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by

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Seiberg-Witten maps and noncommutative Yang-Mills theories for arbitrary gauge groups

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Abstract

Seiberg-Witten maps and a recently proposed construction of noncommutative Yang-Mills theories (with matter fields) for arbitrary gauge groups are reformulated so that their existence to all orders is manifest. The ambiguities of the construction which originate from the freedom in the Seiberg-Witten map are discussed with regard to the question whether they can lead to inequivalent models, i.e., models not related by field redefinitions.

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1 Introduction

The purpose of this work is to elaborate on the construction of noncommutative Yang-Mills theories for arbitrary gauge groups proposed in [1, 2] (see also [3, 4, 5] for related work). The idea in [1, 2] is to use a Seiberg-Witten map for building gauge fields and gauge parameters of the noncommutative theory from Lie algebra valued gauge fields and gauge parameters of a commutative gauge theory. The Seiberg-Witten map, which had been originally established for U(N) theories in [6], is in [1, 2] applied to the universal enveloping algebra of the Lie algebra of the gauge group and yields thus gauge fields for all elements of the enveloping algebra rather than only for the Lie algebra itself. From these enveloping algebra valued gauge fields one constructs the corresponding field strengths of the noncommutative theory, and finally the action in terms of these field strengths.

Our approach is slightly different. As in [6], the starting point is a noncommutative model with a Weyl-Moyal star product but with a suitable finite dimensional space $\mathcal{U}$ of matrices in place of $u(N)$. This model contains unconstrained "noncommutative" gauge fields and gauge parameters for a basis of $\mathcal{U}$. By means of a Seiberg-Witten map we express these gauge fields and gauge parameters in terms of "commutative" gauge fields and gauge parameters that also live in $\mathcal{U}$. Finally we set all commutative gauge fields and gauge parameters to zero except for a subset corresponding to a Lie subalgebra $\mathfrak{g}$ of $\mathcal{U}$. In this way one can easily construct noncommutative gauge theories of the same type as in [1, 2] to all orders in the deformation parameter and for all choices of $\mathfrak{g}$. The inclusion of matter fields is also straightforward, as we shall demonstrate.

The main difference of our approach from [1, 2] is that we use unconstrained commutative gauge fields and gauge parameters for *all* elements of a basis of $\mathcal{U}$ (rather than only of $\mathfrak{g}$) and drop the unwanted fields and parameters only at the very end. An advantage of this approach is that the existence of the models to all orders is manifest. Another advantage is that it allows one to analyse more systematically the ambiguities of the construction, especially those which originate from the freedom in the Seiberg-Witten map pointed out already in [7, 2, 8, 9].

For completeness, we mention a few references where other approaches to Seiberg-Witten maps in the Yang-Mills case are discussed. Existence by explicit construction has been shown in e.g.[10], where commutative and noncommutative versions of Wilson lines were compared and in [11], where the Seiberg-Witten map was computed in the framework of Kontsevich's approach to deformation quantization. An explicit inverse Seiberg-Witten map was given in [12], where further references are discussed.

2 Basic idea

We work in flat n-dimensional spacetime with Minkowski metric. In standard (commutative) Yang-Mills theory, the gauge fields A_μ take values in a finite-dimensional Lie algebra $\mathfrak{h}$. This property is preserved under the gauge transformation

$$\delta_\lambda A_\mu = D_\mu \lambda = \partial_\mu \lambda + [A_\mu, \lambda]\,, \tag{2.1}$$

for gauge parameters λ that are also $\mathfrak{h}$-valued. The construction of the gauge invariant Lagrangian

$$L = \frac{1}{2\kappa}\, g_{AB} F^A_{\mu\nu} F^{B\,\mu\nu}, \tag{2.2}$$

requires in addition the existence of a symmetric bilinear invariant form g on $\mathfrak{h}$. The Lagrangian involves the components of both g and the curvature $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ in a basis of $\mathfrak{h}$.

For noncommutative models, the usual multiplication of functions is replaced by star-multiplication and a straightforward generalization of (2.1) is

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \hat{D}_\mu \hat{\lambda} = \partial_\mu \hat{\lambda} + [\hat{A}_\mu \overset{\star}{,} \hat{\lambda}] \tag{2.3}$$

where $[\hat{A}_\mu \overset{\star}{,} \hat{\lambda}]$ is the star-commutator. To define it, the fields are assumed to take values[1] in a finite-dimensional complex associative algebra $\mathfrak{A}$ and

$$[f \overset{\star}{,} g] = f \star g - (-1)^{|f||g|} g \star f = (f^A * g^B - (-1)^{|f||g|} g^A * f^B) t_A t_B\,,$$

where we have introduced a basis t_A in $\mathfrak{A}$. For our purposes it is sufficient to take as $\mathfrak{A}$ a subalgebra of the algebra of constant matrices with complex entries. We also restrict ourselves to the Weyl-Moyal star-product,

$$f \star g = f[\exp(\overleftarrow{\partial_\mu}\, \tfrac{\mathrm{i}}{2}\vartheta\theta^{\mu\nu}\, \overrightarrow{\partial_\nu})]g\,, \tag{2.4}$$

where ϑ is a real deformation parameter and θ is a real antisymmetric matrix.

Let us note that the condition that the fields take values in an associative algebra is sufficient, but not necessary. Indeed, in order for the transformation (2.3) to make sense, it is sufficient that $\hat{A}_\mu$ and $\hat{\lambda}$ take values in some subset $\mathcal{U} \subset \mathfrak{A}$ with the property that the $\mathcal{U}$ valued fields are closed under star-commutation, i.e., that they form a star-Lie algebra. These star-Lie algebras can either be over $\mathbb{C}$ or over $\mathbb{R}$. In the complex case, $\mathcal{U}$ has to be an associative subalgebra of $\mathfrak{A}$, while in the real case, it has to be a real subalgebra of $\mathfrak{A}$, considered itself as a real Lie algebra. In the latter case, $\mathcal{U}$ is in general neither a vector space over $\mathbb{C}$ (in spite of the fact that its elements are matrices with complex entries) nor an associative subalgebra of $\mathfrak{A}$. The basic examples of real star-Lie algebras are those associated to the familiar noncommmutative $U(N)$ models, where $\mathcal{U}$ is the subspace of antihermitian matrices of $Mat(N, \mathbb{C})$.

For a given (complex or real) star-Lie algebra, it is straightforward to construct a Lagrangian which is invariant up to a total divergence under the gauge transformations (2.3):

$$\hat{L} = \frac{1}{2\kappa}\,\mathrm{Tr}\,(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}) = \frac{1}{2\kappa}\, g_{AB} \hat{F}^A_{\mu\nu} \star \hat{F}^{B\,\mu\nu} \tag{2.5}$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + [\hat{A}_\mu \overset{\star}{,} \hat{A}_\nu], \quad g_{AB} = \mathrm{Tr}\,(t_A t_B)\,, \tag{2.6}$$

[1]More precisely, we consider the tensor product of $\mathfrak{A}$ and the space of functions that are formal power series in the deformation parameter ϑ with coefficients that depend on x^μ, the fields and a finite number of their derivatives.

and Tr denotes ordinary matrix trace. Note that the Lagrangian (2.5) can be also defined for gauge fields taking values in an (appropriate subspace of an) abstract associative algebra $\mathfrak{A}$ equipped with a trace. In general, g_{AB} will be degenerate but this poses no problem in our approach because the noncommutative model (2.5) is only an intermediate stage in the construction.

The basic idea to construct a noncommutative deformation for commutative Yang-Mills models with a given gauge Lie algebra $\mathfrak{g}$ is the following:

1. take as $\mathcal{U}$ an appropriate Lie algebra containing $\mathfrak{g}$ as a subalgebra;
2. reformulate the noncommutative Yang-Mills model in terms of commutative gauge fields A_μ and some effective Lagrangian L^{eff} by using a Seiberg-Witten map;
3. reduce consistently the commutative theory described by $\mathcal{U}$-valued gauge field A_μ to the Lie subalgebra $\mathfrak{g}$ of $\mathcal{U}$.

Let us explain these steps in more details, for definiteness in the (more involved) case where $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$.

1. Let $\{t_a\}$ be a basis of a faithful finite dimensional matrix representation of $\mathfrak{g}$. Furthermore, suppose that $g_{ab} = \operatorname{Tr}(t_a t_b)$ is an invariant symmetric bilinear and non degenerate form on $\mathfrak{g}$. We take $\mathcal{U}$ as an enveloping algebra over $\mathbb{R}$ of the complexified Lie algebra $\mathfrak{g}$. More precisely, we take $\mathcal{U}$ to be the matrix algebra generated (over $\mathbb{C}$) by the matrices t_a and complement the elements t_a by additional elements t_i in such a way that the set $\{t_A\} = \{t_i, t_a\}$ provides a basis of $\mathcal{U}$ considered as a vector space over $\mathbb{R}$. This implies that $\mathcal{U}$ valued fields form a real star-Lie algebra.

2. For the noncommutative theory based on the star-Lie algebra of $\mathcal{U}$ valued fields and described by the Lagrangian (2.5), one can construct, as explained in more details in section 4, a Seiberg-Witten map

$$\begin{aligned} \hat{A}_\mu &= \hat{A}_\mu(\vartheta, A, \partial A, \partial^2 A, \dots) = A_\mu + O(\vartheta)\,, \\ \hat{\lambda} &= \hat{\lambda}(\vartheta, \lambda, \partial\lambda, \dots, A, \partial A, \dots) = \lambda + O(\vartheta)\,, \end{aligned} \tag{2.7}$$

required to map the noncommutative gauge transformations (2.3) to the commutative transformations (2.1) according to

$$\delta_\lambda \hat{A}_\mu(\vartheta, A, \partial A, \partial^2 A, \dots) = (\partial_\mu \hat{\lambda} + [\hat{A}_\mu \stackrel{\star}{,} \hat{\lambda}])(\vartheta, \lambda, \partial\lambda, \dots, A, \partial A, \dots)\,. \tag{2.8}$$

Using such a map, one can reformulate the noncommutative theory described by the Lagrangian (2.5) and the gauge transformations (2.3) in terms of the commutative gauge fields with the gauge transformations (2.1) and an effective Lagrangian

$$L^{\text{eff}}[A;\vartheta] = \hat{L}[\hat{A}[A;\vartheta];\vartheta]. \tag{2.9}$$

By construction, L^{eff} is gauge invariant under the commutative gauge transformations up to a total derivative.

3. One can now consistently reduce the model by dropping all the components of the gauge field and gauge parameter complementary to the Lie subalgebra $\mathfrak{g}$ of $\mathcal{U}$. In

the basis $\{t_a, t_i\}$, this means setting to zero the components A^i_μ and λ^i. Using that $\mathfrak{g}$ is a Lie subalgebra of $\{t_A\}$, one obtains:

$$[\delta_\lambda A_\mu]_{A^i_\mu=\lambda^i=0} = \partial_\mu \lambda^a t_a + A^b_\mu \lambda^c [t_b, t_c]. \tag{2.10}$$

On the one hand, this shows that indeed, it is consistent to set A^i_μ and λ^i to zero. On the other hand, it provides the gauge transformations of A^a_μ in the reduced theory. We denote these transformations by $\delta^{\rm red}_\lambda$.

$$\delta^{\rm red}_\lambda A^a_\mu = \partial_\mu \lambda^a + A^b_\mu \lambda^c f_{bc}{}^a, \tag{2.11}$$

where $f_{bc}{}^a$ are the structure constants of $\mathfrak{g}$ in the basis $\{t_a\}$,

$$[t_b, t_c] = f_{bc}{}^a t_a. \tag{2.12}$$

The Lagrangian of the reduced theory is given by

$$L^{\rm eff}_{\rm red}[A^a; \vartheta] = L^{\rm eff}[A^A; \vartheta]\Big|_{A^i=0}. \tag{2.13}$$

It is gauge invariant under the gauge transformations $\delta^{\rm red}_\lambda$ up to a total derivative. The resulting model is a consistent deformation of a commutative Yang-Mills theory with gauge algebra $\mathfrak{g}$, because for $\vartheta = 0$, one recovers the standard Yang-Mills Lagrangian. Even though $L^{\rm eff}$ is in general complex-valued, in the case of real $\mathfrak{g}$ and real g_{ab}, the real part of the Lagrangian $L^{\rm eff}_{\rm red}$ is a consistent deformation of the commutative Yang-Mills theory, because both the real and imaginary parts of $L^{\rm eff}_{\rm red}$ are separately gauge invariant.

Note that one can also choose to work from the beginning consistently over $\mathbb{R}$, if the Weyl-Moyal star-product is taken to be real, i.e., without the imaginary unit i in formula (2.4).

Finally, we also note that, in terms of the noncommutative gauge fields, the effective model described by $L^{\rm eff}_{\rm red}$ is described by $\hat{L}$ supplemented by the constraints $A^i_\mu[\hat{A}^A; \vartheta] = 0$ involving the inverse Seiberg-Witten map. This will be developed in more details in [13].

3 Compact gauge groups and matter fields

Let us now discuss the construction of models which are based on $u(N)$ valued fields[2]. A physical motivation for considering such models is that they arise naturally when one deals with compact Lie algebras $\mathfrak{g}$. Furthermore, they allow one to include directly fermionic matter fields, as we shall demonstrate below. A mathematical motivation is that they provide a natural arena for constructing real star-Lie algebras for the Weyl-Moyal product because the Weyl-Moyal star-commutator of two antihermitian matrices is again antihermitian [14].

So, in the case that the real Lie algebra $\mathfrak{g}$ admits a faithful representation by antihermitian $N \times N$ matrices t_a, as do the physically important compact Lie algebras, there is no need to take as $\mathcal{U}$ the complexified enveloping algebra of the t_a. It is sufficient to take

[2] We use this term in a somewhat sloppy way here: in the following "$u(N)$ valued fields" live in a space of antihermitian (field dependent) matrices which form a star-Lie algebra over $\mathbb{R}$, see below.

$\mathcal{U}$ equal to $u(N)$, or to a Lie subalgebra of $u(N)$ that contains $\mathfrak{g}$ as a subalgebra and is such that $\mathcal{U}$ valued functions form a real star-Lie algebra. One can then proceed with steps 2 and 3 described in Section **2**, with $u(N)$-valued fields $\hat{A}_\mu = \hat{A}_\mu^A t_A$ and $\hat{\lambda} = \hat{\lambda}^A t_A$ where all $\hat{A}_\mu^A$ and $\hat{\lambda}^A$ are real and the t_A are antihermitian.

To discuss the inclusion of fermionic matter fields, we introduce a set of Dirac spinor fields which make up column vectors $\hat{\psi}$ on which the matrices t_A act, and noncommutative gauge transformations

$$\hat{\delta}_{\hat{\lambda}} \hat{\psi} = -\hat{\lambda} \star \hat{\psi}. \tag{3.1}$$

The Dirac conjugate spinor fields then transform according to

$$\hat{\delta}_{\hat{\lambda}} \bar{\hat{\psi}} = -\bar{\hat{\psi}} \star \hat{\lambda}^\dagger. \tag{3.2}$$

The straightforward noncommutative generalization of standard commutative Yang-Mills actions with fermionic matter fields reads

$$\hat{L} = \frac{1}{2\kappa} \operatorname{Tr} (\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}) + \mathrm{i}\, \bar{\hat{\psi}} \star \gamma^\mu \hat{D}_\mu \hat{\psi} \tag{3.3}$$

where, in view of (3.1),

$$\hat{D}_\mu \hat{\psi} = \partial_\mu \hat{\psi} + \hat{A}_\mu \star \hat{\psi}. \tag{3.4}$$

Using now (3.2), one readily verifies that the Lagrangian (3.3) is invariant up to a total derivative under the gauge transformations (2.3) and (3.1) when $\hat{\lambda}$ is $u(N)$ valued[3].

In section 4, we shall show that the Seiberg-Witten maps can be extended such that

$$\hat{\psi} = \hat{\psi}[\psi, A; \vartheta] = \psi + O(\vartheta)\,, \quad \delta_\lambda \hat{\psi}[\psi, A; \vartheta] = -\hat{\lambda}[\lambda, A; \vartheta] \star \hat{\psi}[\psi, A; \vartheta] \tag{3.5}$$

where δ_λ acts on the commutative fermions ψ in the standard way:

$$\delta_\lambda \psi = -\lambda \psi. \tag{3.6}$$

One then proceeds as described in the items 2) and 3) of the previous section (now with fermions included) to arrive at a reduced effective model with Lagrangian and gauge transformations given by

$$L^{\mathrm{eff}}_{\mathrm{red}}[A, \psi; \vartheta] = \hat{L}[\hat{A}[A; \vartheta], \hat{\psi}[\psi, A; \vartheta]; \vartheta]\Big|_{A^i=0}\,, \tag{3.7}$$

$$\delta^{\mathrm{red}}_\lambda A^a_\mu = \partial_\mu \lambda^a + A^b_\mu \lambda^c f_{bc}{}^a, \quad \delta^{\mathrm{red}}_\lambda \psi = -\lambda^a t_a \psi. \tag{3.8}$$

[3] To include fermionic matter fields with gauge transformations (3.1) for $\hat{\lambda}$'s that are not antihermitian, one may double the fermion content by introducing an additional set of fermions $\hat{\chi}$ with gauge transformations $\hat{\delta}_{\hat{\lambda}} \hat{\chi} = \hat{\lambda}^\dagger \star \hat{\chi}$. The Lagrangian would then contain $\bar{\hat{\chi}} \star \gamma^\mu \hat{D}_\mu \hat{\psi}$ in place of $\bar{\hat{\psi}} \star \gamma^\mu \hat{D}_\mu \hat{\psi}$.

4 Seiberg-Witten maps

In this section we show the existence of Seiberg-Witten maps for Yang-Mills models based on arbitrary star-Lie algebra. For completeness, formulas for fermions are also added. In order for the them to make sense one should either restrict to the case of antihermitian gauge fields or add additional fermions as explained in footnote 3.

The field-antifield formalism [15, 16, 17, 18, 19, 20, 21] (for reviews, see e.g. [22, 23]) is an expedient framework to construct Seiberg-Witten maps [24, 25, 26, 27] because it encodes the action, gauge transformations and the commutator algebra of the gauge transformations in a single functional, the so-called master action which is a solution to the master equation. In the language of the field-antifield formalism, Seiberg-Witten maps are "anticanonical transformations" (i.e., transformations of the fields and antifields preserving the antibracket) which turn the master action $\hat{S}$ of a noncommutative model into the master action S of a corresponding (effective) commutative model,

$$\hat{S}[\hat{\Phi}[\Phi,\Phi^*;\vartheta],\hat{\Phi}^*[\Phi,\Phi^*;\vartheta];\vartheta]=S[\Phi,\Phi^*;\vartheta]. \tag{4.1}$$

Furthermore, the $\hat{\Phi}$'s and $\hat{\Phi}^*$'s are required to agree with their unhatted counterparts at $\vartheta=0$,

$$\hat{\Phi}[\Phi,\Phi^*;0]=\Phi,\quad \hat{\Phi}^*[\Phi,\Phi^*;0]=\Phi^*. \tag{4.2}$$

These maps are generated by a functional $\hat{\Xi}$ satisfying

$$\frac{\partial\hat{\Phi}}{\partial\vartheta}=-\left(\hat{\Phi},\hat{\Xi}[\hat{\Phi},\hat{\Phi}^*;\vartheta]\right),\quad \frac{\partial\hat{\Phi}^*}{\partial\vartheta}=-\left(\hat{\Phi}^*,\hat{\Xi}[\hat{\Phi},\hat{\Phi}^*;\vartheta]\right), \tag{4.3}$$

where $(\ ,\ )$ is the antibracket for the hatted fields and antifields.

Explicitly, the master actions are[4],

$$\hat{S}[\hat{\Phi},\hat{\Phi}^*;\vartheta]=\int\left[\hat{L}+\hat{A}^{*\mu}_A(\hat{D}_\mu\hat{C})^A+\hat{\psi}^*\hat{C}\star\hat{\psi}+\bar{\hat{\psi}}\star\hat{C}\bar{\hat{\psi}}^*+\hat{C}^*_A(\hat{C}\star\hat{C})^A\right]d^nx, \tag{4.4}$$

$$S[\Phi,\Phi^*;\vartheta]=S^{\rm eff}[A,\psi;\vartheta]+\int\left[A^{*\mu}_A(D_\mu C)^A+\psi^*C\psi+\bar{\psi}C\bar{\psi}^*+C^*_A(CC)^A\right]d^nx, \tag{4.5}$$

where the Φ's and Φ^*'s and their hatted counterparts denote collectively the fields and antifields of the respective formulation ($\{\Phi^M\}\equiv\{A^A_\mu,\psi,C^A\}$ etc.), the $\hat{C}^A$ and C^A are ghost fields substituting for the gauge parameters $\hat{\lambda}^A$ and λ^A, respectively, $\hat{L}$ is the Lagrangian (3.3) and $S^{\rm eff}[A,\psi;\vartheta]$ will be determined in the course of the construction.

A differential equation for these Seiberg-Witten maps can be obtained by differentiating equation (4.1) with respect to ϑ. Since the antifield dependent terms in S do not depend on ϑ, the right hand side of equation (4.1) gives just $\partial S^{\rm eff}[A,\psi;\vartheta]/\partial\vartheta$. On the left hand side, $\hat{S}$ depends explicitly on ϑ through the star products and implicitly through $\hat{\Phi}$ and $\hat{\Phi}^*$ because the latter are to expressed in terms of the Φ and Φ^* through

[4]For our purposes they need not be proper solutions of the master equation in the strict sense because eventually we shall set A^i_μ and the corresponding ghosts and antifields to zero.

the Seiberg-Witten map which also involves ϑ. In this way one obtains, by using (4.3), the following condition on the generating functional $\hat{\Xi}$,

$$\frac{\partial \hat{S}[\hat{z}[z;\vartheta];\vartheta]}{\partial\vartheta} - (\hat{S}[\hat{z}[z;\vartheta];\vartheta], \hat{\Xi}[\hat{z}[z;\vartheta];\vartheta])_z = \frac{\partial S^{\text{eff}}[z;\vartheta]}{\partial\vartheta} \ , \tag{4.6}$$

where the notation $\hat{z} \equiv (\hat{\Phi}, \hat{\Phi}^*)$, respectively $z \equiv (\Phi, \Phi^*)$ has been used. When expressed in terms of the noncommutative variables, this condition is equivalent to

$$\frac{\partial \hat{S}[\hat{z};\vartheta]}{\partial\vartheta} = (\frac{\partial S^{\text{eff}}}{\partial\vartheta})[z[\hat{z};\vartheta];\vartheta] + (\hat{S}[\hat{z};\vartheta], \hat{\Xi}[\hat{z};\vartheta])_{\hat{z}} \ . \tag{4.7}$$

Explicitly, differentiation of (4.4) with respect to ϑ yields:

$$\frac{\partial \hat{S}[\hat{\Phi},\hat{\Phi}^*,\vartheta]}{\partial\vartheta} = \frac{\mathrm{i}}{2}\,\theta^{\alpha\beta}\int d^nx\,\Big[\frac{2}{\kappa}\,tr(\hat{F}^{\mu\nu}\partial_\alpha\hat{A}_\mu \star \partial_\beta\hat{A}_\nu) + \mathrm{i}\,\bar{\hat{\psi}}\gamma^\mu\partial_\alpha\hat{A}_\mu \star \partial_\beta\hat{\psi}$$
$$+\hat{A}^{*\mu}_A\{\partial_\alpha\hat{A}_\mu \stackrel{\star}{,} \partial_\beta\hat{C}\}^A + \hat{\psi}^*\partial_\alpha\hat{C}\star\partial_\beta\hat{\psi} + \partial_\alpha\bar{\hat{\psi}}\star\partial_\beta\hat{C}\bar{\hat{\psi}}^* + \hat{C}^*_A(\partial_\alpha\hat{C}\star\partial_\beta\hat{C})^A\Big] \tag{4.8}$$

The fact that the ghost fields occur in this expression only differentiated guarantees the existence of a functional $\hat{\Xi}$ that satisfies (4.6). This follows from BRST cohomological arguments that will be given in [13]. It will also be shown there how one obtains $\hat{\Xi}$ systematically by means of a contracting homotopy. Here we only give the result. One obtains

$$\begin{aligned}
\hat{\Xi}[\hat{\Phi},\hat{\Phi}^*;\vartheta] &= \frac{\mathrm{i}}{4}\,\theta^{\alpha\beta}\int d^nx\,\Big[\hat{A}^{*\mu}_A\{\hat{A}_\alpha \stackrel{\star}{,} \hat{F}_{\beta\mu} + \partial_\beta\hat{A}_\mu\}^A + \hat{C}^*_A\{\hat{A}_\alpha \stackrel{\star}{,} \partial_\beta\hat{C}\}^A \\
&\quad +\hat{\psi}^*\hat{A}_\alpha\star(2\partial_\beta\hat{\psi} + \hat{A}_\beta\star\hat{\psi}) + (2\partial_\beta\bar{\hat{\psi}} - \bar{\hat{\psi}}\star\hat{A}_\beta)\star\hat{A}_\alpha\bar{\hat{\psi}}^*\Big], \quad (4.9) \\
(\frac{\partial S^{\text{eff}}}{\partial\vartheta})[\Phi[\hat{\Phi};\vartheta];\vartheta] &= \theta^{\alpha\beta}\int d^nx\,\Big[\frac{\mathrm{i}}{\kappa}\,tr(\hat{F}^{\mu\nu}\hat{F}_{\alpha\mu}\star\hat{F}_{\beta\nu} - \frac{1}{4}\hat{F}_{\alpha\beta}\hat{F}^{\mu\nu}\star\hat{F}_{\mu\nu}) \\
&\quad +\frac{1}{2}\,\bar{\hat{\psi}}\gamma^\mu\hat{F}_{\mu\alpha}\star\hat{D}_\beta\hat{\psi} + \frac{1}{4}\,\bar{\hat{\psi}}\gamma^\mu\hat{F}_{\alpha\beta}\star\hat{D}_\mu\hat{\psi}\Big]\ . \quad (4.10)
\end{aligned}$$

Using now (4.9) in (4.3) gives the differential equations

$$\begin{aligned}
\frac{\partial\hat{A}_\mu}{\partial\vartheta} &= -\frac{\mathrm{i}}{4}\,\theta^{\alpha\beta}\{\hat{A}_\alpha \stackrel{\star}{,} \hat{F}_{\beta\mu} + \partial_\beta\hat{A}_\mu\}, & (4.11) \\
\frac{\partial\hat{\psi}}{\partial\vartheta} &= -\frac{\mathrm{i}}{4}\,\theta^{\alpha\beta}(2\hat{A}_\alpha\star\partial_\beta\hat{\psi} + \hat{A}_\alpha\star\hat{A}_\beta\star\hat{\psi}), & (4.12) \\
\frac{\partial\hat{C}}{\partial\vartheta} &= -\frac{\mathrm{i}}{4}\,\theta^{\alpha\beta}\{\hat{A}_\alpha \stackrel{\star}{,} \partial_\beta\hat{C}\}, & (4.13)
\end{aligned}$$

which can be integrated to arbitrary order in ϑ for the initial conditions (4.2). They generalize the differential equations derived in [6] for the noncommutative gauge fields and gauge parameters in U(N) models to the more general models considered here. In fact (4.11) and (4.13) take the same form as the corresponding equations in the U(N) case. It is also evident that (4.12) reproduces the first order expressions of the Seiberg-Witten maps for the fermions derived in [1, 2]. Using a solution to these equations in

(4.10), the latter provides the commutative effective action $S^{\rm eff}[A,\psi;\vartheta]$ in (4.5) [notice that (4.10) is still expressed in terms of hatted fields] according to

$$
\begin{aligned}
S^{\rm eff}[A,\psi;\vartheta] &= S_0^{\rm eff}[A,\psi] + \int_0^{\vartheta} d\vartheta' \, (\frac{\partial S^{\rm eff}}{\partial \vartheta})[A,\psi;\vartheta'], &\quad (4.14)\\
S_0^{\rm eff}[A,\psi] &= \int d^n x \, [(2\kappa)^{-1} tr(F_{\mu\nu}F^{\mu\nu}) + {\rm i}\,\bar{\psi}\gamma^{\mu}(\partial_{\mu} + A_{\mu})\psi]. &\quad (4.15)
\end{aligned}
$$

5 Ambiguities of the construction

In what follows, we need a description of the ambiguity in the Seiberg-Witten map[5]. For our purposes it is sufficient to do this in terms of the variables of the commutative theory (for simplicity we shall restrict the discussion to models without matter fields). Given two Seiberg-Witten maps determined respectively by:

$$\hat{A}_{\mu} = f_{\mu}[A;\vartheta]\,, \quad \hat{\lambda} = h[A,\lambda;\vartheta]$$

and

$$\hat{A}'_{\mu} = f'_{\mu}[A;\vartheta]\,, \quad \hat{\lambda}' = h'[A,\lambda;\vartheta]\,,$$

the composition of one of these maps and the inverse of the other is a map

$$f^0_{\mu}[A;\vartheta] = f^{-1}_{\mu}[f'[A;\vartheta];\vartheta]\,, \quad h^0[A,\lambda;\vartheta] = h^{-1}[f'[A;\vartheta], h'[A,\lambda;\vartheta];\vartheta]\,, \qquad (5.1)$$

which preserves the gauge structure of commutative theory. More precisely, it satisfies a commutative counterpart of the gauge equivalence condition (2.8)

$$\delta_{\lambda}(f^0_{\mu}[A;\vartheta]) = (\partial_{\mu}h^0 + [f^0_{\mu}, h^0])[A,\lambda;\vartheta]\,. \qquad (5.2)$$

Hence, any Seiberg-Witten map can be obtained as the composition of a fixed Seiberg-Witten map and a map preserving the gauge structure of the commutative theory.

If, as in the previous section, we introduce in the context of the antifield formalism a generating functional Ξ^0 for the infinitesimal anticanonical transformation associated to (5.1), the condition (5.2) can be shown [13, 27] to be equivalent to

$$\big(S,\, \Xi^0\big) = \delta S^{\rm eff}[A;\vartheta]\,, \qquad (5.3)$$

for some antifield independent term $\delta S^{\rm eff}[A;\vartheta]$ which is the variation of the effective action under an infinitesimal field redefinition of A_{μ}, and a generating functional of the form

$$\Xi^0 = \int d^n x \, \big(A^{*\mu}_A \sigma^A_{\mu} + C^*_A \sigma^A\big) \qquad (5.4)$$

where $\sigma_{\mu} = \sigma_{\mu}[A;\vartheta]$ and $\sigma = \sigma[A,C;\vartheta]$. Under some technical assumptions, the general solution of (5.3) has the following form [13]:

$$\sigma_{\mu} = D_{\mu}\lambda + w_{\mu}\,, \qquad \sigma = s\lambda + [C,\lambda]\,. \qquad (5.5)$$

[5]We are grateful to I.V. Tyutin for a discussion of the ambiguity in the Seiberg-Witten map and explaining his unpublished results on that.

Here, $\lambda = \lambda[A;\vartheta]$, $s = (S,\,\cdot\,)$ is the BRST differential, and w_μ satisfies $sw_\mu + [C, w_\mu] = 0$. The latter implies that $w_\mu = w_\mu[F;\vartheta]$ depends on the gauge potentials and their derivatives only through the curvatures $F^A_{\mu\nu}$ and their covariant derivatives.

The finite transformations $A_\mu = f^0_\mu[A^0;\vartheta]$, $C = h^0[A^0, C^0;\vartheta]$ associated to Ξ^0 are obtained as in section **4** by solving the differential equations

$$\frac{\partial A_\mu}{\partial\vartheta} = -\big(A_\mu,\,\Xi^0\big)\,, \qquad \frac{\partial C}{\partial\vartheta} = -\big(C,\,\Xi^0\big)\,, \tag{5.6}$$

with boundary condition $A_\mu(0) = A^0_\mu$, and $C(0) = C^0$. One can then show [13] that the general solution for f^0_μ, h^0 is given by

$$\begin{aligned} h^0 &= \Lambda_0^{-1} C \Lambda_0 \;+\; \Lambda_0^{-1} s\Lambda_0\,, \\ f^0_\mu &= \Lambda_0^{-1}(A_\mu + W^0_\mu)\Lambda_0 \;+\; \Lambda_0^{-1}\partial_\mu\Lambda_0\,, \end{aligned} \tag{5.7}$$

where $\Lambda_0 = \Lambda_0[A;\vartheta]$ and $W^0_\mu = W^0_\mu[F;\vartheta]$ satisfies $sW^0_\mu + [C, W^0_\mu] = 0$. An arbitrary Seiberg-Witten map f'_μ, h' is given by the composition of such a map with a fixed Seiberg-Witten map $\hat{A}_\mu = f_\mu[A;\vartheta], \hat{C} = h[A, C;\vartheta]$ and can be represented as

$$\begin{aligned} h' &= \Lambda^{-1} \star h \star \Lambda \;+\; \Lambda^{-1} \star s\Lambda \\ f'_\mu &= \Lambda^{-1} \star (f_\mu + W_\mu) \star \Lambda \;+\; \Lambda^{-1} \star \partial_\mu\Lambda\,, \end{aligned} \tag{5.8}$$

with $\Lambda = \Lambda[A;\vartheta]$ and $W_\mu = W_\mu[A;\vartheta]$ satisfying the condition $sW_\mu + [h \stackrel{\star}{,} W_\mu] = 0$. This is closely related to the form of the ambiguity in the Seiberg-Witten map discussed in [9].

Let us now discuss the ambiguities in the construction of the noncommutative Yang-Mills models (i.e., in the construction of $L^{\rm eff}_{\rm red}$ from section **2**) due to the choice of a particular Seiberg-Witten map. First we note that if $w_\mu = 0$ then the remaing arbitrariness in (5.5) due to λ does not affect $L^{\rm eff}_{\rm red}$. Indeed, the infinitesimal transformation $A_\mu \to A_\mu + D_\mu\lambda$ is an infinitesimal gauge transformation which leaves the Lagrangian invariant up to a total derivative. So a variation of $L^{\rm eff}_{\rm red}$ that is not of this type is necessarily due to a non-vanishing w_μ. The associated variation of $L^{\rm eff}_{\rm red}$ is

$$(\delta L^{\rm eff})_{\rm red}[A^a;\vartheta] = \left(\frac{\delta L^{\rm eff}}{\delta A^A_\mu}\, w^A_\mu\right)\Big|_{A^i_\mu = 0} + \partial_\mu k^\mu\,. \tag{5.9}$$

The condition for $L^{\rm eff}_{\rm red} + (\delta L^{\rm eff})_{\rm red}$ and $L^{\rm eff}_{\rm red}$ to determine equivalent theories is the existence of an infinitesimal field redefinition $A^a_\mu \to A^a_\mu + V^a_\mu$ such that

$$\left(\frac{\delta L^{\rm eff}}{\delta A^A_\mu}\, w^A_\mu\right)\Big|_{A^i_\mu = 0} = \frac{\delta L^{\rm eff}_{\rm red}}{\delta A^a_\mu}\, V^a_\mu + \partial_\mu j^\mu\,. \tag{5.10}$$

That is, the infinitesimal variation $(\delta L^{\rm eff})_{\rm red}$ must vanish on the equations of motion for $L^{\rm eff}_{\rm red}$, up to a total derivative. In antifield language, this condition is equivalent to the existence of a ghost number -1 functional $\Xi_{\rm red}$ of the reduced theory such that

$$(\delta S^{\rm eff})\Big|_{A^i_\mu = 0} = \big(S_{\rm red},\,\Xi_{\rm red}\big).$$

We do not have an explicit counterexample, but it seems highly unlikely that Eq. (5.10) has a solution for arbitrary w_μ satisfying $sw_\mu + [C, w_\mu] = 0$. From this perspective one concludes that the definition of the model described by $L^{\rm eff}_{\rm red}$ is in general ambiguous.

Reversing the perspective, (5.10) may be used as a criterion to distinguish between Seiberg-Witten maps. One would then consider two Seiberg-Witten maps as equivalent if their infinitesimal difference is such that (5.10) holds, and otherwise as inequivalent. It seems to us that this provides a useful criterion to classify Seiberg-Witten maps used for reduction.

For the construction of $L^{\rm eff}_{\rm red}$, only the reduced Seiberg-Witten map $f^{\rm red}_\mu = f_\mu|_{A^i_\mu=0}$, $h^{\rm red} = h|_{A^i_\mu=0,\, C^i=0}$ is relevant. In the case where $\mathcal{U}$ is an enveloping algebra, $f^{\rm red}_\mu$ and $h^{\rm red}$ reproduce the enveloping-algebra valued Seiberg-Witten map introduced in [2]. The $so(N)$ and $usp(N)$ noncommutative models introduced in [3] fit in the context of section 3 of the present work ($\mathfrak{g} = so(N)$ respectively $\mathfrak{g} = usp(N)$ considered as a subalgebra of $\mathcal{U} = u(N)$ in the fundamental representation). The antiautomorphism in [3] (see also [4]) that selects a star-Lie subalgebra of $u(N)$ valued functions should in this context be understood as a condition on the reduced Seiberg-Witten map.

We also note that the freedom in the Seiberg-Witten map is not the only source of ambiguities for the construction of noncommutative Yang-Mills theories of the type considered here. Another one is the choice of $\mathcal{U}$ or of the representation matrices t_A respectively. We have treated the latter as initial data for defining a model but, clearly, there is a lot of freedom in this choice. Even in the case without matter fields different choices can result in inequivalent models, whereas standard commutative pure Yang-Mills theory is unique (given $\mathfrak{g}$) up to the choice of the bilinear invariant form g_{ab}.

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