\quad e_7F\gamma_a^T\Theta\gamma_a^T[\hat\psi-\hat\psi^*].$$

Proof of (2). We consider the variation $\Theta(\varepsilon):=\Theta+\varepsilon$. For generic values of ε the kernel of the associated operator $A(\varepsilon)$ is trivial and the boundary condition remains unchanged and thus the invariants $a_3(\varepsilon)$ are unchanged at these values of ε. The invariants $a_3(\varepsilon)$ are locally computable. Thus a_3 is independent of ε. Assertion (2) now follows from the identity:

$$\begin{aligned}0=\partial_\varepsilon a_3|_{\varepsilon=0}&=\textstyle\int_{\partial M}\operatorname{Tr}\{2F(e_0+e_1\gamma_a^T\gamma_a^T)\Theta+Fe_3L_{aa}+F(e_4+e_6\gamma_a^T\gamma_a^T)(\hat\psi+\hat\psi^*)+e_8F_{;m}\}\\&=\textstyle\int_{\partial M}\operatorname{Tr}\{2F(e_0-(m-1)e_1)\Theta+Fe_3L_{aa}+e_8F_{;m}+F(e_4-(m-1)e_6)(\hat\psi+\hat\psi^*)\}.\end{aligned}$$

Proof of (3). We shall show that $\operatorname{Tr}\{W_{ab}\gamma_a^T\gamma_b^T\}=0$ and $\operatorname{Tr}\{W_{am}\gamma_a^T\}=0$ so these invariants play no role. Note that $W_{ab}=-W_{ba}$. Furthermore, $[W,\gamma]=0$ as noted above. We use equation (10) to compute:

$$\begin{aligned}&\operatorname{Tr}\{W_{ab}\gamma_a^T\gamma_b^T\}=\operatorname{Tr}\{\gamma_a^TW_{ab}\gamma_b^T\}=\operatorname{Tr}\{W_{ab}\gamma_b^T\gamma_a^T\}\text{ so}\\&\operatorname{Tr}\{W_{ab}\gamma_a^T\gamma_b^T\}=\tfrac12\operatorname{Tr}\{W_{ab}(\gamma_a^T\gamma_b^T+\gamma_b^T\gamma_a^T)\}=-\operatorname{Tr}\{W_{ab}\delta_{ab}\}=0.\end{aligned}$$

Since $m\neq2$, we may show $\operatorname{Tr}\{W_{am}\gamma_a^T\}=0$ by computing

$$\begin{aligned}&-(m-1)\operatorname{Tr}\{W_{am}\gamma_a^T\}=\operatorname{Tr}\{\gamma_b^T\gamma_b^TW_{am}\gamma_a^T\}=\operatorname{Tr}\{W_{am}\gamma_b^T\gamma_a^T\gamma_b^T\}\\&=\operatorname{Tr}\{W_{am}(-2\delta_{ab}\gamma_b^T-\gamma_a^T\gamma_b^T\gamma_b^T)\}=(-2+m-1)\operatorname{Tr}\{W_{am}\gamma_a^T\}.\end{aligned}$$

Proof of (4). We apply the local index theorem. Let M be the unit ball in $\mathbb{R}^m$ and let $E = E_1 = E_2 = Clif(M)$ be a trivial complex vector bundle of dimension 2^m over M. Let (γ, ∇) be the standard Clifford module structure and flat connection on E. Let ψ_1 be an arbitrary endomorphism of E and set $P_1 := \gamma^i \nabla_i + \psi_1 : C^\infty(E_1) \to C^\infty(E_2)$; the formal adjoint is then given by $P_2 := \gamma^i \nabla_i + \psi_1^*$ so $\psi_2 = \psi_1^*$. Let $D_1 := P_2 P_1$ and $D_2 := P_1 P_2$ with the appropriate boundary conditions $\mathcal{B}_i$. It follows from general principles that

$$\operatorname{Tr}\{e^{-t(D_1)_{\mathcal{B}_1}}\} - \operatorname{Tr}\{e^{-t(D_2)_{\mathcal{B}_2}}\} = \operatorname{index}(P_1, \mathcal{B}_1) \text{ so } a_3(D_1, \mathcal{B}_1) - a_3(D_2, \mathcal{B}_2) = 0. \tag{12}$$

We use Lemma 3 (2) to identify the adjoint boundary conditions and Θ_2 We use the equations of structure derived above and study the terms which are linear in ψ_1 in equation (12). Since $F = 1$, Lemma 3 (1) shows the terms involving d_9 play no role. Thus:

$$\begin{aligned}
&\textstyle\int_{\partial M} \operatorname{Tr}\{d_6(-\gamma_m \psi_{1;m} + \psi_{1;m}^* \gamma_m) + (d_{10} L_{aa} + e_4 \Theta_1 + e_6 \gamma_m \gamma_a \Theta_1 \gamma_m \gamma_a)(-\gamma_m \psi_1 + \psi_1^* \gamma_m)\}\\
= &\textstyle\int_{\partial M} \operatorname{Tr}\{d_6(-\gamma_m \psi_{1;m}^* + \psi_{1;m} \gamma_m) + (d_{10} L_{aa} + e_4 \gamma_m \Theta_1 \gamma_m + e_6 \gamma_m \gamma_a \gamma_m \Theta_1 \gamma_m \gamma_m \gamma_a)(-\gamma_m \psi_1^* + \psi_1 \gamma_m)\}\\
&\quad + \operatorname{Tr}\{e_4 + (1-m) e_6) L_{aa}(-\gamma_m \psi_1^* + \psi_1 \gamma_m)\}.
\end{aligned}$$

The terms which are bilinear in (Θ_1, ψ_1) and (Θ_1, ψ_1^*) agree. Since $e_4 = (m-1)e_6$, the final term vanishes. We set $\psi_1 = f(x_m)\gamma_m$ to conclude that $d_6 = 0$ and that $d_{10} = 0$. $\square$

Proof of (5). We use the method of conformal variations described in [10]. Let $\tilde{P}$ be the Dirac operator on the upper hemisphere. Then $\tilde{A}$ is the Dirac operator S^{m-1}. Since S^{m-1} has a metric of positive scalar curvature, $\ker(\tilde{A}) = \{0\}$ by the Lichnerowicz formula [22]. We now perturb $\tilde{P}$ slightly to define an operator of Dirac type P_0 on the ball which is formally self-adjoint. Let $A := \frac{1}{2}(B_0 + B_0^* + L_{aa})$. Since A is close to $\tilde{A}$, $\ker A = \{0\}$ so the realization of P is self-adjoint by Lemma 3. Let f be a smooth function on M. Let

$$\begin{aligned}
&ds^2(\varepsilon) := e^{2\varepsilon f} ds^2, && dvol(\varepsilon) = e^{m\varepsilon f} dvol,\\
&P(\varepsilon) := e^{-\frac{1+m}{2}\varepsilon f} P_0 e^{-\frac{1-m}{2}\varepsilon f}, && P^*(\varepsilon) := e^{(-\frac{1-m}{2} - m)\varepsilon f} P_0 e^{(m - \frac{1+m}{2})\varepsilon f}.
\end{aligned}$$

We fix the metric on the bundle E. The metric determined by the leading symbol of $P(\varepsilon)$ is $ds^2(\varepsilon)$ and $P(\varepsilon)$ is formally self-adjoint. We assume $f = f(x_m)$ and $f|_{\partial M} = 0$. Since $A(\varepsilon) - A_0 = \frac{m-1}{2}\varepsilon f_{;m}$ we set:

$$\Theta(\varepsilon) = \tfrac{1-m}{2}\varepsilon f_{;m} + \tfrac{1}{2} L_{aa}(0)$$

to ensure that the boundary conditions are unchanged. We use Lemma 3 and compute:

$$\begin{aligned}
L_{aa}(\varepsilon) &= -\tfrac{1}{2}\partial_m g_{aa}(\varepsilon) = L_{aa}(0) + (1-m)\varepsilon f_{;m}, \text{ and}\\
\Theta_2(\varepsilon) &= -\tfrac{1-m}{2}\varepsilon f_{;m} - \tfrac{1}{2}L_{aa}(0) + L_{aa}(\varepsilon) = \Theta_1(\varepsilon) + \varepsilon(-2\tfrac{1-m}{2} + (1-m)) f_{;m}\\
&= \Theta_1(\varepsilon).
\end{aligned}$$

Let $\delta := \partial_\varepsilon|_{\varepsilon=0}$. We compute

$$\begin{aligned}
&\delta \operatorname{Tr}_{L^2}\{e^{-tD(\varepsilon)}\} = -t \operatorname{Tr}_{L^2}\{\delta(D(\varepsilon)) e^{-tD_0}\} = -2t \operatorname{Tr}_{L^2}\{\delta(P(\varepsilon)) P_0 e^{-tD_0}\}\\
&\qquad = 2t \operatorname{Tr}_{L^2}\{f D_0 e^{-tD_0}\} = -2t \partial_t \operatorname{Tr}_{L^2}\{f e^{-tD_0}\}. \text{ Consequently}\\
&\delta a_3(1, D(\varepsilon), \mathcal{B}) = (m-3) a_3(f, D_0, \mathcal{B}).
\end{aligned} \tag{13}$$

We showed in [10] that there exists a compatible family of unitary connections ${}^\varepsilon\nabla$ so that

$$\psi(\varepsilon) = e^{-\varepsilon f}(\psi_0 - \varepsilon \tfrac{m-1}{2} f_{;i} \gamma_i).$$

Since $\hat{\psi}_0(\varepsilon) = -\gamma_m \psi_0 + \frac{1}{2}(1-m) f_{;m}$, we have:

$$\begin{aligned}
\delta \hat{\psi}_0 &= \tfrac{1-m}{2} f_{;m} = \delta \hat{\psi}_0^*,\\
\delta d_0 \operatorname{Tr}\{\hat{\psi}_0 \hat{\psi}_0 + \hat{\psi}_0 \hat{\psi}_0\} &= \tfrac{1-m}{2} f_{;m} 2 d_0 \operatorname{Tr}\{\hat{\psi}_0 + \hat{\psi}_0^*\},\\
\delta d_2 \operatorname{Tr}\{\hat{\psi}_0 \hat{\psi}_0^*\} &= \tfrac{1-m}{2} f_{;m} d_2 \operatorname{Tr}\{\hat{\psi}_0 + \hat{\psi}_0^*\},\\
\delta d_3 \operatorname{Tr}\{\gamma_a^T \hat{\psi}_0 \gamma_a^T \hat{\psi}_0 + \gamma_a^T \hat{\psi}_0^* \gamma_a^T \hat{\psi}_0^*\} &= \tfrac{1-m}{2} f_{;m} 2(1-m) d_3 \operatorname{Tr}\{\hat{\psi}_0 + \hat{\psi}_0^*\},\\
\delta d_5 \operatorname{Tr}\{\gamma_a^T \hat{\psi}_0 \gamma_a^T \hat{\psi}_0^*\} &= \tfrac{1-m}{2} f_{;m}(1-m) d_5 \operatorname{Tr}\{\hat{\psi}_0 + \hat{\psi}_0^*\}.
\end{aligned}$$

We use Lemma 3 to see $\delta\int_{\partial M}\operatorname{Tr}\{\gamma_a^T(\hat\psi_{0:a}-\hat\psi_{0:a}^*)\}=0$. We use computations from [6] to see

$$\begin{aligned}\delta d_{12}\tau&=d_{12}(-2(m-1)f_{;mm}+2(m-1)L_{aa}f_{;m}),\\ \delta d_{13}\rho_{mm}&=d_{13}(L_{aa}f_{;m}+(1-m)f_{;mm}),\\ \delta d_{16}L_{ab}L_{ab}&=-2d_{16}f_{;m}L_{aa}m\text{ and}\\ \delta d_{17}L_{aa}L_{bb}&=-2(m-1)d_{17}f_{;m}L_{aa}.\end{aligned}$$

We use equation (13) to see $\delta a_3(1,D(\varepsilon),\mathcal{B})+(3-m)a_3(f,D_0,\mathcal{B})=0$. We have $\delta\Theta=\frac12(1-m)f_{;m}$. Thus

$$\begin{aligned}\delta e_0\operatorname{Tr}\{\Theta^2\}&=(1-m)e_0f_{;m}\operatorname{Tr}\{\Theta\},\\ \delta e_1\operatorname{Tr}\{\gamma_a^T\Theta\gamma_a^T\Theta\}&=(1-m)(1-m)e_1f_{;m}\operatorname{Tr}\{\Theta\},\\ \delta e_4\operatorname{Tr}\{\Theta(\hat\psi_0+\hat\psi_0^*)\}&=\tfrac12(1-m)e_4f_{;m}\operatorname{Tr}\{\hat\psi_0+\hat\psi_0^*\}+(1-m)e_4f_{;m}\operatorname{Tr}\{\Theta\},\text{ and}\\ \delta e_6f_{;m}\operatorname{Tr}\{\gamma_a^T\Theta\gamma_a^T(\hat\psi_0+\hat\psi_0^*)\}&=\tfrac12(1-m)(1-m)e_6f_{;m}\operatorname{Tr}\{\hat\psi_0+\hat\psi_0^*\}+(1-m)(1-m)e_6f_{;m}\operatorname{Tr}\{\Theta\}.\end{aligned}$$

Since $e_0+(1-m)e_1=e_4+(1-m)e_6=0$, these terms play no role. Furthermore we have assumed $P=\gamma_i\nabla_i+\psi_0$ is self-adjoint. Thus $\psi_0=\psi_0^*$ and $\hat\psi_0+\hat\psi_0^*=-\gamma_m\psi_0+\psi_0\gamma_m$ and $\operatorname{Tr}\{\hat\psi_0+\hat\psi_0^*)=0$. Thus this term yields no information. We complete the proof of assertion (5) by computing:

$$\begin{aligned}0=&\textstyle\int_{\partial M}(3-m)d_{18}f_{;m}\operatorname{Tr}\{\psi_0\}\\ &+\{2(m-1)d_{12}+d_{13}-2d_{16}-2(m-1)d_{17}+(3-m)d_{20}\}f_{;m}\operatorname{Tr}\{L_{aa}\}\\ &+\{-2(m-1)d_{12}+(1-m)d_{13}+(3-m)d_{21}\}\operatorname{Tr}\{f_{;mm}\}.\end{aligned}$$

Proof of (6). We now exploit the fact that the connection ∇ is not canonically defined. We let M be the ball and let $E=E_1=E_2=Clif(\mathbb{R}^m)\otimes V$ where V is an auxiliary trivial vector bundle. Let $\sigma_i:=I\otimes\tilde\sigma_i$ be skew-adjoint endomorphisms of E commuting with the Clifford module structure γ. Let

$$\nabla_i(\varepsilon):=\nabla_i+\varepsilon\sigma_i$$

be a smooth 1 parameter family of unitary connections on E. Since $[\sigma_i,\gamma_j]=0$ for all i,j, we have $\nabla_i(\varepsilon)\gamma=0$ so this is an admissible family of connections. We define

$$\psi(\varepsilon):=\psi_0-\varepsilon\gamma_j\sigma_j$$

to ensure that $P(\varepsilon)=\gamma_i\nabla_i(\varepsilon)+\psi(\varepsilon)=P$ is unchanged during the perturbation. We have

$$\begin{aligned}B(\varepsilon)&=-\gamma_m(\gamma_a\nabla_a+\psi_0+\varepsilon\gamma_a\sigma_a-\varepsilon\gamma_i\sigma_i)=B_0-\varepsilon\sigma_m\text{ so}\\ A(\varepsilon)&=\tfrac12(B(\varepsilon)+B(\varepsilon)^*)+\Theta(\epsilon)=\tfrac12(B_0+B_0^*)+\Theta_0=A_0.\end{aligned}$$

Thus the boundary conditions are unchanged by the perturbation if we set $\Theta(\varepsilon):=\Theta_0$. Consequently, $a_3(F,D,\mathcal{B})$ is independent of the parameter ε. We compute

$$\begin{aligned}&\delta\hat\psi(\varepsilon)=-\gamma_b^T\sigma_b-\sigma_m,\\ &\delta\hat\psi(\varepsilon)^*=-\gamma_b^T\sigma_b+\sigma_m,\\ &\delta d_0\operatorname{Tr}\{\hat\psi(\varepsilon)\hat\psi(\varepsilon)+\hat\psi(\varepsilon)^*\hat\psi(\varepsilon)^*\}=2d_0\operatorname{Tr}\{-\gamma_b^T\sigma_b(\hat\psi_0+\hat\psi_0^*)-\sigma_m(\hat\psi_0-\hat\psi_0^*)\},\\ &\delta d_2\operatorname{Tr}\{\hat\psi(\varepsilon)\hat\psi(\varepsilon)^*\}=d_2\operatorname{Tr}\{-\gamma_b^T\sigma_b(\hat\psi_0+\hat\psi_0^*)+\sigma_m(\hat\psi_0-\hat\psi_0^*)\},\\ &\delta d_3\operatorname{Tr}\{\gamma_a^T\hat\psi(\varepsilon)\gamma_a^T\hat\psi(\varepsilon)+\gamma_a^T\hat\psi(\varepsilon)^*\gamma_a^T\hat\psi(\varepsilon)^*\}=2d_3\operatorname{Tr}\{-\gamma_a^T\gamma_b^T\sigma_b\gamma_a^T(\hat\psi_0+\hat\psi_0^*)-\gamma_a^T\sigma_m\gamma_a^T(\hat\psi_0-\hat\psi_0^*)\}\\ &\qquad=2d_3\operatorname{Tr}\{(m-3)(-\gamma_b^T\sigma_b)(\hat\psi_0+\hat\psi_0^*)+(m-1)\sigma_m(\hat\psi_0-\hat\psi_0^*)\},\\ &\delta d_5\operatorname{Tr}\{\gamma_a^T\hat\psi(\varepsilon)^*\gamma_a^T\hat\psi(\varepsilon)\}=d_5\operatorname{Tr}\{-\gamma_a^T\gamma_b^T\sigma_b\gamma_a^T(\hat\psi_0+\hat\psi_0^*)+\gamma_a^T\sigma_m\gamma_a^T(\hat\psi_0-\hat\psi_0^*)\}\\ &\qquad=d_5\operatorname{Tr}\{-(m-3)\gamma_b^T\sigma_b(\hat\psi_0+\hat\psi_0^*)+(1-m)\sigma_m(\hat\psi_0-\hat\psi_0^*)\},\\ &\delta d_9F_{:a}\operatorname{Tr}\{\gamma_a^T(\hat\psi(\varepsilon)-\hat\psi(\varepsilon)^*)\}=d_9F_{:a}\operatorname{Tr}\{-2\gamma_a^T\sigma_m\}=0,\\ &\delta e_4F\operatorname{Tr}\{\Theta(\hat\psi(\varepsilon)+\hat\psi(\varepsilon)^*)\}=-2e_4\operatorname{Tr}\{\Theta\gamma_b^T\sigma_b\},\text{ and}\\ &\delta e_6F\operatorname{Tr}\{\Theta\gamma_a^T(\hat\psi(\varepsilon)+\hat\psi(\varepsilon)^*)\gamma_a^T\}=-2e_6\operatorname{Tr}\{\Theta(m-3)\gamma_b^T\sigma_b\}.\end{aligned}$$

This yields the relation:

$$\begin{aligned}0=\textstyle\int_{\partial M}&\{2d_0+d_2+(m-3)(2d_3+d_5)\}\mathrm{Tr}\,\{-\gamma_b^T\sigma_b(\hat\psi_0+\hat\psi_0^*)\}\\&+\{-2d_0+d_2+(m-1)(2d_3-d_5)\}\mathrm{Tr}\,\{\sigma_m(\hat\psi_0-\hat\psi_0^*)\}\\&+\{-2e_4-2(m-3)e_6\}\mathrm{Tr}\,\{\Theta\gamma_b^T\sigma_b\}.\end{aligned}$$

To determine d_9 we extend the setting to an endomorphism valued smearing function. We study those terms which involve the tangential covariant derivatives of F. After taking into account the lack of commutativity, we see that these terms take the form:

$$\{u_1\mathrm{Tr}\,(F_{:a}\gamma_a^T(\hat\psi-\hat\psi^*)),u_2\mathrm{Tr}\,(F_{:a}\gamma_a^T(\hat\psi+\hat\psi^*)),u_3\mathrm{Tr}\,(F_{:a}(\hat\psi-\hat\psi^*)\gamma_a^T),u_4\mathrm{Tr}\,(F_{:a}(\hat\psi+\hat\psi^*)\gamma_a^T),u_5\mathrm{Tr}\,(F_{:a}\gamma_a^T\theta),u_6\mathrm{Tr}\,(F_{:a}\theta\gamma_a^T\}.$$

If F is then taken to be scalar, we see that $d_8=-u_2-u_4$, $d_9=-u_1-u_3$, and $e_2=-u_5-u_6$. We set $\psi_0=0$, $\theta=0$, and $\sigma_a=0$. Then $\delta(\hat\psi-\hat\psi^*)=-2\sigma_m$ and $\delta(\hat\psi+\hat\psi^*)=0$. Since σ_m commutes with γ_a^T, we get

$$0=-2(u_1+u_3)\mathrm{Tr}\,(F_{:a}\gamma_a^T\sigma_m)$$

since these are the only terms in the variation involving the covariant derivatives of F. (As $\mathrm{Tr}\,(\gamma_a^T\sigma_m)=0$, it is necessary to take $F_{:a}$ endomorphism valued for this argument to work). We can now conclude that $u_1+u_3=0$. This shows $d_9=0$ and completes the proof of assertion 6d). $\square$

Proof of (7). As in the proof of (5), let P_0 be a small perturbation of the Dirac operator on the upper hemisphere so that $\ker(A_0)=\{0\}$ where $A_0:=\frac12(B_0+B_0^*+L_{aa})$; the realization of P is self-adjoint. We consider a variation of the form $P(\varepsilon):=P+\varepsilon$. We then have $B(\varepsilon)=B_0-\gamma_m\varepsilon$ and thus $A(\varepsilon)=\frac12(B(\varepsilon)+B^*(\varepsilon)+L_{aa})=A_0$ is independent of the parameter ε. Thus $P(\varepsilon)$ is self-adjoint. If $\{\phi_k,\lambda_k\}$ is a spectral resolution of P, then $\{\phi_k,\lambda_k+\varepsilon\}$ will be a spectral resolution of $P(\varepsilon)$. We compute:

$$\begin{aligned}&\textstyle\sum_k\partial_\varepsilon^2\{a_k(1,P(\varepsilon)^2,\mathcal{B})\}|_{\varepsilon=0}t^{(n-m)/2}\sim\partial_\varepsilon^2\mathrm{Tr}\,\{e^{-t(P+\varepsilon)^2}\}|_{\varepsilon=0}\\&=\partial_\varepsilon\mathrm{Tr}\,\{-2t(P+\varepsilon)e^{-tP(\varepsilon)^2}\}|_{\varepsilon=0}=\mathrm{Tr}\,\{(-2t+4t^2P^2)e^{-tP^2}\}\\&=-2t\mathrm{Tr}\,\{(1+2t\partial_t)e^{-tP^2}\}\sim-2t\textstyle\sum_n\{1+(n-m)\}a_n(1,P^2,\mathcal{B})t^{(n-m)/2}.\end{aligned}$$

We take $(k,n)=(3,1)$ and equate the coefficient of $t^{(3-m)/2}$ in the two expansions to see:

$$\partial_\varepsilon^2a_3(1,P(\varepsilon)^2,\mathcal{B})=-2(2-m)a_1(1,P^2,\mathcal{B})\tag{14}$$

We use Theorem 1 to see

$$a_1(1,P^2,\mathcal{B})=(4\pi)^{-(m-1)/2}\tfrac14(\beta(m)-1)\textstyle\int_{\partial M}\mathrm{Tr}\,\{I\}\tag{15}$$

We have $\hat\psi(\varepsilon)=\hat\psi_0-\gamma_m\varepsilon$ and $\hat\psi(\varepsilon)^*=\hat\psi_0+\gamma_m\varepsilon$. Assertion (7) now follows from equations (14), (15), and the following identity:

$$\partial_\varepsilon^2a_3(1,P(\varepsilon)^2,\mathcal{B})=(4\pi)^{-(m-1)/2}\textstyle\int_{\partial M}\{-4d_0+2d_2-4(m-1)d_3+2(m-1)d_5\}\mathrm{Tr}\,\{I\}.\ \square$$

Proof of (8). As in the proof of (5), let P_0 be a small perturbation of the Dirac operator on the upper hemisphere so that P_0 is formally self-adjoint and so that $\ker(A_0)=\{0\}$ where $A_0:=\frac12(B_0+B_0^*+\Theta_0+L_{aa})$. We assume that the realization of P_0 is self-adjoint. We consider a variation of the form $P(\varepsilon):=P_0+\sqrt{-1}\varepsilon$. Then

$$\hat\psi(\varepsilon)=\hat\psi_0-\sqrt{-1}\varepsilon\gamma_m,\quad\hat\psi^*(\varepsilon)=\hat\psi_0^*-\sqrt{-1}\varepsilon\gamma_m,\ \text{so we set }\Theta(\varepsilon)=\Theta_0+\sqrt{-1}\varepsilon\gamma_m.$$

Then $A(\varepsilon)=A_0$ so the boundary condition is unchanged. Thus $P^*(\varepsilon)=P_0-\sqrt{-1}\varepsilon$ and $D=P^2+\varepsilon^2$. Consequently we have

$$\mathrm{Tr}\,\{e^{-tD(\varepsilon)}\}=e^{-t\varepsilon^2}\mathrm{Tr}\,\{e^{-tD_0}\}\ \text{so}\ a_3(1,D(\varepsilon),\mathcal{B})=a_3(1,D_0,\mathcal{B})-\varepsilon^2a_1(1,D_0,\mathcal{B}).\tag{16}$$

We compute:

$$
\begin{aligned}
d_0 \mathrm{Tr}\{\hat\psi\hat\psi+\hat\psi^*\hat\psi^*\}(\varepsilon) &= d_0\mathrm{Tr}\{\hat\psi_0\hat\psi_0+\hat\psi_0^*\hat\psi_0^*\}-2d_0\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*)\}+2d_0\varepsilon^2\mathrm{Tr}\{I\}\\
d_2\mathrm{Tr}\{\hat\psi\hat\psi^*\}(\varepsilon) &= d_2\mathrm{Tr}\{\hat\psi_0\hat\psi_0^*\}-d_2\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*)\}+d_2\varepsilon^2\mathrm{Tr}\{I\}\\
d_3\mathrm{Tr}\{\gamma_a^T\hat\psi\gamma_a^T\hat\psi+\gamma_a^T\hat\psi^*\gamma_a^T\hat\psi^*\}(\varepsilon) &= d_3\mathrm{Tr}\{\gamma_a^T\hat\psi_0\gamma_a^T\hat\psi_0+\gamma_a^T\hat\psi_0^*\gamma_a^T\hat\psi_0^*\}-2d_3(m-1)\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*)\}\\
&\quad +2(m-1)d_3\varepsilon^2\mathrm{Tr}\{I\}\\
d_5\mathrm{Tr}\{\gamma_a^T\hat\psi\gamma_a^T\hat\psi^*\}(\varepsilon) &= d_5\mathrm{Tr}\{\gamma_a^T\hat\psi_0\gamma_a^T\hat\psi_0^*\}-d_5(m-1)\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*)\}\\
&\quad +(m-1)d_5\varepsilon^2\mathrm{Tr}\{I\}\\
e_0\mathrm{Tr}\{\Theta\Theta\}(\varepsilon) &= e_0\mathrm{Tr}\{\Theta_0\Theta_0\}+2e_0\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m\Theta_0\}+e_0\varepsilon^2\mathrm{Tr}\{I\}\\
e_1\mathrm{Tr}\{\gamma_a^T\Theta\gamma_a^T\Theta\}(\varepsilon) &= e_1\mathrm{Tr}\{\gamma_a^T\Theta_0\gamma_a^T\Theta_0\}+2e_1(m-1)\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m\Theta_0\}+e_1(m-1)\varepsilon^2\mathrm{Tr}\{I\}\\
e_4\mathrm{Tr}\{\Theta(\hat\psi+\hat\psi^*)\}(\varepsilon) &= e_4\mathrm{Tr}\{\Theta_0(\hat\psi_0+\hat\psi_0^*)\}+e_4\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*-2\Theta_0)\}-2e_4\varepsilon^2\mathrm{Tr}\{I\}\\
e_6\mathrm{Tr}\{\gamma_a^T\Theta\gamma_a^T(\hat\psi+\hat\psi^*)\}(\varepsilon) &= e_6\mathrm{Tr}\{\gamma_a^T\Theta_0\gamma_a^T(\hat\psi_0+\hat\psi_0^*)+e_6(m-1)\sqrt{-1}\varepsilon\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*-2\Theta_0)\}\\
&\quad -2e_6(m-1)\varepsilon^2\mathrm{Tr}\{I\}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
0=\textstyle\int_{\partial M}&\{-2d_0-d_2-2d_3(m-1)-d_5(m-1)+e_4+(m-1)e_6\}\sqrt{-1}\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*)\}\\
&\{\ 2e_0+(m-1)e_1-2e_4-2(m-1)e_6\}\sqrt{-1}\mathrm{Tr}\{\gamma_m\Theta_0\}.
\end{aligned}
$$

To ensure that P_0 is self-adjoint, we must have $\gamma_m\Theta_0\gamma_m=\Theta_0+L_{aa}$. Thus, in particular $\mathrm{Tr}\{\gamma_m\Theta_0\}=0$. Furthermore $\psi_0=\psi_0^*$. Thus $\mathrm{Tr}\{\gamma_m(\hat\psi_0+\hat\psi_0^*)\}=\mathrm{Tr}\{\gamma_m(-\gamma_m\psi_0+\psi_0\gamma_m)\}=0$. Consequently the coefficient of ε produces no information. We use equation (16) to identify the coefficient of ε^2 and see

$$
\begin{aligned}
&(4\pi)^{-(m-1)/2}\textstyle\int_{\partial M}\mathrm{Tr}\{I\}\cdot\{2d_0+d_2+2(m-1)d_3+(m-1)d_5\\
&\qquad +e_0+e_1(m-1)-2e_4-2e_6(m-1)\}\\
&=-(4\pi)^{-(m-1)/2}\textstyle\int_{\partial M}\frac14(\beta(m)-1)\mathrm{Tr}\{I\}.\ \square
\end{aligned}
$$

Proof of (9). Grubb and Seeley [21] gave a complete description of the singularities of $\Gamma(s)\mathrm{Tr}\{FD_1^{-s}\}$ in the cylindrical case - i.e. when the structures are product near the boundary (see Theorem 2.1 [21] for details). We use the inward geodesic flow to identify a neighborhood of the boundary ∂M in M with the collar $\mathcal{C}=\partial M\times(-\epsilon,0]$. Let (y,x_m) be coordinates on $\mathcal{C}$. We suppose that $P=\gamma_m(\partial_m+A)$ on $\mathcal{C}$ where A is a tangential self-adjoint operator of Dirac type whose coefficients are independent of the normal variable x_m. Thus $A=\gamma_a^T\nabla_a+\hat\psi$ where $\hat\psi$ is self-adjoint. Since d_9 vanishes we may take $F=1$. We use Equation (13) [10] to see that:

$$
a_3(F,D,\mathcal{B})=\frac14\left(\frac{m-1}{m-2}\beta(m)-1\right)a_2(F,A^2). \tag{17}
$$

We use Theorem 4.1 [7] to see that:

$$
a_2(F,A^2)=-\tfrac{1}{12}(4\pi)^{-(m-1)/2}\textstyle\int_{\partial M}F\mathrm{Tr}\{R_{abba}+(12-6(m-1))\hat\psi\hat\psi+6\gamma_a^T\hat\psi\gamma_a^T\hat\psi\}. \tag{18}
$$

Assertion (9) now follows from equations (17), (18) and the computation:

$$
a_3(F,D,\mathcal{B})=(4\pi)^{-(m-1)/2}\textstyle\int_{\partial M}F[(2d_0+d_2)\mathrm{Tr}\{\hat\psi\hat\psi\}+(2d_3+d_5)\mathrm{Tr}\{\gamma_a^T\hat\psi\gamma_a^T\hat\psi\}+d_{12}\mathrm{Tr}\{I\}].\ \square
$$

Proof of (10). This follows from computations on the ball. We follow the description in [10] and extend the results to the ones needed for a_3. If $r\in[0,1]$ is the radial normal coordinate and if $d\Sigma^2$ is the usual metric on the unit sphere S^{m-1}, then $ds^2=dr^2+r^2d\Sigma^2$. The inward unit normal on the boundary is $-\partial_r$. The only nonvanishing components of the Christoffel symbols are

$$
\Gamma_{abc}=\frac1r\tilde\Gamma_{abc}\text{ and }\Gamma_{abm}=\frac1r\delta_{ab};
$$

the second fundamental form is given by $L_{ab}=\delta_{ab}$. We denote by $\tilde\Gamma_{abc}$ the Christoffel symbols associated with the metric $d\Sigma^2$ on the sphere S^{m-1} and tilde will always refer to this metric.

We will consider the Dirac operator $P=\gamma^\nu\partial_\nu$ on the ball; we take the flat connection ∇ and set $\psi=0$. We suppose m even (there is a corresponding decomposition for m odd) and use the following representation of the γ-matrices:

$$\gamma_{a(m)}=\begin{pmatrix}0 & \sqrt{-1}\cdot\gamma_{a(m-1)}\\ -\sqrt{-1}\cdot\gamma_{a(m-1)} & 0\end{pmatrix}\ \text{and}$$
$$\gamma_{m(m)}=\begin{pmatrix}0 & \sqrt{-1}\cdot 1_{m-1}\\ \sqrt{-1}\cdot 1_{m-1} & 0\end{pmatrix}.$$

We stress that $\gamma_{j(m)}$ are the γ-matrices projected along some vielbein system e_j. Decompose $\nabla_j=e_j+\omega_j$ where $\omega_j=\frac{1}{4}\Gamma_{jkl}\gamma_{k(m)}\gamma_{l(m)}$ is the connection 1 form of the spin connection. Note that

$$\nabla_a=\frac{1}{r}\left(\begin{pmatrix}\tilde\nabla_a & 0\\ 0 & \tilde\nabla_a\end{pmatrix}+\frac{1}{2}\gamma^T_{a(m)}\right).$$

Let $\tilde P$ the Dirac operator on the sphere. We have:

$$P=\left(\frac{\partial}{\partial x_m}-\frac{m-1}{2r}\right)\gamma_{m(m)}+\frac{1}{r}\begin{pmatrix}0 & \sqrt{-1}\tilde P\\ -\sqrt{-1}\tilde P & 0\end{pmatrix}.$$

Let d_s be the dimension of the spin bundle on the disk; $d_s=2^{m/2}$ if m is even. The spinor modes $\mathcal{Z}_\pm^{(n)}$ on the sphere are discussed in [8]. We have

$$\tilde P\mathcal{Z}_\pm^{(n)}(\Omega)=\pm\left(n+\frac{m-1}{2}\right)\mathcal{Z}_\pm^{(n)}(\Omega)\ \text{for}\ n=0,1,...;$$
$$d_n(m):=\dim\mathcal{Z}_\pm^{(n)}(\Omega)=\frac{1}{2}d_s\begin{pmatrix}m+n-2\\ n\end{pmatrix}.$$

Let $J_\nu(z)$ be the Bessel functions. These satisfy the differential equation [18]:

$$\frac{d^2J_\nu(z)}{dz^2}+\frac{1}{z}\frac{dJ_\nu(z)}{dz}+\left(1-\frac{\nu^2}{z^2}\right)J_\nu(z)=0.$$

Let $P\varphi_\pm=\pm\mu\varphi_\pm$ be an eigen function of P. Modulo a suitable radial normalizing constant C, we may express:

$$\varphi_\pm^{(+)}=\frac{C}{r^{(m-2)/2}}\begin{pmatrix}iJ_{n+m/2}(\mu r)\,Z_+^{(n)}(\Omega),\\ \pm J_{n+m/2-1}(\mu r)\,Z_+^{(n)}(\Omega)\end{pmatrix},\ \text{and} \tag{19}$$
$$\varphi_\pm^{(-)}=\frac{C}{r^{(m-2)/2}}\begin{pmatrix}\pm J_{n+m/2-1}(\mu r)\,Z_-^{(n)}(\Omega)\\ iJ_{n+m/2}(\mu r)\,Z_-^{(n)}(\Omega)\end{pmatrix}. \tag{20}$$

Let $\nabla_a^T:=\nabla_a-\frac{1}{2}L_{ab}\gamma^T_{b(m)}$. Then ∇^T is a compatible unitary connection for the induced Clifford modules structure γ^T; see [16] for details. The tangential operator B takes the form:

$$B=\gamma^T_{a(m)}\left(\nabla_a^T+\frac{1}{2}L_{ab}\gamma^T_{b(m)}\right)=\begin{pmatrix}-\tilde P-\frac{m-1}{2} & 0\\ 0 & \tilde P-\frac{m-1}{2}\end{pmatrix}.$$

We have in particular $B=B^*$. We take $\Theta=\frac{m-1}{2}1_m$. The operator A used to define spectral boundary conditions then reads

$$A=\begin{pmatrix}-\tilde P & 0\\ 0 & \tilde P\end{pmatrix}.$$

The eigenstates and eigenvalues of A then are easily determined:

$$A\begin{pmatrix}\mathcal{Z}_+^{(n)}(\Omega)\\ \mathcal{Z}_-^{(n)}(\Omega)\end{pmatrix}=-\left(n+\frac{m-1}{2}\right)\begin{pmatrix}\mathcal{Z}_+^{(n)}(\Omega)\\ \mathcal{Z}_-^{(n)}(\Omega)\end{pmatrix}\ \text{and}$$
$$A\begin{pmatrix}\mathcal{Z}_-^{(n)}(\Omega)\\ \mathcal{Z}_+^{(n)}(\Omega)\end{pmatrix}=\left(n+\frac{m-1}{2}\right)\begin{pmatrix}\mathcal{Z}_-^{(n)}(\Omega)\\ \mathcal{Z}_+^{(n)}(\Omega)\end{pmatrix}\ \text{for}\ n=0,1,....$$

The boundary condition suppresses the non-negative spectrum of A. Applying the boundary conditions on the solutions (19) and (20), we see that the non-negative modes of A are associated with the radial factor $J_{n+\frac{m}{2}-1}(\mu r)$. Hence the implicit eigenvalue equation is

$$J_p(\mu) = 0 \text{ where } p = n + \frac{m}{2} - 1. \tag{21}$$

In [4,5,9,12] a method has been developed for calculating the associated heat-kernel coefficients for smearing (or localizing) function $F = 1$; in [11] this has been generalized to $F = F(r)$. We summarize the essential results from these papers briefly; in principal one could calculate any number of coefficients. We first suppose that $F = 1$. Instead of looking directly at the heat-kernel we will consider the zeta-function $\zeta(s)$ of the operator P^2 and use the relationship between the pole structure of the zeta function and the asymptotics of the heat equation:

$$a_k = \text{Res }_{s=\frac{m-k}{2}} \Gamma(s)\zeta(s). \tag{22}$$

Thus to compute a_3, we must determine the residues of the zeta-function $\zeta(s)$ at the value $s = (m-3)/2$. We use the eigenvalue equation (21) to express

$$\zeta(s) = 4\sum_{n=0}^{\infty} d_n(m) \int_{\mathcal{C}} \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln J_p(k), \tag{23}$$

where the contour $\mathcal{C}$ runs counterclockwise and encloses all the solutions of (21) which lie on the positive real axis. The factor of four comes from the four types of solutions in (19) and (20). The representation equation (23) is well defined only for $\Re s > m/2$, so the first task is to construct the analytical continuation to the left. In order to do that, it is convenient to define a modified zeta function

$$\zeta^{(n)}(s) = \int_{\mathcal{C}} \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln k^{-p} J_p(k);$$

the additional factor k^{-p} has been introduced to avoid contributions coming from the origin. Since no additional pole is enclosed, the integral is unchanged.

It is the behaviour of $\zeta^{(n)}(s)$ as $n \to \infty$ which controls the convergence of the sum over n. The different orders in n can be studied by shifting the contour to the imaginary axis and by using the uniform asymptotic expansion of the resulting Bessel function $I_p(k)$. To ensure that the resulting expression converges for some range of s when shifting the contour to the imaginary axis, we add a small positive constant to the eigenvalues. For s in the strip $1/2 < \Re s < 1$, we have:

$$\zeta^{(n)}(s) = \frac{\sin(\pi s)}{\pi} \int_{\epsilon}^{\infty} dk (k^2 - \epsilon^2)^{-s} \frac{\partial}{\partial k} \ln k^{-p} I_p(k).$$

We introduce some additional notation dealing with the uniform asymptotic expansion of the Bessel function. For $p \to \infty$ with $z = k/p$ fixed, we use results of [1] to see that:

$$I_p(zp) \sim \frac{1}{\sqrt{2\pi p}} \frac{e^{p\eta}}{(1+z^2)^{1/4}} \left[1 + \sum_{l=1}^{\infty} \frac{u_l(t)}{p^l}\right] \text{ where} \tag{24}$$
$$t = 1/\sqrt{1+z^2} \text{ and } \eta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})].$$

Let $u_0(t) = 1$. We use the recursion relationship given in [1] to determine the polynomials $u_l(t)$ which appear in equation (24):

$$u_{l+1}(t) = \frac{1}{2} t^2 (1-t^2) u_l'(t) + \frac{1}{8} \int_0^t d\tau (1 - 5\tau^2) u_l(\tau).$$

We also need the coefficients $D_m(t)$ defined by the cumulant expansion:

$$\ln\left[1 + \sum_{l=1}^{\infty} \frac{u_l(t)}{p^l}\right] \sim \sum_{q=1}^{\infty} \frac{D_q(t)}{p^q}. \tag{25}$$

The eigenvalue multiplicities $d_n(m)$ are $\mathcal{O}(n^{m-2})$ as $n \to \infty$. Consequently, the leading behaviour of every term is of the order of $p^{-2s-q+m-2}$; thus on the half plane $\Re s > (m-4)/2$, only the values $q = 1$ and $q = 2$ contribute to the residues of the zeta-function. We have

$$D_1(t) = \frac{1}{8}t - \frac{5}{24}t^3, \text{ and } D_2(t) = \frac{1}{16}t^2 - \frac{3}{8}t^4 + \frac{5}{16}t^6.$$

We use equation (24) to decompose

$$\begin{aligned}
\zeta^{(n)}(s) &= A_{-1}^{(n)}(s) + A_0^{(n)}(s) + A_1^{(n)}(s) + R^{(n)}(s), \text{ where} \\
A_{-1}^{(n)}(s) &= \frac{\sin \pi s}{\pi} \int_{\epsilon/p}^{\infty} dz[(zp)^2 - \epsilon^2]^{-s} \frac{\partial}{\partial z} \ln\left(z^{-p} e^{p\eta}\right), \\
A_0^{(n)}(s) &= \frac{\sin \pi s}{\pi} \int_{\epsilon/p}^{\infty} dz[(zp)^2 - \epsilon^2]^{-s} \frac{\partial}{\partial z} \ln\left(1 + z^2\right)^{-1/4}, \\
A_q^{(n)}(s) &= \frac{\sin \pi s}{\pi} \int_{\epsilon/p}^{\infty} dz[(zp)^2 - \epsilon^2]^{-s} \frac{\partial}{\partial z} \left(\frac{D_q(t)}{p^q}\right).
\end{aligned}$$

The remainder $R^{(n)}(s)$ is such that $\sum_{n=0}^{\infty} d_n(m) R^{(n)}(s)$ is analytic on the half plane $\Re s > (m-4)/2$.

Let ${}_2F_1$ be the hypergeometric function. We have

$$\begin{aligned}
{}_2F_1(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}, \text{ and} \\
\int_{\epsilon/p}^{\infty} dz\ [(zp)^2 - \epsilon^2]^{-s} \frac{\partial}{\partial z} t^l &= -\frac{l}{2} \frac{\Gamma(s + \frac{l}{2})\Gamma(1-s)}{\Gamma(1 + \frac{l}{2})} p^l [\epsilon^2 + p^2]^{-s-l/2}.
\end{aligned}$$

We use the first identity to study $A_{-1}^{(n)}(s)$ and $A_0^{(n)}(s)$; we use the second identity to study $A_1^{(n)}(s)$ and $A_2^{(n)}(s)$. This shows that

$$\begin{aligned}
A_{-1}^{(n)}(s) &= \frac{\epsilon^{-2s+1}}{2\Gamma(\frac{1}{2})} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} {}_2F_1(-\frac{1}{2}, s - \frac{1}{2}; \frac{1}{2}; -(\frac{p}{\epsilon})^2) - \frac{p}{2}\epsilon^{-2s} \\
A_0^{(n)}(s) &= -\frac{1}{4}(p^2 + \epsilon^2)^{-s}, \\
A_1^{(n)}(s) &= \frac{1}{8} \frac{1}{\Gamma(s)} \left[-\frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{1}{2})} (p^2 + \epsilon^2)^{-s-\frac{1}{2}} \right] \\
&\quad - \frac{5}{24} \frac{1}{\Gamma(s)} \left[-2 \frac{\Gamma(s + \frac{3}{2})}{\Gamma(\frac{1}{2})} p^2 (p^2 + \epsilon^2)^{-s-\frac{3}{2}} \right], \\
A_2^{(n)}(s) &= \frac{1}{16} \frac{1}{\Gamma(s)} \left[-\Gamma(s+1)(p^2 + \epsilon^2)^{-s-1} \right] \\
&\quad - \frac{3}{8} \frac{1}{\Gamma(s)} \left[-\Gamma(s+2) p^2 (p^2 + \epsilon^2)^{-s-2} \right] \\
&\quad + \frac{5}{16} \frac{1}{\Gamma(s)} \left[-\frac{1}{2}\Gamma(s+3) p^4 (p^2 + \epsilon^2)^{-s-3} \right].
\end{aligned}$$

In the limit $\epsilon \to 0$, the resulting zeta-function which appears is connected to the spectrum on the sphere. Let $d := m - 1$. We define the base zeta-function ζ_{S^d} and the Barnes zeta-function [3] $\zeta_{\mathcal{B}}$,

$$\zeta_{S^d}(s) = 4 \sum_{n=0}^{\infty} d_n(m) p^{-2s} \text{ and } \zeta_{\mathcal{B}}(s, a) = \sum_{n=0}^{\infty} d_n(m)(n + a)^{-s}.$$

We then have the relation $\zeta_{S^d}(s) = 2 d_s \zeta_{\mathcal{B}}\left(2s, \frac{m}{2} - 1\right)$. For $i = -1$, $i = 0$, $i = 1$ and $i = 2$, we shall define $A_i(s) = 4 \sum_{n=0}^{\infty} d_n(m) A_i^{(n)}(s)$. We take the limit as $\epsilon \to 0$ to see that

$$A_{-1}(s) = \frac{1}{4\Gamma(\frac{1}{2})} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)} \zeta_{S^d}(s-\frac{1}{2}), \tag{26}$$

$$A_0(s) = -\frac{1}{4}\zeta_{S^d}(s), \tag{27}$$

$$A_1(s) = -\frac{1}{\Gamma(s)}\zeta_{S^d}(s+\frac{1}{2}) \left[\frac{1}{8\Gamma(\frac{1}{2})}\Gamma(s+\frac{1}{2}) - \frac{5}{12\Gamma(\frac{1}{2})}\Gamma(s+\frac{3}{2})\right], \tag{28}$$

$$A_2(s) = -\frac{1}{\Gamma(s)}\zeta_{S^d}(s+1) \left[\frac{1}{16}\Gamma(s+1) - \frac{3}{8}\Gamma(s+2) + \frac{5}{32}\Gamma(s+3)\right]. \tag{29}$$

We used the Mellin-Barnes integral representation of the hypergeometric functions [18] to calculate $A_{-1}(s)$:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{\mathcal{C}} dt\ \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)}(-z)^t. \tag{30}$$

The contour of integration is such that the poles of $\Gamma(a+t)\Gamma(b+t)/\Gamma(c+t)$ lie to the left of the contour and so that the poles of $\Gamma(-t)$ lie to the right of the contour. We stress that before interchanging the sum and the integral, we must shift the contour $\mathcal{C}$ over the pole at $t=1/2$ to the left; this cancels the term $-\frac{p}{2}\epsilon^{-2s}$ appearing in the expression for A_{-1} above.

This reduces the analysis of the zeta function on the ball to analysis of a zeta function on the boundary. We compute the residues of $\zeta(s)$ from the residues of $\zeta_{\mathcal{B}}(s,a)$. To compute these residues, we first express $\zeta_{\mathcal{B}}(s,a)$ as a contour integral. Let $\mathcal{C}$ be the Hankel contour.

$$\begin{aligned}
\zeta_{\mathcal{B}}(s,a) &= \sum_{n=0}^{\infty} \begin{pmatrix} d+n-1 \\ n \end{pmatrix} (n+a)^{-s} = \sum_{\vec{m}\in\mathbb{N}_0^d} (a+m_1+...+m_d)^{-s} \\
&= \frac{\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} dt\ (-t)^{s-1} \frac{e^{-at}}{(1-e^{-t})^d}.
\end{aligned}$$

The residues of $\zeta_{\mathcal{B}}(s,a)$ are intimately connected with the generalized Bernoulli polynomials [24],

$$\frac{e^{-at}}{(1-e^{-t})^d} = (-1)^d \sum_{n=0}^{\infty} \frac{(-t)^{n-d}}{n!} B_n^{(d)}(a). \tag{31}$$

We use the residue theorem to see that

$$\text{Res}\ _{s=z}\zeta_{\mathcal{B}}(s,a) = \frac{(-1)^{d+z}}{(z-1)!(d-z)!} B_{d-z}^{(d)}(a), \tag{32}$$

for $z=1,...,d$. The needed leading poles are

$$\begin{aligned}
\text{Res}\ _{s=d}\zeta_{\mathcal{B}}(s,a) &= \frac{1}{(d-1)!}, \\
\text{Res}\ _{s=d-1}\zeta_{\mathcal{B}}(s,a) &= \frac{d-2a}{2(d-2)!}, \\
\text{Res}\ _{s=d-2}\zeta_{\mathcal{B}}(s,a) &= \frac{12a^2-d-12ad+3d^2}{24(d-3)!}, \\
\text{Res}\ _{s=d-3}\zeta_{\mathcal{B}}(s,a) &= \frac{-8a^3+12a^2d+2a-6ad^2-d^2+d^3}{48(d-4)!}.
\end{aligned}$$

We may now determine the residues of $\zeta(s)$. At $s=\frac{m-3}{2}=\frac{d-2}{2}$ we find

$$\begin{aligned}
\text{Res}\ _{s=\frac{m-3}{2}} A_{-1}(s) &= -\frac{d_s}{6} \frac{m-2}{2^m\Gamma((m-1)/2)\Gamma((m-3)/2)}, \\
\text{Res}\ _{s=\frac{m-3}{2}} A_0(s) &= \frac{d_s}{96\Gamma(m-4)}, \\
\text{Res}\ _{s=\frac{m-3}{2}} A_1(s) &= \frac{d_s}{6} \frac{(5m-13)}{2^m\Gamma((m-1)/2)\Gamma((m-3)/2)}, \\
\text{Res}\ _{s=\frac{m-3}{2}} A_2(s) &= -\frac{d_s}{256} \frac{(m-3)^2(5m-9)}{\Gamma(m-1)}.
\end{aligned}$$

To get these representations, the 'doubling formula' $\frac{\Gamma(z)}{\Gamma(2z)} = \frac{\sqrt{2\pi}2^{1/2-2z}}{\Gamma(z+1/2)}$ for the Γ function and its functional relation $\Gamma(z+1) = z\Gamma(z)$ has been used. Summing up, using again the given properties of the Γ-functions and (22) for the heat-kernel coefficient a_3, we find

$$\begin{aligned} a_3 &= 2^{-5-m}(m-1)d_s \frac{8(4m-11)\Gamma(m/2) + (17-7m)\Gamma(1/2)\Gamma((m+1)/2)}{3\Gamma(m/2)\Gamma((m+1)/2)} \\ &= (4\pi)^{-(m-1)/2} \int_{S^{m-1}} \mathrm{Tr} \left[\frac{(4m-11)(m-1)\Gamma(m/2)}{48\Gamma(1/2)\Gamma((m+1)/2)} + \frac{(17-7m)(m-1)}{384} \right] \\ &= (4\pi)^{-(m-1)/2} \int_{S^{m-1}} \mathrm{Tr} \left[d_{16}(m-1) + d_{17}(m-1)^2 \right]. \end{aligned}$$

Form here, equation (10a) is immediate.

To get equations (10b) and (10c) we need to introduce a smearing function. For our purposes a smearing function of the form $F(r) = f_0 + f_1 r^2 + f_2 r^4$ is suitable. We note that the radial normalization constant is given by $C = 1/J_{p+1}(\mu)$. We denote the normalized Bessel function by

$$\bar{J}_p(\mu r) := J_p(\mu r)/J_{p+1}(\mu).$$

Instead of the zeta function we consider now the smeared analogue:

$$\zeta(F;s) = \sum_\lambda \int_{B^m} F(x)\varphi^*(x)\varphi(x)\frac{1}{\lambda^{2s}}. \tag{33}$$

Since F depends only on the normal variable, the integral in equation (33) over the sphere S^{m-1} behaves as in the case $F = 1$ so that

$$\zeta(F;s) = 4\sum_{n=0}^{\infty} d_n(m) \int_{\mathcal{C}} \frac{dk}{2\pi i} k^{-2s} \tag{34}$$

$$\cdot \int_0^1 dr F(r) r(\bar{J}^2_{p+1}(kr) + \bar{J}^2_p(kr)) \frac{\partial}{\partial k} \ln J_p(k). \tag{35}$$

The radial integrals may be computed using Schafheitlin's reduction formula [25]:

$$\begin{aligned} &(j+2)\int^z dx x^{j+2} J^2_\nu(x) = (j+1)\left\{\nu^2 - \frac{(j+1)^2}{4}\right\}\int^z dx x^j J^2_\nu(x) \\ &+\frac{1}{2}\left[z^{j+1}\left\{zJ'_\nu(z) - \frac{1}{2}(j+1)J_\nu(z)\right\}^2 + z^{j+1}\left\{z^2 - \nu^2 + \frac{1}{4}(j+1)^2\right\}J^2_\nu(z)\right]. \end{aligned}$$

For the case at hand, using $J_p(\mu) = 0$, we find the radial integrals

$$\begin{aligned} \int_0^1 dr\ r^3 \left[\bar{J}^2_p(\mu r) + \bar{J}^2_{p+1}(\mu r)\right] &= \frac{2p^2+3p+1}{3\mu^2} + \frac{1}{3}, \\ \int_0^1 dr\ r^5 \left[\bar{J}^2_p(\mu r) + \bar{J}^2_{p+1}(\mu r)\right] &= \frac{8p^4+20p^3-20p-8}{15\mu^4} + \frac{4p^2+10p+4}{15\mu^2} + \frac{1}{5}. \end{aligned}$$

Substituting these into (35) the contour integral representations for $\zeta(r^2;s)$ and $\zeta(r^4;s)$ are easily given. The resulting expressions are evaluated using equation (23); simple substitutions suffice to evaluate all relevant terms analogous to (26)—(29). The factors of $1/\mu^2$ and $1/\mu^4$ are absorbed by using $s+1$ and $s+2$ instead of s in equations (26)—(29). The powers of p lower the argument of the base zeta function by 2, by 3/2, by 1, by 1/2 and by 0. It is now a straightforward matter to compute:

$$\begin{aligned} A_{-1}(r^2;s) = &\frac{1}{4\Gamma(\frac{1}{2})}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)}\zeta_{S^d}(s-\frac{1}{2})\left[\frac{1}{3} + \frac{2}{3}\frac{s-\frac{1}{2}}{s+1}\right] \\ &+\frac{1}{4\Gamma(\frac{1}{2})}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+2)}\left[\zeta_{S^d}(s) + \frac{1}{3}\zeta_{S^d}(s+\frac{1}{2})\right], \end{aligned}$$

$$
\begin{aligned}
A_0(r^2;s) &= -\frac{1}{4}\zeta_{S^d}(s) - \frac{1}{4}\zeta_{S^d}(s+\frac{1}{2}) - \frac{1}{12}\zeta_{S^d}(s+1),\\
A_1(r^2;s) &= -\frac{2}{3\Gamma(s+1)}\zeta_{S^d}(s+\frac{1}{2})\left[\frac{1}{8\Gamma(\frac{1}{2})}\Gamma(s+\frac{3}{2}) - \frac{5}{12\Gamma(\frac{1}{2})}\Gamma(s+\frac{5}{2})\right]\\
&\quad -\frac{1}{3\Gamma(s)}\zeta_{S^d}(s+\frac{1}{2})\left[\frac{1}{8\Gamma(\frac{1}{2})}\Gamma(s+\frac{1}{2}) - \frac{5}{12\Gamma(\frac{1}{2})}\Gamma(s+\frac{3}{2})\right]\\
&\quad -\frac{1}{\Gamma(s+1)}\zeta_{S^d}(s+1)\left[\frac{1}{8\Gamma(1/2)}\Gamma(s+3/2) - \frac{5}{12\Gamma(1/2)}\Gamma(s+5/2)\right] + \dots\ ,\\
A_2(r^2;s) &= -\frac{2}{3\Gamma(s+1)}\zeta_{S^d}(s+1)\left[\frac{1}{16}\Gamma(s+2) - \frac{3}{8}\Gamma(s+3) + \frac{5}{32}\Gamma(s+4)\right]\\
&\quad +\frac{1}{3\Gamma(s)}\zeta_{S^d}(s+1)\left[\frac{1}{16}\Gamma(s+1) - \frac{3}{8}\Gamma(s+2) + \frac{5}{32}\Gamma(s+3)\right] + \dots
\end{aligned}
$$

This exemplifies very well the rules of substitution and we spare to write down the associated terms for $\zeta(r^4;s)$ explicitly.

Although lengthy, it is again easy to add up all contributions to find $a_3(F,D,\mathcal{B})$ for the smearing function given by $F(r) = f_0 + f_1 r^2 + f_2 r^4$. We derive equations (10b) and (10c) by identifying the boundary invariants:

$$
\begin{aligned}
F(1) &= F\,|_{\partial M} = f_0 + f_1 + f_2,\\
F'(1) &= -F_{;m}\,|_{\partial M} = 2f_1 + 4f_2,\\
F''(1) &= F_{;mm}\,|_{\partial M} = 2f_1 + 12f_2.\ \square
\end{aligned}
$$

Proof of (11). We give the ball B^2 the usual metric $ds_B^2 = dr^2 + r^2 d\theta^2$. Let N be a compact Riemannian manifold without boundary and let $M = B^2 \times N$ with the product metric. The extrinsic curvature is $L_{\theta\theta} = 1$, $L_{ab} = 0$ otherwise. Let $\tilde{P}$ be the Dirac operator on N. The Dirac operator P on M reads

$$
P = \left(\frac{\partial}{\partial x_m} - \frac{1}{2r}\right)\gamma_{m(m)} + \frac{1}{r}\begin{pmatrix} 0 & \sqrt{-1}\gamma_{\theta(m-1)}\\ -\sqrt{-1}\gamma_{\theta(m-1)} & 0\end{pmatrix}\partial_\theta + \begin{pmatrix} 0 & \sqrt{-1}\tilde{P}\\ -\sqrt{-1}\tilde{P} & 0\end{pmatrix}. \tag{36}
$$

Write the eigenfunction φ of P, $P\varphi = \mu\varphi$, in the form $\varphi = \binom{\psi_1}{\psi_2}$. Let $\mathcal{Z}_n$ be an eigenfunction of $\tilde{P}$. An ansatz of the form $\psi_1 = f(r)e^{i(m+1/2)\theta}\mathcal{Z}_n$ is not possible because $\gamma_{\theta(m-1)}$ and $\tilde{P}$ anticommute. A simultaneous set of eigenfunctions of ∂_θ and $\tilde{P}$ thus does not exist. However, $\gamma_{\theta(m-1)}$ plays the role of 'γ^5' for the γ-matrices on N. Therefore, define $\mathcal{Z}_n^\pm$ to be the upper and lower chirality parts of $\mathcal{Z}_n$,

$$
\mathcal{Z}_n^\pm := \frac{1}{\sqrt{2}}\left(1 \pm \sqrt{-1}\gamma_{\theta(m-1)}\right)\mathcal{Z}_n.
$$

Consequently

$$
\tilde{P}\mathcal{Z}_n^\pm = \lambda_n \mathcal{Z}_n^\mp \text{ and } \tilde{P}^2\mathcal{Z}_n^\pm = \lambda_n^2\mathcal{Z}_n^\pm,
$$

and $\psi_1 = f(r)e^{i(m+1/2)\theta}\mathcal{Z}_n^\pm$ might be chosen. A full set of eigenfunctions is then found to read

$$
\varphi_1^{(\pm)} = e^{i(m+1/2)\theta}\begin{pmatrix} J_{m+1}(\sqrt{\mu^2-\lambda_n^2}r)\mathcal{Z}_n^+\\ \mp\frac{i}{\mu}\sqrt{\mu^2-\lambda_n^2}J_m(\sqrt{\mu^2-\lambda_n^2}r)\mathcal{Z}_n^+ \mp \frac{i\lambda_n}{\mu}J_{m+1}(\sqrt{\mu^2-\lambda_n^2}r)\mathcal{Z}_n^-\end{pmatrix}, \tag{37}
$$

$$
\varphi_2^{(\pm)} = e^{i(m+1/2)\theta}\begin{pmatrix} J_m(\sqrt{\mu^2-\lambda_n^2}r)\mathcal{Z}_n^-\\ \pm\frac{i}{k}\sqrt{\mu^2-\lambda_n^2}J_{m+1}(\sqrt{\mu^2-\lambda_n^2}r)\mathcal{Z}_n^- \mp \frac{i\lambda_n}{\mu}J_m(\sqrt{\mu^2-\lambda_n^2}r)\mathcal{Z}_n^+\end{pmatrix}. \tag{38}
$$

We need to impose spectral boundary conditions. We choose $\theta = 1/2$. the boundary operator reads

$$
A = \begin{pmatrix} -\gamma_{\theta(m-1)} & 0\\ 0 & +\gamma_{\theta(m-1)}\end{pmatrix}\partial_\theta + \begin{pmatrix} -\tilde{P} & 0\\ 0 & \tilde{P}\end{pmatrix}
$$

and we need the projection on its non-negative spectrum. Obviously one chooses the ansatz $\alpha = \binom{\alpha_1}{\alpha_2}$ as eigenspinor of A and gets the equations

$$
\begin{aligned}
-\gamma_{\theta(m-1)}\partial_\theta\alpha_1 - \tilde{P}\alpha_1 &= E_t\alpha_1, \\
\gamma_{\theta(m-1)}\partial_\theta\alpha_2 + \tilde{P}\alpha_2 &= E_t\alpha_2,
\end{aligned}
\tag{39}
$$

Define $b_\pm = \frac{m+1/2\pm\sqrt{\lambda_n^2+(m+1/2)^2}}{\lambda}$. Expand α_1 and α_2 in terms of $\mathcal{Z}_n^\pm$. Then eigenfunctions are given by:

$$
\begin{aligned}
&\alpha_1^{(\mp)} = e^{i(m+1/2)\theta}(b_\pm \mathcal{Z}_n^+ + \mathcal{Z}_n^-) \text{ and } \alpha_2^{(\mp)} = e^{i(m+1/2)\theta}(b_\mp \mathcal{Z}_n^+ + \mathcal{Z}_n^-), \text{ where} \\
&A\alpha^\mp = \mp\sqrt{\lambda_n^2 + (m+1/2)^2}\alpha^\mp.
\end{aligned}
\tag{40}
$$

Imposing spectral boundary conditions so means that the projection on all eigenfunctions α^+ has to vanish. Boundary conditions can not be imposed on $\varphi_1^{(\pm)}$ and $\varphi_2^{(\pm)}$, but instead on suitable linear combinations. Define

$$
a_\mp = \frac{\lambda_n \mp \mu}{\sqrt{\mu^2 - \lambda_n^2}},
$$

and impose boundary conditions on $\varphi_1 + a_\mp\varphi_2$. This gives the conditions, using $b_- b_+ = -1$,

$$
\begin{aligned}
J_m(\sqrt{\mu^2 - \lambda_n^2}) + \frac{b_-}{a_-} J_{m+1}(\sqrt{\mu^2 - \lambda_n^2}) &= 0, \\
J_m(\sqrt{\mu^2 - \lambda_n^2}) + \frac{b_-}{a_+} J_{m+1}(\sqrt{\mu^2 - \lambda_n^2}) &= 0.
\end{aligned}
$$

With $a_- a_+ = -1$ this can be combined to read

$$
J_m^2(\sqrt{\mu^2 - \lambda_n^2}) - \frac{2\lambda_n b_-}{\sqrt{\mu^2 - \lambda_n^2}} J_m(\sqrt{\mu^2 - \lambda_n^2}) J_{m+1}(\sqrt{\mu^2 - \lambda_n^2}) - b_-^2 J_{m+1}(\sqrt{\mu^2 - \lambda_n^2}) = 0.
$$

So the starting point for the zeta function with smearing function $F = 1$ is

$$
\zeta(s) = \sum_{m=-\infty}^{\infty} \sum_n \int_{\mathcal{C}} \frac{dk}{2\pi i} (k^2 + \lambda_n^2)^{-s} \times \frac{\partial}{\partial k} \ln\left\{ J_m^2(k) - \frac{2\lambda_n b_-}{k} J_m(k) J_{m+1}(k) - b_-^2 J_{m+1}^2(k) \right\}.
$$

Using for $l \in \mathbb{N}$, $J_{-l}(k) = (-1)^l J_l(k)$ and shifting the contour to the imaginary axis we find

$$
\begin{aligned}
\zeta(s) = \frac{2\sin(\pi s)}{\pi} &\sum_{m=0}^{\infty} \sum_n \int_{|\lambda_n|} dk \ (k^2 - \lambda_n^2)^{-s} \times \\
&\frac{\partial}{\partial k} \ln\left\{ k^{-2m} \left[I_m^2(k) - \frac{2\lambda_n b_-}{k} I_m(k) I_{m+1}(k) + b_-^2 I_{m+1}^2(k) \right] \right\}.
\end{aligned}
$$

The role of the base zeta function will here be played by the zeta function associated with A^2. We thus define (actually, this is only 1/2 the zeta function because the sum over m runs from $m = 0$ only instead of $m = -\infty$)

$$
\zeta_A(s) = \sum_{m=0}^{\infty} \sum_n \left[(m+1/2)^2 + \lambda_n^2 \right]^{-s}
\tag{41}
$$

and will need furthermore

$$
\zeta_A^l(s) = \sum_{m=0}^{\infty} \sum_n \frac{(m+1/2)^l}{[(m+1/2)^2 + \lambda_n^2]^s}.
\tag{42}
$$

This suggests, that a suitable expansion parameter is $\nu = m + 1/2$. We define

$$
\delta = \frac{\nu}{\sqrt{\nu^2 + \lambda_n^2}}
$$

and have the following relations,

$$\delta = \frac{1-b_-^2}{1+b_-^2}, \quad \frac{\delta-1}{\delta} = \frac{\lambda_n}{\nu} b_-, \quad b_-^2 = \frac{1-\delta}{1+\delta}, \quad 1+b_-^2 = \frac{2}{1+\delta}.$$

In addition, the zeta function associated with the spectrum λ_n of the manifold N will naturally appear in the calculations,

$$\zeta_N(s) = \sum_n (\lambda_n^2)^{-s}.$$

After a lengthy calculation using the expansion (24) we find for the relevant expression the following asymptotic expansion for $\nu \to \infty$:

$$\ln\left\{ z^{-2\nu+1}\left[I^2_{\nu-1/2}(z\nu) + b_-^2 I^2_{\nu+1/2}(z\nu) - \frac{2\lambda_n b_-}{\nu z} I_{\nu-1/2}(z\nu) I_{\nu+1/2}(z\nu)\right]\right\} \sim$$
$$\ln\left\{ z^{-2\nu}\frac{e^{2\nu\eta}}{2\pi\nu}(1+b_-^2)\left(1+t\frac{\sqrt{\lambda^2+\nu^2}}{\nu}\right)\right\} + \frac{1}{\nu}M_1(t) + \frac{1}{\nu^2}M_2(t) + \mathcal{O}(1/\nu^3).$$

The polynomials are

$$M_1(t) = \frac{\delta}{2}t^2 - \frac{5}{12}t^3,$$
$$M_2(t) = \frac{1}{2}\frac{\delta^2}{\delta+t}t^3 + \frac{1}{8}\frac{\delta}{\delta+t}t^4 - \frac{1}{8}\frac{\delta^3}{\delta+t}t^4 - \frac{1}{2}\frac{1}{\delta+t}t^5 - \frac{5}{8}\frac{\delta^2}{\delta+t}t^5 + \frac{5}{8}\frac{1}{\delta+t}t^7.$$

In analogy to the treatment in the proof of (10), this suggests the definitions

$$A_{-1}(s) = \frac{2\sin(\pi s)}{\pi}\sum_{m=0}^\infty \sum_n \int_{|\lambda_n|/\nu}^\infty dz\ (z^2\nu^2 - \lambda_n^2)^{-s}\frac{\partial}{\partial z}\ln\left(z^{-2\nu}e^{2\nu\eta}\right),$$
$$A_0(s) = \frac{2\sin(\pi s)}{\pi}\sum_{m=0}^\infty \sum_n \int_{|\lambda_n|/\nu}^\infty dz\ (z^2\nu^2 - \lambda_n^2)^{-s}\frac{\partial}{\partial z}\ln\left(1+t\frac{\sqrt{\lambda_n^2+\nu^2}}{\nu}\right),$$
$$A_q(s) = \frac{2\sin(\pi s)}{\pi}\sum_{m=0}^\infty \sum_n \int_{|\lambda_n|/\nu}^\infty dz\ (z^2\nu^2 - \lambda_n^2)^{-s}\frac{\partial}{\partial z}\frac{M_q(t)}{\nu^q}.$$

We use (30) to see

$$A_{-1}(s) = -\frac{2}{\sqrt{\pi}\Gamma(s)}\int_{\mathcal{C}}\frac{dt}{2\pi i}\ \frac{\Gamma(s-1/2+t)\Gamma(-t)}{t-1/2}\zeta_H(-2t;1/2)\zeta_N(s+t-1/2), \tag{43}$$

where the contour lies to the left of $\Re t = -1/2$. If we denote the heat-kernel coefficients of $\tilde{P}^2$ on N as $a_j^{(N)}$, we have the relations [7]:

$$\Gamma((m-2)/2)\ \mathrm{Res}\ _{s=(m-2)/2}\zeta_N(s) = a_0^{(N)} = (4\pi)^{-(m-2)/2}\int_N \mathrm{Tr}\ 1,$$
$$\Gamma((m-4/2)\ \mathrm{Res}\ _{s=(m-4)/2}\zeta_N(s) = a_1^{(N)} = (4\pi)^{-(m-2)/2}\left(-\frac{1}{12}\right)\int_N \mathrm{Tr}\, R(N).$$

For later use, in the same way we define $a_j^{(S^1\times N)}$ associated with A^2. Using $\zeta_A(s)$ instead of $\zeta_N(s)$ in the above equations, the results with obvious replacements remain valid.

Shifting the contour in (43) to the left we pick up the poles of $A_{-1}(s)$. To provide checks of the calculation, we also present the residues to the right of $s=(m-3)/2$. E.g. we find that

$$\Gamma(m/2)\ \mathrm{Res}\ _{s=m/2}A_{-1}(s) = \frac{1}{2}a_0^{(N)},$$
$$\Gamma((m-1)/2)\ \mathrm{Res}\ _{s=(m-1)/2}A_{-1}(s) = 0,$$
$$\Gamma((m-2)/2)\ \mathrm{Res}\ _{s=(m-2/2}A_{-1}(s) = \frac{1}{2}a_1^{(N)} - \frac{1}{12}a_0^{(N)},$$
$$\Gamma((m-3)/2)\ \mathrm{Res}\ _{s=(m-3)/2}A_{-1}(s) = 0.$$

We continue with $A_0(s)$. It may be casted into the form

$$A_0(s) = -\frac{1}{\Gamma(s)}\zeta_A(s)\left(\Gamma(s) - \frac{\Gamma(s+1/2)}{\sqrt{\pi}s}\right).$$

At the values of s needed the k-sum can be given in closed form and one finds

$$\begin{aligned}
&\Gamma((m-1)/2)\ \mathrm{Res}\ _{s=(m-1)/2}A_0(s) = -a_0^{(S^1\times N)}\left\{1 - \frac{\Gamma(m/2)}{\Gamma(1/2)\Gamma((m+1)/2)}\right\},\\
&\Gamma((m-2)/2)\ \mathrm{Res}\ _{s=(m-2)/2}A_0(s) = 0,\\
&\Gamma((m-3)/2)\ \mathrm{Res}\ _{s=(m-3)/2}A_0(s) = -a_1^{(S^1\times N)}\left\{1 - \frac{\Gamma(m/2-1)}{\Gamma(1/2)\Gamma((m-1)/2)}\right\}.
\end{aligned}$$

Similarly, $A_1(s)$ and $A_2(s)$ can be represented in terms of $\zeta_A^l(s)$, equation (42). The relevant residues of $\zeta_A^l(s)$ can be determined from $\zeta_A(s)$ by a suitable scaling of the circle S^1. One has

$$\sum_{m=0}^{\infty}\sum_n \frac{(\nu^2)^l}{(\lambda_n^2+\nu^2)^{s+l+1}} = (-1)^l\frac{\Gamma(s+1)}{\Gamma(s+l+1)}\times\left(\frac{d}{db}\right)^l\sum_{m=0}^{\infty}\sum_n(\lambda_n^2+\nu^2 b)^{-s-1}\,|_{b=1}.$$

The residues of the right hand side can be obtained from $a_j^{(S^1\times N)}$. E.g.

$$\mathrm{Res}\ _{s=(m-3)/2}\sum_{m=0}^{\infty}\sum_n(\lambda_n^2+\nu^2 b)^{-s-1} = \frac{1}{\Gamma((m-1)/2)}\frac{a_0^{(S^1\times N)}}{\sqrt{b}}.$$

It follows

$$\mathrm{Res}\ _{s=(m-3)/2}\sum_{m=0}^{\infty}\sum_n \frac{(\nu^2)^l}{(\lambda_n^2+\nu^2)^{s+l+1}} = \frac{\Gamma(l+1/2)}{\Gamma(1/2)\Gamma(m/2+l-1/2)}a_0^{(S^1\times N)}.$$

This, and a similar equation for $s=(m-2)/2$, allows one to find the remaining contributions to the leading pole:

$$\begin{aligned}
&\Gamma(\tfrac{m-2}{2})\ \mathrm{Res}\ _{s=(m-2)/2}A_1(s) = \tfrac{1}{3}\left(1 - \tfrac{3}{4}\tfrac{\Gamma(1/2)\Gamma(m/2)}{\Gamma((m+1)/2)}\right)(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\,1,\\
&\Gamma((m-3)/2)\ \mathrm{Res}\ _{s=(m-3)/2}A_2(s) =\\
&\left(-\frac{3m-4}{16\Gamma(1/2)(m^2-1)}\frac{\Gamma(m/2)}{\Gamma((m+1)/2)} + \frac{m^2+8m-17}{128(m^2-1)}\right)(4\pi)^{-(m-1)/2}\int_{\partial M}\mathrm{Tr}\,1.
\end{aligned}$$

Putting things together, we can use a_0, a_1 and a_2 as a check of the calculation. The value we compute for d_{12} agrees with our previous calculation. Finally, we complete the proof of assertion (11) of Lemma 4. ☐

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