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The set of gradients of a bump
by

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# THE SET OF GRADIENTS OF A BUMP 

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#### Abstract

The range of the gradient of a differentiable real-valued function with a non-empty and bounded support (i.e., a bump) is investigated. For a smooth bump $f$ on $\mathbb{R}^{2}$ it is shown that the interior $\operatorname{int} \nabla f\left(\mathbb{R}^{2}\right)$ is connected and dense in $\nabla f\left(\mathbb{R}^{2}\right)$. A purely topological characterization of such gradient ranges is however impossible. We give an example of a compact set $K \subset \mathbb{R}^{2}$ that is homeomorphic to the closed unit disk, but such that no affine image of $K$ is the gradient range of a smooth bump on $\mathbb{R}^{2}$. For smooth bumps on $\mathbb{R}^{n}$ we show that the gradient range cannot be separated by a totally disconnected set. The proof relies on a Morse-Sard type result involving irreducible separators of $\mathbb{R}^{n}$. Proofs are carried out for a class of $\mathcal{C}^{1}$ functions containing all those whose first order derivatives are Lipschitz or of bounded variation.

Finally, we present an example of a $\mathcal{C}^{1}$-smooth bump on $\ell_{2}$, which has a gradient range with non-empty and disconnected interior, and a $\mathcal{C}^{\infty}$-smooth weak bump on $\ell_{2}$ with the same property.


## 1. Introduction

A real-valued function $f: X \rightarrow \mathbb{R}$ defined on a Banach space is called a bump if its support $\operatorname{spt} f$ (i.e., the closure of $\{x \in X: f(x) \neq 0\}$ ) is non-empty and bounded. When $X$ is a Hilbert space we identify the (Fréchet) derivative $f^{\prime}(x) \in X^{*}$ in the usual way with the gradient $\nabla f(x) \in X$.

The range of the gradient $\nabla f\left(\mathbb{R}^{n}\right)$ of a $\mathcal{C}^{1}$-smooth bump $f$ on $\mathbb{R}^{n}$ is a locally connected continuum that contains $0 \in \mathbb{R}^{n}$ in its interior (see Lemma 2.5). On the other side, explicit constructions (see [6]), show that the gradient range of a $\mathcal{C}^{1}$-smooth bump on $\mathbb{R}^{n}$ may fail to be simply connected. Furthermore, in a Banach space $X$ with separable dual $X^{*}$, any analytic subset $A$ of $X^{*}$ can be realized as the range of the derivative of a $\mathcal{C}^{1}$-smooth bump on $X$ provided the interior of $A$ is connected, contains the origin, and every point $x \in A$ is accessible by an arc from $\operatorname{int} A$ (see [14]; the special case of sets with convex interior is treated in [15]; both papers contain further material on the finite- and infinite-dimensional case).

It is natural to ask whether there is a topological condition which, together with the condition that $0 \in \mathbb{R}^{n}$ belongs to the interior, characterize the range of the derivative. For example we would like to know the answers to the following two questions; the first has been raised previously in [6].
Question. Is the gradient range of a $\mathcal{C}^{1}$-smooth bump on $\mathbb{R}^{n}$ regularly closed (i.e., equal to the closure of its interior)?
Question. Does the gradient range of a $\mathcal{C}^{1}$-smooth bump on $\mathbb{R}^{n}$ have a connected interior?

[^0]In [15] it is shown that the gradient range of a $\mathcal{C}^{2}$-smooth bump on the plane is regularly closed, and in [5] it is shown that there exists a $\mathcal{C}^{1}$-smooth and Lipschitz continuous bump on $\ell_{2}$ for which the gradient range has an empty interior. (Note that $0 \in \operatorname{int} \overline{\left(\nabla f\left(\ell_{2}\right)\right)}$ always holds for a $\mathcal{C}^{1}$-smooth bump $f$ on $\ell_{2}$, see e.g. [6].)

The authors are not aware of any previous attempts to answer the question about connectedness of the interior.

The main goal of this paper is to show that the range of the gradient of a smooth bump on the plane has connected interior. Under the same assumptions, it is also shown that the gradient range is regularly closed. However, we have proved this under much weaker smoothness assumptions on the bump in [19]. Another objective is to show that the class of gradient ranges of smooth bumps on the plane is not invariant under homeomorphisms of the plane that have the origin as fixed point. This clearly excludes existence of a topological condition which, together with the condition that the origin belongs to the interior, characterize such gradient ranges.

For smooth bumps on higher dimensional spaces the situation is less clear. Our main result in this direction asserts that they have gradient ranges that cannot be separated by totally disconnected sets. (For $\mathcal{C}^{n}$-smooth bumps this has been shown already in [15].) However, we cannot even give partial answers to the above questions; for example, it is unclear whether the gradient range of a $\mathcal{C}^{\infty}$-smooth bump on $\mathbb{R}^{3}$ is regularly closed, and whether it has a connected interior.

The infinite-dimensional case is very different. We show that given any regularly open set $G$ in $\ell_{2}$, there exists a $\mathcal{C}^{1}$-smooth and Lipschitz continuous bump $f$ on $\ell_{2}$, such that $G=\operatorname{int} \nabla f\left(\ell_{2}\right)$. (An open set $G$ is regularly open if $G=\operatorname{int} \bar{G}$.) If we allow for so-called weak bumps (cf. [10]), then we can improve the smoothness to $\mathcal{C}^{\infty}$. Starting with examples from [5] and [10], the key procedure here is an inexpensive modification of a bump so that it becomes affine on an open set (Proposition 6.1).

Related references for bumps on finite-dimensional spaces are [6] and [15]. The literature on bumps in the infinite-dimensional case is substantially larger, and includes [1], [2], [3], [5], [6], [7], [10], [14], [15] and [24]. For more information about the ranges of derivatives of bumps and related matter we refer to the survey paper [4].

We proceed with the precise statements of our main results.
Theorem 1.1. Let $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$-smooth bump, such that either
(i) $\nabla f$ has a modulus of continuity $\omega=\omega(t)$ with the property $\int_{0}^{1} \frac{d t}{\omega(t)}=\infty$, or
(ii) $\nabla f$ is of bounded variation.

Then the interior $\operatorname{int} \nabla f\left(\mathbb{R}^{2}\right)$ is connected and dense in $\nabla f\left(\mathbb{R}^{2}\right)$.
In particular, the gradient range of a $\mathcal{C}^{1,1}$-smooth bump on the plane cannot have the figure-of-eight shape, which was proven by P. Hájek and M. Johanis [16] by a different method.
Theorem 1.2. There exists a set $K \subset \mathbb{R}^{2}$ which is homeomorphic to the closed unit disk, and which has the property that no affine transformation of $K$ is the gradient range of a $\mathcal{C}^{1,1}$-smooth bump on $\mathbb{R}^{2}$.

The closed unit disk is the gradient range of a $\mathcal{C}^{\infty}$-smooth bump on $\mathbb{R}^{2}$. For example, when $b(x)=\varphi\left(\|x\|^{2}\right)$, where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{C}^{\infty}$ function with bounded
support, and $2 \max _{t \in \mathbb{R}}\left|\varphi^{\prime}(t) t\right|=1$, then $\nabla b\left(\mathbb{R}^{2}\right)=\mathrm{B}[0,1]$. (On $\mathbb{R}^{n},\|\cdot\|$ always denotes the euclidean norm. $\mathrm{B}(x, r)$ denotes the open ball with centre $x$ and radius $r, \mathrm{~B}[x, r]$ the corresponding closed ball.) With $K$ as above, let $u \in \operatorname{int} K$ and put $\tilde{K}=-u+K$. There exists a homeomorphism $H: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$, such that $H(0)=0$ and $H(\mathrm{~B}[0,1])=\tilde{K}(c f .[22], \S 61 . \mathrm{V}$, Theorem 3). Note that $\mathrm{B}[0,1]$ is the gradient range of a smooth bump, whereas $\tilde{K}$ is not.

Recall that a topological space is totally disconnected if it is not connected between any pair of points. For subsets of $\mathbb{R}^{n}$ this is equivalent to the property that each (connected) component be a singleton.
Theorem 1.3. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$-smooth bump which satisfies either
(i) $\nabla f$ has a modulus of continuity $\omega=\omega(t)$ with the property $\int_{0}^{1} \frac{d t}{\omega(t)}=\infty$, or
(ii) $\nabla f$ is of bounded variation.

Then the gradient range $\nabla f\left(\mathbb{R}^{n}\right)$ cannot be separated by a totally disconnected set, i.e., whenever $C \subset \mathbb{R}^{n}$ is totally disconnected, then $\nabla f\left(\mathbb{R}^{n}\right) \backslash C$ is connected.

In particular, if $n \geq 2$ then there is no such a bump $f$ with $\nabla f\left(\mathbb{R}^{n}\right)=\mathrm{B}[0,1] \cup$ $\mathrm{B}\left[2 e_{1}, 0\right]$. In fact, there is no such $f$ with $\nabla f\left(\mathbb{R}^{n}\right)=F_{8} \times[0,1]^{n-2} \subset \mathbb{R}^{n}$, where $F_{8}=$ $\mathrm{B}[0,1] \cup \mathrm{B}\left[2 e_{1}, 0\right] \subset \mathbb{R}^{2}$, since the function $g(x, y)=f\left(x, y, z_{0}\right)$ would contradict Theorem 1.1 for a suitable $z_{0} \in \mathbb{R}^{n-2}$. There is more which can be derived using restrictions and projections in a similar manner, but it is not our goal to deal with it here.

A weak bump (cf. [10]) on a Banach space $X$ is a non-constant, real-valued function $f: X \rightarrow \mathbb{R}$ for which there exists a continuous norm $\omega$ on $X$, such that the support of $f$ is bounded with respect to $\omega$. The following is a special case of Example 6.3.
Theorem 1.4. Let $G$ be a regularly open set in $\ell_{2}$. There exists a $\mathcal{C}^{1}$-smooth and Lipschitz continuous bump $f$ on $\ell_{2}$ with int $\nabla f\left(\ell_{2}\right)=G$. Furthermore, there exists a $\mathcal{C}^{\infty}$-smooth and Lipschitz continuous weak bump $g$ on $\ell_{2}$ with $\operatorname{int} \nabla g\left(\ell_{2}\right)=G$.

The proofs of Theorems 1.1 and 1.3 both contain a step involving a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and a continuum $C \subset \mathbb{R}^{n}$, such that $\nabla f=0$ on $C$. It is crucial for the strategy of the proof that we can conclude that $f$ is constant on $C$. If $f$ is a $\mathcal{C}^{n}$ function, then this is a consequence of the classical Morse-Sard theorem. However, for less smooth functions this conclusion is known to fail (cf. [25]), and it is precisely this fact that forces us to work with the smoothness assumptions (i) or (ii). Actually, H . Whitney [25] gave an example of a simple arc in $\mathbb{R}^{3}$ and a $\mathcal{C}^{2}$ function which is critical, but not constant, on the arc. That the assumptions (i) or (ii) still suffice is because the continuum $C$ is special, it is by construction an irreducible separator of $\mathbb{R}^{n}$. (A set $C$ is an irreducible separator of $\mathbb{R}^{n}$ if $\mathbb{R}^{n} \backslash C$ is disconnected, but $\mathbb{R}^{n} \backslash C^{\prime}$ is connected whenever $C^{\prime} \subset C$ is a proper subset of $C$.)
Proposition 1.1. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$ function, such that either
(i) $\nabla f$ has a modulus of continuity $\omega=\omega(t)$ with the property $\int_{0}^{1} \frac{d t}{\omega(t)}=\infty$, or
(ii) $\nabla f$ is of bounded variation.

Then $f$ is constant on each bounded irreducible separator of $\mathbb{R}^{n}$, which is contained in the critical set $\left\{x \in \mathbb{R}^{n}: \nabla f(x)=0\right\}$.

We reemphasize that in the stated generality the result does not follow from a general Morse-Sard theorem. However, for functions on $\mathbb{R}^{2}$ we do not know any counter-example to the conclusion of the Morse-Sard theorem under the smoothness hypothesis (i). The conclusion of the Morse-Sard theorem for $\mathcal{C}^{1}$ functions on $\mathbb{R}^{2}$ satisfying (ii) was obtained in [23]. For functions on $\mathbb{R}^{n}$ with $n \geq 3$ we have already mentioned that the conclusion of the Morse-Sard theorem cannot cover functions that merely satisfy (i) or (ii), and that it is then crucial to consider irreducible separators. On the other side, the conditions (i) or (ii) cannot be omitted, even when only irreducible separators of $\mathbb{R}^{2}$ are considered. Again, this follows from [25] (or see [20]).

The organization of the paper: in Section 2 we recall some elementary facts from Topology and derive some preliminary results that are used in the subsequent sections. Section 3 contains an elementary proof for regular closedness of the gradient range of $\mathcal{C}^{2}$-smooth bumps on the plane. The proof strategy is very different from those of the other proofs for regular closedness that we are aware of. Section 4 contains the proof of the main result, Theorem 1.1. In Section 5 we prove Theorems 1.2 and 1.3 , but in the opposite order. The proof of Theorem 1.2 relies on a quantitative version of Theorem 1.3, and hence we found it expedient to prove the latter first. Section 6 contains the results about bumps and weak bumps on infinite-dimensional spaces, and finally we present the proof of Proposition 1.1 in Section 7.

## 2. Preliminaries

In this section we have collected some elementary results that are used throughout the paper. Furthermore we recall some terminology and results from [22]. Let $\mathcal{X}$ be a topological space, $S \subset \mathcal{X}$ and $a, b \in \mathcal{X} \backslash S$.

- $\quad S$ separates (the space $\mathcal{X}$ between) $a$ and $b$ if there exist sets $M, N \subset \mathcal{X}$ with $M \cup N=\mathcal{X} \backslash S, \bar{M} \cap N=\emptyset=M \cap \bar{N}$ and $a \in M, b \in N$. In this case $S$ is also called a separator. (See [22], §46.VII.)
- $\quad S$ is an irreducible separator (of $\mathcal{X}$ ) between $a$ and $b$ if $S$ is a separator between these points, but any proper subset $R \subset S$, is not.

When $\mathcal{X}$ is connected, $S$ is a separator (of $\mathcal{X}$ ) if $\mathcal{X} \backslash S$ is disconnected. (This is of course the same as $S$ being a separator between two points of $\mathcal{X}$.)

A separator always contains a separator which is also a closed set. This is the content of the next result, which is a special case of the results in [22], §46.VII.

Lemma 2.1. If $S$ separates $a$ and $b$, then there exists a closed set $C$ in $\mathcal{X}$, such that $C \subseteq S$ and $C$ separates $a$ and $b$.

Proof. By definition there are sets $M, N \subset \mathcal{X}$, such that $M \cup N=\mathcal{X} \backslash S, \bar{M} \cap N=$ $\emptyset=M \cap \bar{N}$ and $a \in M, b \in N$. It is not hard to show that $C=\partial M \cap \partial N$ has the stated properties.

The following result recalls some useful properties of irreducible separators of locally connected metric spaces.

Lemma 2.2. Let $\mathcal{X}$ be a locally connected metric space.
(1) Let $C$ be a closed set, $A$ and $B$ two distinct components of $\mathcal{X} \backslash C$, and let $a \in A$ and $b \in B$. The set $C$ is an irreducible separator between $a$ and $b$ if and only if $\partial A=C=\partial B$.
(2) If $A$ is open and connected and $B$ is a component of $\mathcal{X} \backslash \bar{A}$, the set $\partial B$ is an irreducible separator between each pair of points $a \in A$ and $b \in B$.
(3) Every closed separator between a and $b$ contains a closed irreducible separator between $a$ and $b$.

Proof. (1), (2) and (3) are Theorems 1, 2 and 3, respectively, of [22], §49.VI.
Lemma 2.3. Every irreducible separator of $\mathbb{R}^{n}$ is closed and connected.
Proof. It follows from Lemma 2.1 that every irreducible separator is closed. That it is connected then follows from Theorem 3 of [22], §59.IV.

The following elementary lemma is used to show that a given point $u \in \mathbb{R}^{n}$ is an interior point in the gradient set. A slight extension is derived in Section 5.

Lemma 2.4. Let $G \subset \mathbb{R}^{n}$ be bounded and open. Assume $f: \bar{G} \rightarrow \mathbb{R}$ is continuous, differentiable on $G, f=0$ on $\partial G$ and $f$ is not identically zero on $\bar{G}$. Then $0 \in \mathbb{R}^{n}$ is an interior point of $\nabla f(G)$.
Proof. Let $M=\max f(\bar{G})>0$. (In the opposite case $f$ should be replaced by $-f$.) Find $x_{0} \in \bar{G}$ such that $f\left(x_{0}\right)=M$. Obviously $x_{0} \in G$, and therefore $\nabla f\left(x_{0}\right)=0$, so that $0 \in \nabla f(G)$. More generally, let $\tilde{f}(x)=f(x)-u \cdot x$, where $u \in \mathbb{R}^{n}$ is arbitrary, such that $\|u\|<M / \operatorname{diam} G$. Find $\tilde{x}_{0} \in \bar{G}$, such that $\tilde{f}\left(\tilde{x}_{0}\right)=\max \tilde{f}(\bar{G})$. If $\tilde{x}_{0} \in \partial G$, then $\tilde{f}\left(x_{0}\right)-\tilde{f}\left(\tilde{x}_{0}\right)=f\left(x_{0}\right)-f\left(\tilde{x}_{0}\right)-u \cdot\left(x_{0}-\tilde{x}_{0}\right)=M-0-u \cdot\left(x_{0}-\tilde{x}_{0}\right) \geq$ $M-\|u\| \operatorname{diam} G>0$, hence $\tilde{f}\left(\tilde{x}_{0}\right)<\tilde{f}\left(x_{0}\right)$, which contradicts $\tilde{f}\left(\tilde{x}_{0}\right)=\max \tilde{f}(\bar{G})$. Hence $\tilde{x}_{0} \in G$, and thus $\nabla \tilde{f}\left(\tilde{x}_{0}\right)=0$, that is, $\nabla f\left(\tilde{x}_{0}\right)=u$. This shows that $\mathrm{B}(0, M / \operatorname{diam} G) \subseteq \nabla f(G)$.

Without additional smoothness hypotheses on the bump we cannot say so much about its gradient range.
Lemma 2.5. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$-smooth bump. The gradient range $\nabla f\left(\mathbb{R}^{n}\right)$ is a locally connected continuum, which contains $0 \in \mathbb{R}^{n}$ in its interior.

Proof. The support of $f$ is contained in an open ball B , so $0 \in \operatorname{int} \nabla f(\mathrm{~B})=$ $\operatorname{int} \nabla f\left(\mathbb{R}^{n}\right)$ by Lemma 2.4. The closure $\overline{\mathrm{B}}$ of the ball is a locally connected continuum, so the same is true of $\nabla f(\overline{\mathrm{~B}})=\nabla f\left(\mathbb{R}^{n}\right)$ (see [22], §50.II, Theorem 5).

## 3. Regular closedness from the inverse function theorem

We are aware of different types of arguments for regular closedness of the gradient range. The following is an elementary and, we believe, elegant argument that was developed in discussions with Bernd Kirchheim. It is based on the inverse function theorem and a trick that has previously been used in [9] and [18]. Using less elementary tools it works also for $\mathcal{C}^{1,1}$-smooth bumps on the plane. As mentioned in the Introduction this result was also obtained for $\mathcal{C}^{2}$-smooth bumps in [15] by a different method, and it is also contained in our Theorems 1.1 and 1.3.
Lemma 3.1. Assume $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is a $\mathcal{C}^{2}$-smooth bump. Then $\nabla f\left(\mathbb{R}^{2}\right)$ is regularly closed. The same is true also for $\mathcal{C}^{1,1}$-smooth bump functions.

Proof. Let $f$ be a $\mathcal{C}^{2}$-smooth bump and put $K=\nabla f\left(\mathbb{R}^{2}\right)$. Then $K$ is clearly a continuum, and $0 \in \operatorname{int} K$ by Lemma 2.4. If $x \in \mathbb{R}^{2}$ is a point where $\operatorname{det} \nabla^{2} f(x) \neq 0$, then, in view of the inverse function theorem, there exists an open neighbourhood
$U$ of $x$ such that the restriction of $\nabla f$ to $U$ is a $\mathcal{C}^{1}$ diffeomorphism. It follows in particular, that $\left.\nabla f\right|_{U}$ is an open map, and thus that $\nabla f(x) \in$ int $K$. Hence obviously $\nabla f(\bar{M}) \subseteq \overline{\operatorname{int} K}$, where $M=\left\{x: \operatorname{det} \nabla^{2} f(x) \neq 0\right\}$. So we may focus on gradients corresponding to $\Omega=\{x: \nabla f(x) \neq 0\} \backslash \bar{M}$. We know already that $\nabla f(\partial \Omega) \subseteq \overline{\operatorname{int} K}$. Since int $\nabla f(\Omega) \subseteq \operatorname{int} K$, it is enough to show that $\partial \nabla f(\Omega) \subseteq$ $\nabla f(\partial \Omega)$. For that purpose, define for each $\varepsilon>0$ the auxiliary map $F_{\varepsilon}(x)=$ $\nabla f(x)+\varepsilon\left(-x_{2}, x_{1}\right), x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$. Now

$$
\nabla F_{\varepsilon}(x)=\nabla^{2} f(x)+\varepsilon\left(\begin{array}{rr}
0 & -1 \\
1 & 0
\end{array}\right)
$$

and hence using the formula $\operatorname{det}(A+B)=\operatorname{det} A+\operatorname{cof} A \cdot B+\operatorname{det} B$ and that $\operatorname{cof} A \cdot B=0$ when $A$ is symmetric and $B$ is anti-symmetric, we $\operatorname{deduce} \operatorname{det} \nabla F_{\varepsilon}(x)=$ $\operatorname{det} \nabla^{2} f(x)+\varepsilon^{2}$ for all $x$. If therefore we restrict $x \in \Omega$, then $\operatorname{det} \nabla F_{\varepsilon}(x)=\varepsilon^{2} \neq 0$, so that $F_{\varepsilon}: \Omega \rightarrow \mathbb{R}^{2}$ is an open map. (Notice that this argument does not work when $n \geq 3$.) Because $\Omega$ is bounded, $F_{\varepsilon}(\bar{\Omega})$ is compact, and therefore $F_{\varepsilon}(\bar{\Omega}) \supseteq$ $\overline{F_{\varepsilon}(\Omega)}=F_{\varepsilon}(\Omega) \cup \partial F_{\varepsilon}(\Omega)$, and thus $F_{\varepsilon}(\Omega) \cup \partial F_{\varepsilon}(\Omega) \subseteq F_{\varepsilon}(\Omega) \cup F_{\varepsilon}(\partial \Omega)$. Since $F_{\varepsilon}(\Omega)$ is open, it follows that $\partial F_{\varepsilon}(\Omega) \subseteq F_{\varepsilon}(\partial \Omega)$.

We assert that letting $\varepsilon \searrow 0, \partial \nabla f(\Omega) \subseteq \nabla f(\partial \Omega)$ results. Indeed, let $u \in$ $\partial \nabla f(\Omega)$ and $r>0$. Take $x \in \Omega$ with $\nabla f(x) \in \mathrm{B}\left(u, \frac{r}{2}\right)$ and $v \in \mathrm{~B}\left(u, \frac{r}{2}\right) \backslash \nabla f(\Omega)$. If $v \in \nabla f(\partial \Omega)$, then obviously $\mathrm{B}(u, r) \cap \nabla f(\partial \Omega) \neq \emptyset$. Suppose $v \notin \nabla f(\partial \Omega)$, so that $v \in \mathrm{~B}\left(u, \frac{r}{2}\right) \backslash \nabla f(\bar{\Omega})$. Because $F_{\varepsilon}$ converges uniformly to $\nabla f$ on $\bar{\Omega}$, we have for sufficiently small $\varepsilon>0$ that $v \in \mathrm{~B}\left(u, \frac{r}{2}\right) \backslash F_{\varepsilon}(\bar{\Omega})$ and $F_{\varepsilon}(x) \in \mathrm{B}\left(u, \frac{r}{2}\right)$. Therefore, for such $\varepsilon$, we can select $u_{\varepsilon} \in \mathrm{B}\left(u, \frac{r}{2}\right) \cap \partial F_{\varepsilon}(\Omega)$. But $\partial F_{\varepsilon}(\Omega) \subseteq F_{\varepsilon}(\partial \Omega)$, so there is $x_{\varepsilon} \in \partial \Omega$ with $F_{\varepsilon}\left(x_{\varepsilon}\right) \in \mathrm{B}\left(u, \frac{r}{2}\right)$. Again by uniform convergence, $\nabla f\left(x_{\varepsilon}\right) \in \mathrm{B}(u, r)$ for sufficiently small $\varepsilon$, and consequently $\mathrm{B}(u, r) \cap \nabla f(\partial \Omega) \neq \emptyset$ also in this case. Since $\nabla f(\partial \Omega)$ is closed, this proves the assertion and concludes the proof.

Now, let us consider the more general case $f \in \mathcal{C}^{1,1}\left(\mathbb{R}^{2}\right)$. Then $F(y)=\nabla f(y)$ is Lipschitz continuous and hence differentiable almost everywhere. First let $x \in \mathbb{R}^{2}$ be such that $F$ is differentiable at $x$ and $\operatorname{det} \nabla F(x) \neq 0$. We will show that $F(x) \in \operatorname{int} K$. Let $L=\nabla F(x)$ and let $\tilde{L}(y)=F(x)+L \cdot(y-x)$. By the assumption, $L$ is an invertible matrix, and $\tilde{L}$ is an affine bijection, and $\|L v\| \geq\|v\| /\left\|L^{-1}\right\|$ for $v \in \mathbb{R}^{2}$. Obviously, $|\operatorname{deg}(\tilde{L}, \mathrm{~B}[x, r], z)|=1$ for every $r>0$ and every $z \in$ $\mathrm{B}\left(F(x), r /\left\|L^{-1}\right\|\right) \subseteq \tilde{L}(\mathrm{~B}(x, r))$. Since $\tilde{L}$ is a good approximation of $F$, it is easy to show that there is $\varepsilon>0$ such that also $|\operatorname{deg}(F, \mathrm{~B}[x, \varepsilon], z)|=1$ for every $z$ in $\mathrm{B}\left(F(x), \varepsilon / 2\left\|L^{-1}\right\|\right)$. The last set is therefore a subset of $K=F\left(\mathbb{R}^{2}\right)$ and $\nabla f(x)=$ $F(x)$ is its interior point. Because $F$ is differentiable in a dense set, it remains only to consider points of the open set
$\Omega=\{x: F(x) \neq 0$ and there is a neighbourhood of $x$ where $\operatorname{det} \nabla F=0$

$$
\text { whenever } \nabla F \text { exists\}. }
$$

The rest of the proof is the same, we just have to observe that, since $\nabla F$ is symmetric at almost all points where it exists, $\operatorname{det} \nabla F_{\varepsilon}=\varepsilon^{2}$ a.e. in $\Omega$, hence $F_{\varepsilon}$ is a mapping of bounded distortion. Therefore $F_{\varepsilon}$ is open on $\Omega$.

## 4. Proof of Theorem 1.1

The proof relies in an essential way upon the notion of index for planar closed curves and related properties of the topology of the plane. The absence of these features in $\mathbb{R}^{n}$ when $n \geq 3$ forces us to work exclusive with bumps on $\mathbb{R}^{2}$. It is
at present not at all clear whether or not a similar result is true for bumps on $\mathbb{R}^{n}$ when $n \geq 3$. (As mentioned in the Introduction the result fails for bumps on $\ell_{2}$.) Another issue is the smoothness of the bump. It is far from clear to us whether or not the result is true for bumps on $\mathbb{R}^{2}$ that are merely $\mathcal{C}^{1}$-smooth. In the proof the additional smoothness is required only when using Proposition 1.1. As mentioned in the Introduction such a result fails without additional smoothness assumptions.

In this section a (planar) curve is a continuous map $\gamma:[0,1] \rightarrow \mathbb{R}^{2}$. With an abuse of the notation we denote by $\gamma$ also its image $\gamma([0,1])$. The points $\gamma(0)$ and $\gamma(1)$ are called the end-points of the curve $\gamma$, and $\gamma$ is closed when $\gamma(0)=\gamma(1)$. When $\gamma$ is a closed curve and $x \notin \gamma$ the index of $\gamma$ with respect to $x$ is denoted by $\operatorname{ind}_{\gamma}(x)$. We refer to [22] for the definition and the properties of the index. In the definition below, which is central to the proof, we use the index to distinguish a class of points on a curve.
Definition 4.1. A point $u \in \mathbb{R}^{2}$ is an essential point of the curve $\gamma \subseteq \mathbb{R}^{2}$ with respect to the set $K \subseteq \mathbb{R}^{2}$ if $u$ is an end-point of $\gamma$ or if for every neighbourhood $U$ of $u$ the set

$$
I_{\gamma-\gamma_{0}}(U)=\left\{\operatorname{ind}_{\gamma-\gamma_{0}}(v): v \in U \backslash\left(\gamma \cup \gamma_{0} \cup K\right)\right\}
$$

contains at least two elements, where $\gamma_{0}$ is any curve connecting the end-points of $\gamma$ and such that $u \notin \gamma_{0}$.

Remarks. (1) The definition is independent of the choice of $\gamma_{0}$ : Let $\tilde{\gamma}_{0}$ be another curve that connects the end-points of $\gamma$ and for which $u \notin \tilde{\gamma}_{0}$. Since $\gamma_{0} \cup \tilde{\gamma}_{0}$ is compact we can find a connected neighbourhood $\tilde{U}$ of $u$ that does not intersect $\gamma_{0} \cup \tilde{\gamma}_{0}$. On $\tilde{U}$ the index of the closed curve $\gamma_{0}-\tilde{\gamma}_{0}$ is constant, say equal to $c$, so $I_{\gamma-\tilde{\gamma}_{0}}(\tilde{U})=I_{\gamma-\gamma_{0}}(\tilde{U})+c$. Now, if $U$ is a neighbourhood of $u$ such that $I_{\gamma-\gamma_{0}}(U)$ contains less than two points, then the same is true for $I_{\gamma-\tilde{\gamma}_{0}}(U \cap \tilde{U})$. Similarly for $\gamma_{0}$ and $\tilde{\gamma}_{0}$ interchanged.
(2) Obviously, if $u$ is an essential point of $\gamma$ with respect to $K$, then $u \in \gamma$ and $u \notin \operatorname{int} K$.

For the proof of Theorem 1.1 we need some properties of essential points. We proceed to establish these.

Lemma 4.1. Let $F: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ be continuous and $\hat{\gamma}$ be a curve in $\mathbb{R}^{2}$. Assume $u$ is an essential point of $\gamma=F \circ \hat{\gamma}$ with respect to $K=F\left(\mathbb{R}^{2}\right)$. If $\hat{\gamma}_{0}$ is any curve connecting the end-points of $\hat{\gamma}$, then $u \in \gamma_{0}=F \circ \hat{\gamma}_{0}$.

Proof. Assume for a contradiction that $u \notin \gamma_{0}$. This means that $\gamma_{0}$ can be used in the Definition 4.1 of essential point. Let $U$ be a neighbourhood of $u$. For $s, t \in[0,1]$, define the homotopies $\hat{\Gamma}_{s}(t)=s \hat{\gamma}_{0}(t)+(1-s) \hat{\gamma}(t)$ and

$$
\Gamma_{s}(t)=\left\{\begin{array}{cl}
F\left(\hat{\Gamma}_{s}(2 t)\right) & \text { for } 0 \leq t<\frac{1}{2} \\
\gamma_{0}(2-2 t) & \text { for } \frac{1}{2} \leq t \leq 1
\end{array}\right.
$$

Here $\Gamma$ is a homotopy in $K$ carrying $\Gamma_{0}=\gamma-\gamma_{0}$ to $\Gamma_{1}=\gamma_{0}-\gamma_{0}$, so that, by homotopy invariance of the index, $\operatorname{ind}_{\gamma-\gamma_{0}}(v)=\operatorname{ind}_{\gamma_{0}-\gamma_{0}}(v)=0$ for every $v \in \mathbb{R}^{2} \backslash K=\mathbb{R}^{2} \backslash\left(\gamma \cup \gamma_{0} \cup K\right)$. Thus $I_{\gamma-\gamma_{0}}(U) \subseteq\{0\}$, and $u$ is not an essential point of $\gamma$ with respect to $K$. This contradiction concludes the proof.

Recall that a set is said to cut between two points if it intersects every continuum that contains the two points (see [22], §47.IX). The next lemma summarizes additional properties of cuts and separators of $\mathbb{R}^{n}$ that are used below.

Lemma 4.2. For a closed set $H \subset \mathbb{R}^{n}$ and points $a, b \in \mathbb{R}^{n}$ the following are equivalent:
(1) $H$ separates $a$ and $b$;
(2) there exist disjoint open sets $G_{0}, G_{1}$, such that $G_{0} \cup G_{1}=\mathbb{R}^{n} \backslash H, a \in G_{0}$, $b \in G_{1}$;
(3) $H$ cuts between $a$ and $b$;
(4) $H$ intersects every curve connecting $a$ and $b$;
(5) $a$ and $b$ are in different components of $\mathbb{R}^{n} \backslash H$.

Proof. (1) $\Longleftrightarrow(2)$ follows from the definition. $(2) \Longrightarrow(3) \Longrightarrow(4)$ is obviously true (for any set $H$ ). (4) $\Longrightarrow(5)$ since the components of the open set $\mathbb{R}^{n} \backslash H$ are path-connected. For $(5) \Longrightarrow$ (2) let $G_{0}$ be the component of $\mathbb{R}^{n} \backslash H$ containing $a$ and let $G_{1}=\mathbb{R}^{n} \backslash\left(H \cup G_{0}\right)$. Then obviously $\partial G_{0} \subseteq H$, hence $G_{0} \cup H$ is closed and $G_{1}$ open.

Lemma 4.3. Let $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$-smooth bump, which satisfies either
(i) $\nabla f$ has a modulus of continuity $\omega=\omega(t)$ with the property $\int_{0}^{1} \frac{d t}{\omega(t)}=\infty$, or
(ii) $\nabla f$ is of bounded variation.

If $\hat{\gamma}$ is a curve in $\mathbb{R}^{2}$ and $\gamma=\nabla f(\hat{\gamma})$, then, besides the end-points, $\gamma$ has no essential points with respect to $K=\nabla f\left(\mathbb{R}^{2}\right)$.

Proof. We can assume without loss in generality that $a=\hat{\gamma}(0) \neq \hat{\gamma}(1)=b$. Assume $u \in \gamma \backslash\{a, b\}$ is an essential point of $\gamma$ with respect to $K$. Obviously, $u \notin \operatorname{int} K$, so that in particular, $u \neq 0$ by Lemma 2.5. Put $H=(\nabla f)^{-1}(u)$; then $H \subset \operatorname{spt} f$, so $H$ is bounded. Furthermore, by Lemma 4.1, $H$ intersects every curve that has $a$ and $b$ as end-points, so according to Lemma 4.2, $H$ has the properties (1)-(5). In particular, $H$ separates $a$ and $b$, hence by Lemma 2.2 it contains a closed subset $C$, which is the common boundary of two disjoint open and connected sets $A, B \subset \mathbb{R}^{2}$. Since $C \subseteq H$ and $H$ is bounded, at least one of the sets $A, B$ must be bounded too. Choose one and call it $\Omega$. Note that $\Omega$ is then an open, bounded and connected set and $\partial \Omega$ is an irreducible separator. Now $\partial \Omega \subseteq H$, so $\nabla f=u$ on $\partial \Omega$, and it therefore follows from Proposition 1.1 applied to the function $x \mapsto f(x)-u \cdot x$ and the irreducible separator $\partial \Omega$, that $f(x)-u \cdot x=c$ on $\partial \Omega$ for some constant c. Finally, use Lemma 2.4 to the function $x \mapsto f(x)-u \cdot x-c$ and the set $\Omega$, and infer that $u \in \operatorname{int} \nabla f(\Omega) \subseteq \operatorname{int} K$. This contradiction concludes the proof.

We identify $\mathbb{R}^{2}$ with $\mathbb{C}$ in the usual way and write $\mathbb{S}^{1}=\{z \in \mathbb{C}:|z|=1\}$ and $\mathbb{S}^{2}=\mathbb{C} \cup\{\infty\}$. When $f: W \rightarrow \mathbb{S}^{1}$ is a continuous map from a subset $W \subseteq \mathbb{S}^{2}$ to $\mathbb{S}^{1}$ and $K \subseteq W$ we write $f \sim 1$ on $K$ if there exists a continuous function $\phi: K \rightarrow \mathbb{R}$, such that $f(w)=e^{i \phi(w)}$ for $w \in K$. The following result is instrumental in the proof of Theorem 1.1.

Theorem 4.1 (Eilenberg [13], pages 75 and 88). Let $u, v \in \mathbb{C}$ be two distinct points. A subset $K \subseteq \mathbb{S}^{2} \backslash\{u, v\}$ does not cut $\mathbb{S}^{2}$ between $u$ and $v$ if and only if $r \sim 1$ on $K$, where

$$
r(w)=\left\{\begin{array}{cl}
\frac{u-w}{|u-w|} \frac{|v-w|}{v-w} & \text { if } w \in \mathbb{C} \backslash\{u, v\} \\
1 & \text { if } w=\infty .
\end{array}\right.
$$

Obviously, if $K$ is bounded, then $K$ does not cut $\mathbb{R}^{2}=\mathbb{C}$ between $u$ and $v$ if and only if $r \sim 1$ on $K$. See also [22], $\S 62$.II.

Proof of Theorem 1.1. Let $K=\nabla f\left(\mathbb{R}^{2}\right)$, and recall from Lemma 2.5 that $0 \in \operatorname{int} K$. Let $G$ be the component of $K$ that contains 0 . It suffices to show that $G$ is dense in $K$. We argue by contradiction, and assume that we can find a $u \in K \backslash \bar{G}$. Since $0 \in G$ it follows in particular that $\partial G$ separates $\mathbb{S}^{2}$ between 0 and $u$.

Take $x_{0}, x \in \mathbb{R}^{2}$ with $\nabla f\left(x_{0}\right)=0$ and $\nabla f(x)=u$, and choose a point $\tilde{u}$ near $u$, not on the straight line through 0 and $u$, and so that the triangle $0 \tilde{u} u$ is oriented clockwise. Denoting by $\overline{v w}$ the linear parameterization of the oriented segment $[v, w]$, define $\gamma_{0}=\overline{0 u}$ and $\gamma_{1}=\overline{0 \tilde{u}}+\overline{\tilde{u} u}$. Since $\mathbb{R}^{2} \backslash \gamma_{0}$ and $\mathbb{R}^{2} \backslash \gamma_{1}$ do not cut $\mathbb{S}^{2}$ between $v=0$ and $u$, we have by Theorem 4.1 the representations

$$
\begin{equation*}
r(w)=e^{i \phi_{0}(w)} \text { on } \mathbb{R}^{2} \backslash \gamma_{0} \quad \text { and } \quad r(w)=e^{i \phi_{1}(w)} \text { on } \mathbb{R}^{2} \backslash \gamma_{1}, \tag{4.1}
\end{equation*}
$$

where $\phi_{0}: \mathbb{R}^{2} \backslash \gamma_{0} \rightarrow \mathbb{R}$ and $\phi_{1}: \mathbb{R}^{2} \backslash \gamma_{1} \rightarrow \mathbb{R}$ are continuous functions. (It is not difficult to find $\phi_{0}, \phi_{1}$ explicitly.)

For each $w \in \mathbb{R}^{2} \backslash\left(\gamma_{0} \cup \gamma_{1}\right)$, $\operatorname{ind}_{\gamma_{0}-\gamma_{1}}(w) \in\{0,1\}$. Choose an arbitrary point $w_{0} \in \mathbb{R}^{2} \backslash\left(\gamma_{0} \cup \gamma_{1}\right)$ with $\operatorname{ind}_{\gamma_{0}-\gamma_{1}}\left(w_{0}\right)=0$. Because $e^{i \phi_{0}\left(w_{0}\right)}=r\left(w_{0}\right)=e^{i \phi_{1}\left(w_{0}\right)}$, $\phi_{0}\left(w_{0}\right)-\phi_{1}\left(w_{0}\right)=2 k_{0} \pi$ for some $k_{0} \in \mathbb{Z}$, and since $\left(\phi_{0}-\phi_{1}\right) / 2 \pi: \mathbb{R}^{2} \backslash\left(\gamma_{0} \cup \gamma_{1}\right) \rightarrow \mathbb{Z}$ is continuous, it follows that

$$
\phi_{0}(w)-\phi_{1}(w)=2 \pi k_{0} \quad \text { for all } w, \text { such that } \operatorname{ind}_{\gamma_{0}-\gamma_{1}}(w)=0 .
$$

Similarly, there is a $k_{1} \in \mathbb{Z}$, such that

$$
\phi_{0}(w)-\phi_{1}(w)=2 \pi k_{1} \quad \text { for all } w, \text { such that } \operatorname{ind}_{\gamma_{0}-\gamma_{1}}(w)=1,
$$

and hence we have shown that for $w \in \mathbb{R}^{2} \backslash\left(\gamma_{0} \cup \gamma_{1}\right)$

$$
\begin{equation*}
\phi_{0}(w)-\phi_{1}(w)=2 \pi k_{0}+2 \pi\left(k_{1}-k_{0}\right) \operatorname{ind}_{\gamma_{0}-\gamma_{1}}(w) \tag{4.2}
\end{equation*}
$$

holds. Now fix a curve $\hat{\gamma}$ with $\hat{\gamma}(0)=x_{0}$ and $\hat{\gamma}(1)=x$ (e.g. $\hat{\gamma}=\overline{x_{0} x}$ ), and let $\gamma=(\nabla f) \circ \hat{\gamma}$. Note that $\gamma \subseteq K$ and that $\gamma(0)=0$ and $\gamma(1)=u$. Take $v \in \partial G \backslash \gamma_{0}$. Clearly, $v$ is not an end-point of $\gamma$ and it is not an interior point of $K$ (since $G$ was a component of int $K$ ). Hence $I_{\gamma-\gamma_{0}}(V) \neq \emptyset$ for every neighbourhood $V$ of $v$ (we use $K=\nabla f\left(\mathbb{R}^{2}\right)$ in Definition 4.1). By Lemma 4.3, $v$ is not an essential point of $\gamma$ with respect to $K$ either. We can therefore find $\varepsilon=\varepsilon\left(v, \gamma_{0}\right)>0$, such that $I_{\gamma-\gamma_{0}}(\mathrm{~B}(v, \varepsilon))$ contains at most one element, and thus precisely one element. Denote the unique element by $\widetilde{\operatorname{ind}}_{\gamma-\gamma_{0}}(v) \in \mathbb{Z}$. As indicated the unique element is independent of the choice of $\varepsilon=\varepsilon\left(v, \gamma_{0}\right)$ since $I_{\gamma-\gamma_{0}}\left(V^{\prime}\right) \subseteq I_{\gamma-\gamma_{0}}\left(V^{\prime \prime}\right)$ when $V^{\prime} \subseteq V^{\prime \prime}$. Hence we have hereby defined a function $\operatorname{ind}_{\gamma-\gamma_{0}}: \partial G \backslash \gamma_{0} \rightarrow \mathbb{Z}$, and by the properties of the index, it is a locally constant, continuous function. Define ind $_{\gamma-\gamma_{1}}$ similarly on $\partial G \backslash \gamma_{1}$.

Take $v \in \partial G \backslash\left(\gamma_{0} \cup \gamma_{1}\right)$ and let $\varepsilon=\min \left\{\varepsilon\left(v, \gamma_{0}\right), \varepsilon\left(v, \gamma_{1}\right)\right.$, $\left.\operatorname{dist}\left(v, \gamma_{0} \cup \gamma_{1}\right)\right\}$. Since $v \notin \operatorname{int} K$, the set $\mathrm{B}(v, \varepsilon) \backslash\left(\gamma \cup \gamma_{0} \cup \gamma_{1}\right)$ is non-empty, and for each of its elements $w$ we have

$$
\operatorname{ind}_{\gamma-\gamma_{1}}(w)=\operatorname{ind}_{\gamma-\gamma_{0}}(w)+\operatorname{ind}_{\gamma_{0}-\gamma_{1}}(w)
$$

Now $\operatorname{ind}_{\gamma-\gamma_{1}}(w)=\widetilde{\operatorname{ind}}_{\gamma-\gamma_{1}}(v), \operatorname{ind}_{\gamma-\gamma_{0}}(w)=\widetilde{\operatorname{ind}}_{\gamma-\gamma_{0}}(v)$ and $\operatorname{ind}_{\gamma_{0}-\gamma_{1}}$ is constant on $\mathrm{B}(v, \varepsilon)$, so it follows that

$$
\begin{equation*}
\widetilde{\operatorname{ind}}_{\gamma-\gamma_{1}}(v)=\widetilde{\operatorname{ind}}_{\gamma-\gamma_{0}}(v)+\operatorname{ind}_{\gamma_{0}-\gamma_{1}}(v) \tag{4.3}
\end{equation*}
$$

Define

$$
\phi(v)= \begin{cases}\phi_{0}(v)+2 \pi\left(k_{1}-k_{0}\right) \widetilde{\operatorname{ind}}_{\gamma-\gamma_{0}}(v) & \text { for } v \in \partial G \backslash \gamma_{0} \\ \phi_{1}(v)+2 \pi\left(k_{1}-k_{0}\right) \widetilde{\operatorname{ind}}_{\gamma-\gamma_{1}}(v)+2 \pi k_{0} & \text { for } v \in \partial G \backslash \gamma_{1}\end{cases}
$$

Observe that the definition is consistent by (4.2) and (4.3). Since $\gamma_{0} \cap \gamma_{1}=\{0, u\} \subset$ $\mathbb{R}^{2} \backslash \partial G, \phi(v)$ is defined for every $v \in \partial G$, and because $\widetilde{\mathrm{ind}}_{\gamma-\gamma_{0}}, \widetilde{\operatorname{ind}}_{\gamma-\gamma_{1}}$ are continuous, also $\phi: \partial G \rightarrow \mathbb{R}$ is continuous. Finally, we check from (4.1) and the definition of $\phi$ that $r(v)=e^{i \phi(v)}$ for all $v \in \partial G$, so that, according to Theorem 4.1, $\partial G$ does not cut $\mathbb{S}^{2}$ between $v=0$ and $u$. This contradiction concludes the proof.

## 5. Separators of the gradient set. Proofs of Theorems 1.2 and 1.3.

In terms of topological dimension Theorem 1.3 states that any separator of the gradient range of a suitably smooth bump on $\mathbb{R}^{n}$ must at least be of dimension one, that is, the gradient range is two-dimensionally connected (cf. [22], §46.XI). It is not hard to show that a particular consequence is that the topological dimension of the gradient range at each of its points is at least two, i.e. if $f$ denotes the bump, $\operatorname{dim}_{u} \nabla f\left(\mathbb{R}^{n}\right) \geq 2$ for all $u \in \nabla f\left(\mathbb{R}^{n}\right)$. When $n=2$ this is easily seen to imply that $\nabla f\left(\mathbb{R}^{2}\right)$ is regularly closed. However, it does not follow from Theorem 1.3 that $\nabla f\left(\mathbb{R}^{2}\right)$ has connected interior, as is obvious from the following

Example 5.1. Let

$$
K=[0,1]^{2} \backslash \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{2^{j}-1} Q_{j, k},
$$

where $Q_{j, k}=x_{j, k}+\left(0,4^{-j}\right)^{2}$ and $x_{j, k}=\left(2^{-1}+2^{-j}, k 2^{-j}\right)$. Then $K$ is a locally connected, compact and two-dimensionally connected subset of $\mathbb{R}^{2}$, and it is not hard to see that the interior int $K$ has two components.

Proof of Theorem 1.3. Suppose the theorem is false, so that there exists a bump $f$ satisfying all the hypotheses, but for which the gradient set $K=\nabla f\left(\mathbb{R}^{n}\right)$ can be separated by a totally disconnected set $C \subset \mathbb{R}^{n}$. By Lemma 2.1 we can assume that $C$ is closed and therefore 0-dimensional, cf. [17], Chapter II, $\S 4$ A.

We assert that $C \cap \partial K$ separates $K$. To prove this, use Lemma 2.2 on the locally connected metric space $K$ (cf. Lemma 2.5), to find an irreducible separator $F \subseteq C \cap K$. Assume there is $x \in F \cap$ int $K$ and $\varepsilon>0$ such that $\mathrm{B}(x, \varepsilon) \subset K$. Since $F \cap \mathrm{~B}(x, \varepsilon)$ is 0 -dimensional, it does not separate $\mathrm{B}(x, \varepsilon)$ (see [17], Chapter IV, $\S 5$, Corollary 2). Therefore $F \backslash\{x\}$ separates $K$, which contradicts its irreducibility. Consequently, $F \subseteq \partial K$ and $C \cap \partial K$ separates $K$ as asserted.

Henceforth we assume that $C \subseteq \partial K$. Let $K_{0}$ denote a component of $K \backslash C$ that does not contain $0 \in \mathbb{R}^{n}$. Since $K$ is locally connected, $K_{0}$ is open relative to $K$ (cf. [22], §49.II, Theorem 4) and its boundary relative to $K$ is $\partial_{K} K_{0}=C \cap \bar{K}_{0}$. The set $U=(\nabla f)^{-1}\left(K_{0}\right)$ is therefore non-empty, bounded and open, and $\partial U \subseteq$ $(\nabla f)^{-1}(C)$. Select a component $V$ of $U$, let $W$ denote the unbounded component of $\mathbb{R}^{n} \backslash \bar{V}$ and let $\Omega$ be the component of $\mathbb{R}^{n} \backslash \bar{W}$ that contains $V$. By Lemma 2.2(2), $\partial \Omega$ is an irreducible separator between any pair of points $x \in \Omega$ and $y \in W$, and thus, $\partial \Omega$ is connected by Lemma 2.3. It is also a subset of $\partial U$, so that $\nabla f(\partial \Omega)$ is a connected subset of $C$ and hence is a singleton: $\nabla f(x) \equiv u \in C$ on $\partial \Omega$. Apply Proposition 1.1 to the function $x \mapsto f(x)-u \cdot x$ and the irreducible separator
$\partial \Omega$ and deduce that $f(x)-u \cdot x=c$ on $\partial \Omega$ for some constant $c \in \mathbb{R}$. Next use the Lemma 2.4 on the function $x \mapsto f(x)-u \cdot x-c$ and the set $\Omega$ to infer that $u \in \operatorname{int} \nabla f(\Omega) \subseteq \operatorname{int} K$. Thus $C \cap \operatorname{int} K \neq \emptyset$, which is the desired contradiction.

The following result is a quantitative version of Theorem 1.3. It is used in the proof of Theorem 1.2.
Proposition 5.1. Let $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1,1}$-smooth bump and put $K=\nabla f\left(\mathbb{R}^{2}\right)$, $L=\operatorname{Lip}(\nabla f)$ and $A=\mathcal{L}^{2}(\operatorname{cospt}(f))$ (i.e., the area of the convex hull of the support). Suppose that $C$ is a closed set that separates $K$ and let $K_{0}$ denote a component of $K \backslash C$ that does not contain 0 . Define the numbers $r=\sup \left\{\operatorname{diam} C_{0}\right.$ : $C_{0}$ component of $\left.C \cap \bar{K}_{0}\right\}$ and $R=\sup \left\{\operatorname{dist}\left(u, C \cap \bar{K}_{0}\right): u \in K_{0}\right\}$. If each component of $C \cap \bar{K}_{0}$ intersects $\partial K$, then

$$
\begin{equation*}
\frac{r}{R} \geq \frac{e^{-\frac{A L^{2}}{R^{2}}}}{\sqrt{2 e\left(\frac{A L^{2}}{R^{2}}+1\right)}} \tag{5.1}
\end{equation*}
$$

Proof. We shall derive the inequality by establishing first a stronger, but less appealing inequality. We may assume that $r<R$, since otherwise (5.1) is trivially satisfied. Observe that for the rescaled function $\tilde{f}(x)=\frac{L}{R^{2}} f\left(\frac{R}{L} x\right)$ we have the relations $\tilde{r}=\frac{r}{R}, \tilde{R}=1, \tilde{L}=1$ and $\tilde{A}=\frac{A L^{2}}{R^{2}}$, and consequently that we can assume without loss in generality that $0<r<R=1$ and $L=1$. With these simplifications in place, select $\bar{u} \in K_{0}$ such that $\operatorname{dist}\left(\bar{u}, C \cap \bar{K}_{0}\right)=1$, and let $\bar{x}$ be a point where $\nabla f(\bar{x})=\bar{u}$. Define the open set $U=(\nabla f)^{-1}\left(K_{0}\right) \subset \operatorname{spt}(f)$. Let $V$ denote the component of $U$ that contains $\bar{x}$, and define the open set $\Omega$ as in the proof of Theorem 1.3. Note that $\Omega \subset \operatorname{cospt}(f)$, and also that $\nabla f(\partial \Omega)$ is a connected subset of $C \cap \bar{K}_{0}$, which is therefore contained in a single component $C_{0}$ of $C \cap \bar{K}_{0}$. By assumption we can take $u_{0} \in C_{0} \cap \partial K$. Define $g(x)=f(x)-u_{0} \cdot x, x \in \mathbb{R}^{2}$, and observe that $\|\nabla g(x)\| \leq r$ on $\partial \Omega$. The proof of inequality (5.1) relies on the following elementary observation, which we formulate in a slightly more general form than is strictly needed for the proof.

Lemma 5.1. Let $G \subset \mathbb{R}^{n}$ be a bounded open set. If $h: \bar{G} \rightarrow \mathbb{R}$ is continuous, differentiable on $G$ and $\operatorname{osc}(h, \partial G)<\operatorname{osc}(h, G)$, then $0 \in \mathbb{R}^{n}$ is an interior point for $\nabla h(G)$.

Proof. Because $G$ is bounded, there exists $\varepsilon>0$, such that if $h_{u}(x)=h(x)-u \cdot x$, $x \in \bar{G}$, then $\operatorname{osc}\left(h_{u}, \partial G\right)<\operatorname{osc}\left(h_{u}, G\right)$ for all $u \in \mathrm{~B}(0, \varepsilon)$. Fix $u \in \mathrm{~B}(0, \varepsilon)$. By continuity and compactness we can write

$$
\operatorname{osc}\left(h_{u}, \bar{G}\right)=\max _{\bar{G}} h_{u}-\min _{\bar{G}} h_{u}
$$

and

$$
\operatorname{osc}\left(h_{u}, \partial G\right)=\max _{\partial G} h_{u}-\min _{\partial G} h_{u} .
$$

Therefore the assumption on the oscillations can be restated as

$$
\left(\max _{\bar{G}} h_{u}-\max _{\partial G} h_{u}\right)+\left(\min _{\partial G} h_{u}-\min _{\bar{G}} h_{u}\right)>0,
$$

and it follows from this that there is $x \in G$ with $\nabla h_{u}(x)=0$ and hence $u \in$ $\nabla h(G)$.

Because $u_{0} \notin \operatorname{int} K$, the above lemma yields $\operatorname{osc}(g, \partial \Omega) \geq \operatorname{osc}(g, \Omega)$. However, it appears to be difficult to estimate the oscillation over $\partial \Omega$ directly, so instead we estimate the oscillation over smoother sets close by. To facilitate this, let $d(x)=$ $\operatorname{dist}(x, \partial \Omega)=\inf \{\|x-y\|: y \in \partial \Omega\}$ denote the distance from $x$ to the boundary $\partial \Omega$. There exists a regularized distance $\delta: \Omega \rightarrow \mathbb{R}$, which is $\mathcal{C}^{\infty}$ on $\Omega$ and has the properties $\frac{1}{2} d(x) \leq \delta(x) \leq d(x),\|\nabla \delta(x)\| \leq 1$ and $\left|\partial^{\alpha} \delta(x)\right| \leq M_{|\alpha|} d(x)^{1-|\alpha|}$ for all $x \in \Omega$ and any multi-index $\alpha$, where $M_{|\alpha|} \in[1, \infty)$ are constants. We refer to [8], Theorem 10 on page 78, for an explicit construction.

For $\varepsilon \in(0,1)$, let

$$
\begin{equation*}
\Omega_{\varepsilon}=\{x \in \Omega: \delta(x)>\varepsilon\} . \tag{5.2}
\end{equation*}
$$

By the Morse-Sard theorem $\mathcal{L}^{1}(\delta(\{x \in \Omega: \nabla \delta(x)=0\}))=0$, and so for almost every $\varepsilon \in(0,1), \nabla \delta(x) \neq 0$ for all $x \in \partial \Omega_{\varepsilon}$. Hence $\Omega_{\varepsilon}$ has a $\mathcal{C}^{1}$ boundary for almost all $\varepsilon \in(0,1)$. We have to deal with the possibility that $\partial \Omega_{\varepsilon}$ can be disconnected. Note that $\mathrm{B}(\bar{x}, 1) \subseteq U$, and therefore that $\mathrm{B}(\bar{x}, 1) \subseteq V \subseteq \Omega$. Hence, $\mathrm{B}(\bar{x}, 1-\varepsilon) \subseteq \Omega_{\varepsilon}$ for $\varepsilon \in(0,1)$. In particular, $\bar{x} \in \Omega_{\varepsilon}$, and $\partial \Omega_{\varepsilon}$ separates $\mathbb{R}^{2}$ between $\bar{x}$ and any point $x \in \partial \Omega$. By Lemma 2.2 there exists an open, connected set $W$, such that $\bar{x} \in W$, $W \subseteq \Omega_{\varepsilon}, \partial W \subseteq \partial \Omega_{\varepsilon}$ and $\partial W$ is an irreducible separator of $\mathbb{R}^{2}$. In view of Lemma 2.3, $\partial W$ is connected. Because $W$ is connected, $\partial W \subseteq \partial \Omega_{\varepsilon}, \bar{x} \in W$ and $\mathrm{B}(\bar{x}, 1-\varepsilon) \subseteq \Omega_{\varepsilon}$, we deduce that also $\mathrm{B}(\bar{x}, 1-\varepsilon) \subseteq W$. In the following we can therefore assume without loss in generality that $\partial \Omega_{\varepsilon}$ is connected, since otherwise we can replace $\Omega_{\varepsilon}$ by the set $W$. The boundary $\partial \Omega_{\varepsilon}$ is then a simple closed $\mathcal{C}^{1}$ curve for almost all $\varepsilon \in(0,1)$. Since $\|\nabla g(x)\| \leq r+\varepsilon$ on $\partial \Omega_{\varepsilon}$, we can for such $\varepsilon$ estimate

$$
\operatorname{osc}\left(g, \partial \Omega_{\varepsilon}\right) \leq \frac{1}{2}(r+\varepsilon) \mathcal{H}^{1}\left(\partial \Omega_{\varepsilon}\right)
$$

For $0<\varrho<\sigma<1$ it follows from the coarea formula that

$$
A \geq \mathcal{L}^{2}(\Omega)>\int_{\varrho}^{\sigma} \mathcal{H}^{1}\left(\partial \Omega_{t}\right) d t
$$

and hence

$$
\underset{\varrho<t<\sigma}{\operatorname{ess} \inf }\left(t \mathcal{H}^{1}\left(\partial \Omega_{t}\right)\right)<\frac{A}{\ln \frac{\sigma}{\varrho}}
$$

Let $\alpha \in(0,1)$, and take $\varepsilon \in\left(r, r^{\alpha}\right)$, such that

$$
\varepsilon \mathcal{H}^{1}\left(\partial \Omega_{\varepsilon}\right) \leq \frac{A}{\ln \frac{r^{\alpha}}{r}}=\frac{A}{1-\alpha} \cdot \frac{1}{\ln \frac{1}{r}} .
$$

With this choice of $\varepsilon$,

$$
\operatorname{osc}\left(g, \partial \Omega_{\varepsilon}\right) \leq \varepsilon \mathcal{H}^{1}\left(\partial \Omega_{\varepsilon}\right) \leq \frac{A}{1-\alpha} \cdot \frac{1}{\ln \frac{1}{r}}
$$

Next, to estimate the oscillation over $\Omega_{\varepsilon}$ from below, let $\lambda \in(0,1)$ and assume that $r^{\alpha}<\lambda$. Then $\varepsilon<\lambda$ too, so $\mathrm{B}(\bar{x}, 1-\lambda) \subset \mathrm{B}(\bar{x}, 1-\varepsilon) \subseteq \Omega_{\varepsilon}$. In particular it follows that $\Omega_{\varepsilon}$ contains the segment $\gamma=[\bar{x}-(1-\lambda) v, \bar{x}+(1-\lambda) v]$, where $v=\left(\bar{u}-u_{0}\right) /\left\|\bar{u}-u_{0}\right\|$. Moreover, $\nabla g(\bar{x})=\bar{u}-u_{0}$ and $\operatorname{Lip}(\nabla g)=1$. Hence,

$$
\operatorname{osc}\left(g, \Omega_{\varepsilon}\right) \geq \int_{\gamma} \nabla_{v} g \geq \int_{-(1-\lambda)}^{1-\lambda}(1-|s|) d s=1-\lambda^{2}
$$

and consequently,

$$
1-\lambda^{2} \leq \operatorname{osc}\left(g, \Omega_{\varepsilon}\right) \leq \operatorname{osc}\left(g, \partial \Omega_{\varepsilon}\right) \leq \frac{A}{1-\alpha} \cdot \frac{1}{\ln \frac{1}{r}}
$$

when $r<\lambda^{\frac{1}{\alpha}}$. Together with the opposite case, we can summarize this as

$$
r \geq \min \left\{\lambda^{\frac{1}{\alpha}}, \exp \left(-\frac{A}{1-\alpha} \cdot \frac{1}{1-\lambda^{2}}\right)\right\}
$$

for $\alpha, \lambda \in(0,1)$. We will take $\alpha, \lambda \in(0,1)$, such that

$$
\lambda^{\frac{1}{\alpha}}=\exp \left(-\frac{A}{1-\alpha} \cdot \frac{1}{1-\lambda^{2}}\right) \text {, i.e. } \lambda^{2}=\exp \left(-\frac{\alpha}{1-\alpha} \cdot \frac{2 A}{1-\lambda^{2}}\right)
$$

This can be solved for $\alpha$ :

$$
\frac{1}{\alpha}=1+\frac{2 A}{(y-1) \ln y}>1, \quad \text { where } y=\lambda^{2}
$$

Hence

$$
\lambda^{\frac{1}{\alpha}}=y^{\frac{1}{2 \alpha}}=\exp \left(\frac{A}{y-1}+\frac{1}{2} \ln y\right),
$$

where $y=\lambda^{2} \in(0,1)$. We choose $y \in(0,1)$ where this quantity is maximal, that is,

$$
y=A+1-\sqrt{A^{2}+2 A} \quad(\in(0,1))
$$

Inserting this in the estimate and rearranging terms we arrive at

$$
r \geq \sqrt{A+1-\sqrt{A^{2}+2 A}} \exp \left(-\frac{1}{2}\left(A+\sqrt{A^{2}+2 A}\right)\right)
$$

Because $\sqrt{A^{2}+2 A} \leq A+1$ and

$$
\sqrt{A+1-\sqrt{A^{2}+2 A}}=\left(A+1+\sqrt{A^{2}+2 A}\right)^{-\frac{1}{2}} \geq \frac{1}{\sqrt{2(A+1)}}
$$

we deduce that

$$
r \geq \frac{1}{\sqrt{2 e(A+1)}} e^{-A}
$$

which is the desired inequality.
Proof of Theorem 1.2. Let

$$
h(x)=\frac{e^{-x}}{\sqrt{2 e(x+1)}}, \quad x \in(0, \infty)
$$

Then $h$ is strictly decreasing and $h(x) \rightarrow 0$ as $x \rightarrow \infty$. For given values of $0<$ $r<R$ we can therefore find a unique $x=x(r, R)>0$, such that $h\left(x / R^{2}\right)=r / R$. In particular, for each positive integer $j$ we can take $r_{j} \in\left(0,2^{-1-j}\right)$, such that $x\left(r_{j}, 2^{-j}\right) \geq j$.

Let $s_{j}=2^{-1}+2^{-2}+\cdots+2^{-j}$ and define the function

$$
f_{j}(t)=\max \left\{2^{-j}-\frac{2^{1-j}}{r_{j}}\left|t-s_{j}\right|, 0\right\}, \quad t \in \mathbb{R},
$$

and put

$$
f(t)=\sum_{j=1}^{\infty} f_{j}(t), \quad t \in \mathbb{R}
$$

It is clear that $f$ is a continuous function. Let

$$
\gamma_{1}(t)=(t, f(t)), \quad t \in[0,1]
$$

and let $\gamma_{2}$ denote any curve with end-points $(0,0)$ and $(1,0)$, which, except for the end-points, lies in the open lower half-plane. Define $\gamma=\gamma_{1}+\gamma_{2}$. Then $\gamma$ is a Jordan curve, and if we take $K$ to be the closure of the interior domain bounded by $\gamma$, then it follows from Proposition 5.1 that $K$ has the desired property.

## 6. Examples

Recall that a bump is a function with non-empty and bounded support. In this section $\omega$ denotes a norm on $X$ which is $\|$.$\| -continuous. (Therefore we may$ assume that $\omega(x) \leq\|x\|$ and $\mathrm{B}_{\| \|}(0,1) \subseteq \mathrm{B}_{\omega}(0,1)$.) An $\omega$-bump is a function whose support is non-empty and bounded with respect the the norm $\omega$, cf. [10]. (The Fréchet differentiability and continuity of functions is always regarded with respect to $\|\|.$.$) If \omega()=.\|\cdot\|$ then $\omega$-bumps are, of course, bumps; in the general case we also refer to $\omega$-bumps as weak bumps.

In recent years there has been an increasing interest in the ranges of derivatives of bumps on infinite dimensional spaces (see [7], [2], [14], [5], [1] and the literature cited therein). For the previously constructed bumps, ranges of their derivatives have connected interiors. Using an example of [5] we construct a $\mathcal{C}^{1}$-smooth bump on $\ell_{2}$ whose gradient set has disconnected interior, and also a $\mathcal{C}^{\infty}$-smooth weak bump (on any Banach space, which has separable dual, and that admits a $\mathcal{C}^{\infty}$ _ smooth and Lipschitz continuous bump, e.g. $\ell_{2}$ ) with the same property. This makes strong contrast with our result in $\mathbb{R}^{2}$; note that in $\mathbb{R}^{n}$ weak bumps and bumps coincide.

We remark that if $X$ has separable dual, then $X$ has an equivalent Fréchet differentiable norm and therefore admits a $\mathcal{C}^{1}$-smooth Lipschitz continuous bump (see [12]), which reduces the number of assumptions in Examples 6.2-6.4 in the case $p=1$.

First we need two technical lemmas.
Lemma 6.1. For every $a, b, \alpha, \beta \in \mathbb{R}$ with $a<b$ and for every $\varepsilon>0$ there is a function $f \in \mathcal{C}^{\infty}(\mathbb{R})$ with $f(x)=\alpha$ for $x \leq a, f(x)=\beta$ for $x \geq b$ and $\left|f^{\prime}(x)\right|<$ $\frac{|\beta-\alpha|}{b-a}+\varepsilon$ for $x \in \mathbb{R}$. Furthermore, $\left|f(x)-\left(\alpha+\frac{\beta-\alpha}{b-a}(x-a)\right)\right|<\varepsilon(b-a)$ holds for $x \in[a, b]$.

Proof. Consider a mollification of a function $g$, which is linear in $\left[a_{1}, b_{1}\right] \subset(a, b)$ and $g(x)=\alpha$ for $x \leq a_{1}$ and $g(x)=\beta$ for $x \geq b_{1}$.

Lemma 6.2. For every $\eta>0$ and $\delta>0$ there is $a \in(0, \delta)$ and a function $\phi \in$ $\mathcal{C}^{\infty}(\mathbb{R}), 0 \leq \phi \leq 1$, such that $\phi(t)=1$ for $t \leq a, \phi(t)=0$ for $t \geq \delta$ and, for $t>0$,

$$
\begin{equation*}
\left|\phi^{\prime}(t)\right| \leq \frac{\eta}{t} \tag{6.1}
\end{equation*}
$$

Proof. Since $\lim _{t \rightarrow 0+}-\frac{\eta}{4} \ln \frac{t}{\delta}=\infty$, there exists $a \in(0, \delta)$, such that $-\frac{\eta}{4} \ln \frac{a}{\delta}=1$. Let $f$ be as in the previous lemma with $\beta=b=\delta, \alpha=a$, and with $\varepsilon$ chosen so that $\left|f^{\prime}(t)\right| \leq 2$ and, for $t \in[\alpha, \delta],|f(t)-t| \leq \frac{\alpha}{2} \leq \frac{t}{2}$ and hence $f(t) \geq \frac{t}{2}$.

Let $\phi(t)=-\frac{\eta}{4} \ln \frac{f(t)}{\delta}$. Then $\left|\phi^{\prime}(t)\right|=\left|\frac{\eta}{4} \frac{f^{\prime}(t)}{f(t)}\right| \leq \frac{\eta}{t}, \phi(t)=-\frac{\eta}{4} \ln \frac{a}{\delta}=1$ for $t \leq a$, $\phi(t)=-\frac{\eta}{4} \ln \frac{\delta}{\delta}=0$ for $t \geq \delta$.

Lemma 6.3. Let $X$ be a Banach space and $p \in \mathbb{N} \cup\{\infty\}$. Then the following assertions are equivalent:
(1) there is a Lipschitz function $\mathcal{N}: X \rightarrow \mathbb{R}, \mathcal{N} \in \mathcal{C}^{p}(X \backslash\{0\})$ and $K>0$, such that $\|x\| \leq \mathcal{N}(x) \leq K\|x\|$ for all $x \in X$;
(2) there is a Lipschitz $\mathcal{C}^{p}$-smooth bump $g$ on $X$;
(3) there is a Lipschitz $\mathcal{C}^{p}$-smooth bump $h$ on $X$, such that $0 \in \operatorname{int} h^{\prime}(X)$.

Proof. (1) $\Longrightarrow(2)$ : Let $g(x)=\phi(\mathcal{N}(x))$ with $\phi$ from Lemma 6.1. (2) $\Longrightarrow(3)$ : See [1], Corollary 3.3. $(3) \Longrightarrow(2)$ is obvious. $(2) \Longrightarrow(1)$ is due to Leduc [24], see also [11], Proposition 3.2, and [12], Proposition II.5.1.

Proposition 6.1. Let $p \in \mathbb{N} \cup\{\infty\}$ and let $X$ be a Banach space which admits a Lipschitz $\mathcal{C}^{p}$-smooth bump. Let $b \in \mathcal{C}^{p}(X), x_{A} \in X, A=b^{\prime}\left(x_{A}\right) \in X^{*}$ and $\varepsilon>0$. Then there exists $f \in \mathcal{C}^{p}(X)$, such that $f^{\prime}(x)=A$ in a neighbourhood $U$ of $x_{A}$, $f(x)=b(x)$ whenever $\left\|x-x_{A}\right\| \geq \varepsilon$ and $f^{\prime}(X) \subseteq b^{\prime}(X) \cup \mathrm{B}(A, \varepsilon)$.

Proof. Let $K$ and $\mathcal{N}$ be as in (1). Choose $0<\delta<\varepsilon / 3$, such that $\left\|b^{\prime}(x)-A\right\| \leq \varepsilon / 3$ when $\left\|x-x_{A}\right\|<\delta$. Let $\eta=1 / \operatorname{Lip} \mathcal{N}$. Let $\phi$ and $a$ be as in Lemma 6.2, $\psi_{0}(x)=$ $\phi(\mathcal{N}(x)), \psi_{1}(x)=1-\phi(\mathcal{N}(x))$ and

$$
f(x)=b\left(x_{A}+\left(x-x_{A}\right) \psi_{1}\left(x-x_{A}\right)\right)+\psi_{0}\left(x-x_{A}\right) A\left(x-x_{A}\right) .
$$

Then $f(x)=b(x)$ and $f^{\prime}(x)=b^{\prime}(x)$ whenever $\left\|x-x_{A}\right\|>\delta$ and also $f(x)=b\left(x_{A}\right)+$ $A\left(x-x_{A}\right)$ and $f^{\prime}(x)=A$ whenever $x \in U=\mathrm{B}\left(x_{A}, a / K\right)$. Obviously, $0 \leq \psi_{1} \leq 1$ and, by (6.1), $\left\|\psi_{1}^{\prime}(x)\right\| \leq \eta \operatorname{Lip} \mathcal{N} / \mathcal{N}(x) \leq 1 /\|x\|$ for $x \neq 0$. Furthermore,

$$
\begin{aligned}
& f^{\prime}(x)=b^{\prime}\left(x_{A}+\left(x-x_{A}\right) \psi_{1}\left(x-x_{A}\right)\right) \psi_{1}\left(x-x_{A}\right) \\
& +\left\langle b^{\prime}\left(x_{A}+\left(x-x_{A}\right) \psi_{1}\left(x-x_{A}\right)\right), x-x_{A}\right\rangle \psi_{1}^{\prime}\left(x-x_{A}\right) \\
& \\
& \quad+A \psi_{0}\left(x-x_{A}\right)+\left\langle A, x-x_{A}\right\rangle \psi_{0}^{\prime}\left(x-x_{A}\right)
\end{aligned}
$$

and therefore if $\left\|x-x_{A}\right\|<\delta$, then

$$
\begin{aligned}
\left\|f^{\prime}(x)-A\right\| \leq & \left\|\left\langle b^{\prime}\left(x_{A}+\left(x-x_{A}\right) \psi_{1}\left(x-x_{A}\right)\right)-A, x-x_{A}\right\rangle \psi_{1}^{\prime}\left(x-x_{A}\right)\right\| \\
& +\left\|b^{\prime}\left(x_{A}+\left(x-x_{A}\right) \psi_{1}\left(x-x_{A}\right)\right)-A\right\| \psi_{1}\left(x-x_{A}\right) \\
\leq & \frac{\varepsilon}{3}\left\|x-x_{A}\right\|\left\|\psi_{1}^{\prime}\left(x-x_{A}\right)\right\|+\frac{\varepsilon}{3} \psi_{1}\left(x-x_{A}\right)<\varepsilon .
\end{aligned}
$$

Hence $f^{\prime}(X) \subseteq f^{\prime}\left(\mathrm{B}\left(x_{A}, \delta\right)\right) \cup f^{\prime}\left(X \backslash \mathrm{~B}\left(x_{A}, \delta\right)\right) \subseteq \mathrm{B}(A, \varepsilon) \cup b^{\prime}(X)$.
Example 6.1. Let $p \in \mathbb{N} \cup\{\infty\}$ and let $X$ be a Banach space which admits a Lipschitz $\mathcal{C}^{p}$-smooth bump. Assume there is a $\mathcal{C}^{p}$-smooth $\omega$-bump b on $X$, such that $\operatorname{int} b^{\prime}(X)=\emptyset$. Then there is a $\mathcal{C}^{p}$-smooth $\omega$-bump $f$ on $X$, such that int $f^{\prime}(X)$ is disconnected.
(Note that if $b$ is Lipschitz then $f$ can also be taken to be Lipschitz.)
Proof. Using Proposition 6.1 twice, we get a function $f_{1} \in \mathcal{C}^{p}(X)$ with bounded support, such that $f_{1}^{\prime}(X) \subseteq b^{\prime}(X) \cup \mathrm{B}\left(A_{1}, \varepsilon\right) \cup \mathrm{B}\left(A_{2}, \varepsilon\right), A_{1}, A_{2} \in b^{\prime}(X), \varepsilon<$ $\left\|A_{1}-A_{2}\right\|$ and $f_{1}^{\prime}(x)=A_{i}$ on a non-empty open set $U_{i}$. Let $h$ be as in (3) of Lemma 6.3. Obviously there are rescaled translations $h_{1}, h_{2}$ of $h, h_{i}(x)=c_{i} h\left(r_{i} x+a_{i}\right)$, such that spt $h_{i} \subseteq U_{i}$ and $h_{i}^{\prime}(X) \subseteq \mathrm{B}(0, \varepsilon)$. Let $f(x)=f_{1}(x)+h_{1}(x)+h_{2}(x)$. Then the interior of $f^{\prime}(X) \subseteq b^{\prime}(X) \cup \mathrm{B}\left(A_{1}, \varepsilon\right) \cup \mathrm{B}\left(A_{2}, \varepsilon\right)$ has at least one component in each $\mathrm{B}\left(A_{i}, \varepsilon\right)$.

Example 6.2. Let $p \in \mathbb{N} \cup\{\infty\}$ and let $X$ be a Banach space with a separable dual, and which admits a $\mathcal{C}^{p}$-smooth and Lipschitz continuous bump. Assume $b \in \mathcal{C}^{p}(X)$ is an $\omega$-bump. Let

$$
\mathcal{C}(b)=\left\{\lambda b^{\prime}(x): x \in X, \lambda \geq 0\right\}
$$

Let $G \subset X^{*}$ be an open set, such that $\left(X^{*} \backslash \mathcal{C}(b)\right) \cap\left(X^{*} \backslash G\right)$ is dense in $X^{*} \backslash G$. (This is true e.g. when $G$ is regularly open and $\operatorname{int} \mathcal{C}(b)=\emptyset$ ). Then there is a $\mathcal{C}^{p}$-smooth $\omega$-bump $f$ on $X$, such that int $f^{\prime}(X)=G$.

Proof. As a consequence of Ekeland's variational principle, $b^{\prime}(X)$ is dense in a neighbourhood of 0 in $X^{*}$ (cf. [12] page 58, Proposition 5.2) and hence $\mathcal{C}(b)$ is dense in $X^{*}$. Using the separability of $X^{*}$, choose $\left\{a_{i}=s_{i} b_{i}\right\}$ dense in $X^{*}$ with $s_{i}>0$ and $b_{i} \in b^{\prime}(X) \backslash\{0\}$. Fix $R>0$ so that $\operatorname{spt} b \subseteq \mathrm{~B}_{\omega}(0, R)$.

Put $r_{j}=\frac{1}{j}$, and note that for any bounded set $U_{2} \subset X^{*}$ and open set $U_{1} \subseteq U_{2}$, $G=\bigcup\left\{a_{i}+r_{j} U_{1}: a_{i}+r_{j} U_{2} \subseteq G\right\}$, because for every $x \in G$ and $j$ there is $a_{i} \in x-r_{j} U_{1}$; and $x-r_{j} U_{1}+r_{j} U_{2} \subseteq G$ for large $j$. Let $h$ be as in Lemma 6.3 (3), $U_{1}=\operatorname{int} h^{\prime}(X)$ and $U_{2}=h^{\prime}(X)$.

For each $i$ and $j$, using Proposition 6.1 (with $A=a_{i}=s_{i} b_{i}$ and $s_{i} b$ instead of $b$ ), we get $f_{i, j} \in \mathcal{C}^{p}(X)$, such that $f_{i, j}^{\prime}(x)=a_{i}$ on a non-empty open set $G_{i, j}$ and $f_{i, j}^{\prime}(X) \subseteq \mathcal{C}(b) \cup\left(a_{i}+r_{j} U_{2}\right)$. Now we can scale $h$, shift its support into $G_{i, j}$ and add the result to $f_{i, j}$ : For a suitable $s_{i, j}, y_{i, j}, h_{i, j}(x)=r_{j} s_{i, j} h\left(x / s_{i, j}+y_{i, j}\right)$, and $g_{i, j}=f_{i, j}+h_{i, j}$ we then have spt $g_{i, j} \subseteq \mathrm{~B}_{\omega}(0, R)$ and $a_{i}+r_{j} U_{1} \subseteq g_{i, j}^{\prime}(X) \subseteq$ $\mathcal{C}(b) \cup\left(a_{i}+r_{j} U_{2}\right)$.

Now, find $z_{i, j} \in X, i, j \in \mathbb{N}$ with $\omega\left(z_{i, j}\right) \leq 30 R$ and $\omega\left(z_{i, j}-z_{i^{\prime}, j^{\prime}}\right)>3 R$ whenever $(i, j) \neq\left(i^{\prime}, j^{\prime}\right)$ (we have $\operatorname{dim} X=\infty$ ) and let $f(x)=\sum\left\{g_{i, j}\left(x+z_{i, j}\right): a_{i}+r_{j} U_{2} \subseteq\right.$ $G\}$. Then $f \in \mathcal{C}^{p}(X)$ and $G \subseteq f^{\prime}(X) \subseteq \mathcal{C}(b) \cup G$. Hence int $f^{\prime}(X)=G$ by the assumptions on $G$.

Example 6.3. Let $p \in \mathbb{N} \cup\{\infty\}$ and let $X$ be a separable infinite-dimensional Banach space which admits a $\mathcal{C}^{p}$-smooth and Lipschitz continuous bump. Let $G$ be a regularly open subset of $X^{*}$. Then there is a weak bump $f \in \mathcal{C}^{p}(X)$ with int $f^{\prime}(X)=G$.

Furthermore, for every regularly open set $G$ in $\ell_{2}$, there is a $\mathcal{C}^{1}$-smooth bump $g$ on $\ell_{2}$ with int $\nabla g\left(\ell_{2}\right)=G$.
Proof. On any separable infinite-dimensional Banach space, there is, by [10], page 345 , a weak bump $b \in \mathcal{C}^{\infty}(X)$ with $\operatorname{int} \mathcal{C}(b)=\emptyset$.
[5] gives an example of a Lipschitz $\mathcal{C}^{1}$-smooth bump $b: \ell_{2} \rightarrow \mathbb{R}$, such that $\mathcal{C}(b)$ has empty interior.

In both cases, $f$ is then obtained from Example 6.2.
Note that, in the case of $\ell_{2}$, it is enough to assume, say, that $G$ is open and for every $x=\left(x_{i}\right) \in \ell_{2} \backslash G$ there is $n \in \mathbb{N}$, such that $x$ is in the closure of the relative interior of $\ell_{2} \backslash G$ in the hyperplane $\left\{z=\left(z_{i}\right): z_{n}=x_{n}\right\}$. This depends on the fact that $\mathcal{C}(b) \subseteq Z=\{0\} \cup\left\{\left(z_{i}\right) \in \ell_{2}: z_{i} \neq 0\right.$ for infinitely many $\left.i \in \mathbb{N}\right\}$ for the particular $\omega$-bump from [5] or [10].

Example 6.4. In $\ell_{2}$, let $G_{1}=\left\{\left(x_{i}\right) \in \ell_{2}:\left\|\left(x_{i}\right)\right\|<1, x_{1} \neq 0\right\}, Z_{1}=\left\{\left(x_{i}\right) \in \ell_{2}\right.$ : $\left\|\left(x_{i}\right)\right\|<1, x_{1}=0$ and $x_{i} \neq 0$ for infinitely many $\left.i \in \mathbb{N}\right\} \cup\{0\}$, and $G=G_{1} \cup Z_{1}$. Then, for every $y \in G$, there is a $\mathcal{C}^{1}$-smooth and Lipschitz continuous bump $f$ on $\ell_{2}$, such that $\nabla f\left(\ell_{2}\right)=-y+G$.

Proof ingredients. The set $G$ consists of two open hemispheres $G_{1}$ and interconnection $Z_{1}$ which is disjoint with int $G=G_{1}$. The bump from [5] (see above) allows crossing of the interconnection with the derivatives staying in the set $Z$. It has bounded derivatives, therefore we may stay in $B(0,1)$. Shifting the derivative range using a composition of functions is standard. One component of $G_{1}$ together with $Z_{1}$ is to be filled using [14] or [15] ( $Z_{1}$ is analytic). Proposition 6.1 allows to fill the other component of $G_{1}$.

## 7. Proof of Proposition 1.1

The proposition is naturally divided in two parts that we state and prove separately.

Proposition 7.1. Let $\omega:(0, \infty) \rightarrow(0, \infty)$ be a continuous nondecreasing function satisfying

$$
\begin{equation*}
\int_{0}^{1} \frac{d t}{\omega(t)}=\infty \tag{7.1}
\end{equation*}
$$

If $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a $\mathcal{C}^{1}$ function satisfying

$$
\begin{equation*}
\|\nabla f(x)-\nabla f(y)\| \leq \omega(\|x-y\|) \tag{7.2}
\end{equation*}
$$

for $x, y \in \mathbb{R}^{n}$, and $\nabla f=0$ on an irreducible separator $C$ of $\mathbb{R}^{n}$, then $f$ is constant on $C$.

Proof. In view of Lemma 2.1, $C$ is closed, and by Lemma 2.2 we can take distinct components $\Omega, \Lambda$ of $\mathbb{R}^{n} \backslash C$, such that $\partial \Omega=C=\partial \Lambda$. The conclusion follows if we can show that the vector field

$$
V= \begin{cases}\nabla f & \text { in } \Omega \\ 0 & \text { in } \mathbb{R}^{n} \backslash \Omega\end{cases}
$$

is curl-free (in the distributional sense) on $\mathbb{R}^{n}$. That is,

$$
V_{, i}^{j}-V_{, j}^{i}=0 \quad(i, j=1, \ldots, n)
$$

holds in the distributional sense on $\mathbb{R}^{n}$. Indeed, suppose that this has been done. Then, by a standard result, $V=\nabla g$ for some $\mathcal{C}^{1}$ function $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$. Because $\Omega$ is open and connected, we can assume that $g=f$ on $\bar{\Omega}$. Now $\nabla g=0$ on $\mathbb{R}^{n} \backslash \Omega$, so in particular, $\nabla g=0$ on $\Lambda$. Since $\Lambda$ is open and connected, $g$ must be constant on $\bar{\Lambda}=\Lambda \cup C$. But $f=g$ on $\bar{\Omega}=\Omega \cup C$, so $f$ is constant on $C$ too.

It remains to show that the vector field $V$ is curl-free. To this end let $\varphi \in$ $\mathcal{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right), i, j \in\{1, \ldots, n\}, i \neq j$ and compute

$$
\left\langle V_{, i}^{j}-V_{, j}^{i}, \varphi\right\rangle=\int_{\Omega}\left(f_{, i} \varphi_{, j}-f_{, j} \varphi_{, i}\right) .
$$

As in the proof of Proposition 5.1, see (5.2), we consider the auxiliary sets $\Omega_{\varepsilon}=$ $\{x \in \Omega: \delta(x)>\varepsilon\}$, where $\delta(x)$ is a regularized distance from $x$ to $\partial \Omega$ and $\varepsilon>0$. Recall that $\Omega_{\varepsilon}$ has a $\mathcal{C}^{1}$ boundary for almost all $\varepsilon \in(0,1)$. If for such $\varepsilon$ we let $N=-\nabla \delta /\|\nabla \delta\|$, then $N$ is the outward unit normal on $\partial \Omega_{\varepsilon}$ and we have the Gauss-Green formula

$$
\int_{\Omega_{\varepsilon}} \operatorname{div} W=\int_{\partial \Omega_{\varepsilon}} W \cdot N
$$

for any $\mathcal{C}^{1}$ vector field $W$ defined on a neighbourhood of $\bar{\Omega}_{\varepsilon}$. If $\left\{\rho_{t}\right\}$ denotes a standard mollifier we get for almost every $\varepsilon$ that

$$
\begin{gathered}
\int_{\Omega_{\varepsilon}}\left(f_{, i} \varphi, j-f_{, j} \varphi_{, i}\right)=\lim _{t \rightarrow 0^{+}} \int_{\Omega_{\varepsilon}} \operatorname{div}\left(\varphi\left(\rho_{t} \star f_{, i}\right) e_{j}-\varphi\left(\rho_{t} \star f_{, j}\right) e_{i}\right)= \\
\lim _{t \rightarrow 0^{+}} \int_{\partial \Omega_{\varepsilon}}\left(\varphi\left(\rho_{t} \star f_{, i}\right) N^{j}-\varphi\left(\rho_{t} \star f_{, j}\right) N^{i}\right)=\int_{\partial \Omega_{\varepsilon}} \varphi\left(f_{, i} N^{j}-f_{, j} N^{i}\right) .
\end{gathered}
$$

In view of (7.2) it therefore holds for almost all $\varepsilon$ that

$$
\left|\int_{\Omega_{\varepsilon}}\left(f_{, i} \varphi_{, j}-f_{, j} \varphi_{, i}\right)\right| \leq \int_{\partial \Omega_{\varepsilon}}\|\nabla f\||\varphi| \leq \omega(\varepsilon) \mathcal{H}^{n-1}\left(\mathrm{~B} \cap \partial \Omega_{\varepsilon}\right) \max |\varphi|,
$$

where B is a ball containing the support of $\varphi$. For $0<\varrho<\sigma<1$ we proceed as in the proof of Proposition 5.1 and find that

$$
\mathcal{L}^{n}(\mathrm{~B} \cap \Omega) \geq \underset{\varrho \leq t \leq \sigma}{\operatorname{essinf}}\left(\omega(t) \mathcal{H}^{n-1}\left(\mathrm{~B} \cap \partial \Omega_{t}\right)\right) \int_{\varrho}^{\sigma} \frac{d t}{\omega(t)}
$$

By virtue of (7.1) this implies that

$$
\underset{0<t \leq \sigma}{\operatorname{ess} \inf }\left(\omega(t) \mathcal{H}^{n-1}\left(\mathrm{~B} \cap \partial \Omega_{t}\right)\right)=0
$$

and consequently

$$
\liminf _{\varepsilon \rightarrow 0^{+}}\left|\int_{\Omega_{\varepsilon}}\left(f_{, i} \varphi_{, j}-f_{, j} \varphi_{, i}\right)\right|=0
$$

On the other hand we also have by the dominated convergence theorem that

$$
\lim _{\varepsilon \rightarrow 0^{+}} \int_{\Omega_{\varepsilon}}\left(f_{, i} \varphi_{, j}-f_{, j} \varphi_{, i}\right)=\int_{\Omega}\left(f_{, i} \varphi_{, j}-f_{, j} \varphi_{, i}\right)
$$

hence

$$
\int_{\Omega}\left(f_{, i} \varphi_{, j}-f_{, j} \varphi_{, i}\right)=0
$$

and $V$ is curl-free.
Proposition 7.2. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$ function. If $\nabla f$ is locally of bounded variation (i.e., the distributional partial derivatives $f_{, i j}$ are Radon measures on $\mathbb{R}^{n}$ ) and $\nabla f=0$ on a bounded irreducible separator $C$ of $\mathbb{R}^{n}$, then $f$ is constant on $C$.

Proof. Define the sets $\Omega, \Lambda$ as in the proof of Proposition 7.1. Because $C$ is bounded, the complement $\mathbb{R}^{n} \backslash C$ has precisely one unbounded component. We can therefore assume that $\Omega$ is bounded.

As in the proof of Proposition 7.1 we denote by $\delta$ a regularized distance function to the boundary $\partial \Omega$. Let $\rho$ be an even $\mathcal{C}^{\infty}$ function with the properties $\rho(x) \geq 0$ for all $x, \rho(x)=0$ for $\|x\|>1$ and $\int_{\mathbb{R}^{n}} \rho=1$. Define $\Delta(x)=\frac{1}{4 \operatorname{diam} \Omega} \delta(x)^{2}$ and the auxiliary function

$$
\begin{equation*}
F(x)=\int_{\mathbb{R}^{n}} \rho(y) f(x-\Delta(x) y) d y, \quad x \in \Omega \tag{7.3}
\end{equation*}
$$

Then $F$ is a $\mathcal{C}^{\infty}$ function on $\Omega$, and it is not hard to show that

$$
\lim _{y \rightarrow x, y \in \Omega} F(y)=f(x) \quad \text { and } \quad \lim _{y \rightarrow x, y \in \Omega} \nabla F(y)=0
$$

for each $x \in \partial \Omega$. In the next step we establish an integral inequality for the second order partial derivatives of the function $F$. As it does not rely on the specific
behaviour of $f$ on the boundary of $\Omega$, and since it could be of independent interest we state it as a lemma.
Lemma 7.1. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$ function for which the first order partial derivatives are locally of bounded variation, and for a bounded open set $\Omega \subset \mathbb{R}^{n}$ define the function $F$ as in (7.3). For all indices $i, j$ it then holds that

$$
\int_{\Omega}\left|F_{, i j}(x)\right| d x \leq c \sum_{1 \leq k, l \leq n}\left|f_{, k l}\right|(\bar{\Omega}),
$$

where $\left|f_{, k l}\right|(\bar{\Omega})$ is the total variation of the measure $f_{, k l}$ on $\bar{\Omega}$ and $c$ is a constant depending only on $M_{2}$.

Proof of Lemma 7.1. First we assert that the general case of Lemma 7.1 can be deduced from the special case, where $f$ is $\mathcal{C}^{2}$. This follows by approximation using mollifiers: let $f^{(t)}=\rho_{t} \star f$, where $\left\{\rho_{t}\right\}$ denotes a standard smooth mollifier, and let $F^{(t)}$ denote the corresponding auxiliary function. We assume that the lemma holds for the pairs $f^{(t)}, F^{(t)}$. From standard properties of mollifiers (see e.g. [26], Chapter 5) we have, for all pairs $i, j$ and $k, l$ of indices from $\{1, \ldots, n\}$, that as $t \searrow 0$,

$$
\begin{gathered}
\left|F_{, i j}^{(t)}(x)\right| \rightarrow\left|F_{, i j}(x)\right| \quad \text { for each } x \in \Omega, \quad \text { and } \\
\left|f_{, k l}^{(t)}\right| \mathcal{L}^{n}\left\llcorner\mathrm { B } \stackrel { * } { \rightharpoonup } | f _ { , k l } | \left\llcorner\mathrm{~B} \quad \text { in } \mathcal{C}_{0}^{0}(\mathrm{~B})^{*},\right.\right.
\end{gathered}
$$

where $\mathrm{B} \subset \mathbb{R}^{n}$ is any open ball, and $\mathcal{C}_{0}^{0}(\mathrm{~B})$ is the Banach space of real-valued continuous functions on $B$ that vanish on the boundary $\partial B$. The assertion of the lemma for $f, F$ now follows by use of Fatou's lemma and standard properties of weak* convergence of measures.

Henceforth in establishing the lemma we can assume that $f$ is $\mathcal{C}^{2}$. We then have, for every $x \in \Omega$,

$$
\begin{aligned}
F_{, i j}(x)= & \int_{\mathbb{R}^{n}} \rho(y)\left(f_{, i j}(x-\Delta(x) y)-f_{, i k}(x-\Delta(x) y) \Delta_{, j}(x) y_{k}-f_{, k j}(x-\Delta(x) y) \Delta_{, i}(x) y_{k}\right. \\
& \left.+f_{, k l}(x-\Delta(x) y) \Delta_{, j}(x) \Delta_{, i}(x) y_{k} y_{l}-f_{, k}(x-\Delta(x) y) \Delta_{, i j}(x) y_{k}\right) d y
\end{aligned}
$$

where we have used the usual summation convention of summing over repeated indices. Before we go any further let us remark that for each fixed $y \in \mathrm{~B}(0,1)$ the $\operatorname{map} \Phi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ defined as $\Phi(x)=x-\Delta(x) y$ for $x \in \Omega$ and $\Phi(x)=x$ otherwise, is a $\mathcal{C}^{1}$ map. For $x \in \Omega$ its $n$-dimensional Jacobian is $J \Phi(x)=|\operatorname{det}(I-y \otimes \nabla \Delta(x))|=$ $|1-\operatorname{trace}(y \otimes \nabla \Delta(x))|$, so $J \Phi(x) \geq 1-\|\nabla \Delta(x)\| \geq \frac{1}{2}$. It follows that $\Phi$ is a $\mathcal{C}^{1}$ diffeomorphism, and hence in particular that $\Phi$ is a diffeomorphism of $\Omega$ onto itself. If $\Psi=\Phi^{-1}$ is the inverse map, then $\Psi$ maps $\Omega$ diffeomorphically onto itself, and its Jacobian $J \Psi(x)$ is not larger than 2.

We now proceed with the estimations and integrate $\left|F_{, i j}(x)\right|$ over $x \in \Omega$. The first term is estimated from above by interchanging the order of integration and changing the coordinates $x \mapsto \Phi(x)$ :

$$
\left|\int_{\Omega} \int_{\mathbb{R}^{n}} \rho(y) f_{, i j}(x-\Delta(x) y) d y d x\right| \leq 2 \int_{\Omega}\left|f_{, i j}(x)\right| d x
$$

Using that $\left|\Delta_{, i}(x)\right| \leq \frac{1}{2}$ the following terms are estimated by similar means:

$$
\mid \int_{\Omega} \int_{\mathbb{R}^{n}} \rho(y)\left(-f_{, i k}(x-\Delta(x) y) \Delta_{, j}(x) y_{k}-f_{, k j}(x-\Delta(x) y) \Delta_{, i}(x) y_{k}+\right.
$$

$$
\begin{gathered}
\left.+f_{, k l}(x-\Delta(x) y) \Delta_{, j}(x) \Delta_{, i}(x) y_{k} y_{l}\right) d y d x \mid \leq \\
3 \sum_{1 \leq k, l \leq n} \int_{\Omega}\left|f_{, k l}(x)\right| d x
\end{gathered}
$$

To estimate the last term, observe that

$$
f_{, k}(x-\Delta(x) y)-f_{, k}(x)=-\int_{0}^{1} f_{, k l}(x-t \Delta(x) y) y_{l} d t \Delta(x)
$$

and use that $\int y_{k} \rho(y) d y=0$ to obtain

$$
\begin{gathered}
\left|\int_{\Omega} \int_{\mathbb{R}^{n}} f_{, k}(x-\Delta(x) y) \Delta_{, i j}(x) y_{k} \rho(y) d y d x\right| \leq \\
\sum_{1 \leq k, l \leq n} \int_{\Omega} \int_{\mathbb{R}^{n}} \rho(y) \int_{0}^{1}\left|f_{, k l}(x-t \Delta(x) y)\right| d t \Delta(x)\left|\Delta_{, i j}(x)\right| d y d x
\end{gathered}
$$

Because $\left|\Delta_{, i j}(x)\right| \leq \frac{1}{2 \operatorname{diam} \Omega}\left(M_{2}+1\right)$, this is not larger than

$$
\frac{M_{2}+1}{2 \operatorname{diam} \Omega} \sum_{1 \leq k, l \leq n} \int_{\Omega} \int_{\mathbb{R}^{n}} \rho(y) \int_{0}^{1}\left|f_{, k l}(x-t \Delta(x) y)\right| d t \Delta(x) d y d x
$$

To finish the proof of the lemma we proceed similarly to above by interchanging the orders of integrations and changing variables in the $x$-integral.

Now we may continue the proof of Proposition 7.2. Because $f=F$ on $\partial \Omega$, it suffices to show that the vector field

$$
V=\left\{\begin{array}{cl}
\nabla F & \text { on } \Omega \\
0 & \text { else }
\end{array}\right.
$$

is curl-free. The reason for this is exactly the same as in Proposition 7.1. Suppose we knew that $V$ is of Sobolev class $W^{1,1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$. Then the distributional partial derivatives $V_{, i}^{j}$ can be represented by the approximate partial derivatives ap- $\frac{\partial V^{j}}{\partial x_{i}}$, which exist almost everywhere, and it is then easy to see that $V$ is curl-free on $\mathbb{R}^{n}$. (Obviously ap- $\frac{\partial V^{j}}{\partial x_{i}}=0$ at every point of density of $\partial \Omega$.)

Hence the proof is complete if we can show that $V$ is of class $W^{1,1}$. To that end we show that each coordinate function $V^{j}$ is of class ACL (see [26], Theorem 2.1.4). Fix $x_{0}^{\prime} \in \mathbb{R}^{n-1}$ and put $v(t)=V^{j}\left(x_{0}^{\prime}, t\right), t \in \mathbb{R}$. It is clear that $v$ is continuous and that it vanishes outside a bounded set. Put

$$
\ell=\left\{t \in \mathbb{R}:\left(x_{0}^{\prime}, t\right) \in \partial \Omega\right\}
$$

Then $\ell$ is closed and $v \equiv 0$ on $\ell$. Since $F_{, j}$ is $\mathcal{C}^{\infty}$ and $W^{1,1}$ on $\Omega, v$ is $\mathcal{C}^{\infty}$ and $W^{1,1}$ on $\mathbb{R} \backslash \ell$. Let

$$
w(t)=\left\{\begin{array}{cl}
v^{\prime}(t) & \text { if } t \notin \ell \\
0 & \text { if } t \in \ell .
\end{array}\right.
$$

Clearly, $w \in L^{1}(\mathbb{R})$ and for $a<b$ with $a, b \notin \ell$

$$
\int_{a}^{b} w(t) d t=\int_{(a, b) \backslash \ell} v^{\prime}(t) d t
$$

We can write $(a, b) \backslash \ell=\bigcup_{i \in I}\left(a_{i}, b_{i}\right)$, where $\left(a_{i}, b_{i}\right), i \in I$, is an at most countable collection of disjoint intervals. We can assume that $a_{1}=a$ and $b_{2}=b$. Since $v=0$ at all the remaining $a_{i}$ and $b_{i}$ 's we get

$$
\int_{(a, b) \backslash \ell} v^{\prime}(t) d t=\sum_{i \in I} \int_{a_{i}}^{b_{i}} v^{\prime}(t) d t=\sum_{i \in I}\left(v\left(b_{i}\right)-v\left(a_{i}\right)\right)=v(b)-v(a) .
$$

This implies that $v$ is absolutely continuous on $\mathbb{R}$, and concludes the proof, since $e_{n}$ could be replaced by any direction.
Remark. A function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is delta-convex if it can be written as a difference of two convex real-valued functions. Because convex real-valued functions are locally of class $B V^{2}$, also delta-convex functions are locally of class $B V^{2}$ (see e.g. [23]). The squared distance function to a subset $A \subseteq \mathbb{R}^{n}$, i.e. $d^{2}(x)=\inf \left\{\|x-y\|^{2}: y \in A\right\}$, is an example of a delta-convex function (because $d^{2}(x)=\|x\|^{2}-\sup _{y \in A}\left(2 x \cdot y-\|y\|^{2}\right)$ ).

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