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Affine diffeomorphisms of translation surfaces: Periodic points, Fuchsian groups, and arithmeticity

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AFFINE DIFFEOMORPHISMS OF TRANSLATION SURFACES: PERIODIC POINTS, FUCHSIAN GROUPS, AND ARITHMETICITY

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Abstract. We study translation surfaces with rich groups of affine diffeomorphisms. We introduce the notion of "prelattice" translation surfaces. They include the lattice translation surfaces studied by W. Veech. Our results characterize arithmetic surfaces among prelattice translation surfaces by the infinity of the set of periodic points under the action of this group. We show that there are prelattice but nonlattice translation surfaces, negatively answering a question of Veech.

We study periodic points of hyperelliptic translation surfaces. In particular, we give explicit examples of translation surfaces whose sets of periodic points and Weierstrass points coincide.

1. Introduction

INTRODUCTION

A translation surface is a two dimensional real manifold with an atlas whose transition functions are translations. The simplest example is produced by identifying by translations the opposite sides of a parallelogram. This yields a flat torus, $\mathcal{T}$. The four vertices of the parallelogram are glued together into the "origin", $o \in \mathcal{T}$. The total angle at o is 2π, thus o is a *regular point.* A closed translation surface, $\mathcal{S}$, of genus greater than one necessarily has points where the angle is a multiple of 2π. These are the *singular points*, or the *cone points* of $\mathcal{S}$. Let $C(\mathcal{S})$ be the finite set of these points. The abovementioned atlas covers the complement $\mathcal{S} \setminus C(\mathcal{S})$, i.e., the regular points of $\mathcal{S}$. Translation surfaces (and closely related *half-translation surfaces* [GJ00]) arise in several contexts: mathematical billiards, Riemann surfaces and their moduli, classification of surface diffeomorphisms and measured foliations, etc. The present paper studies them from the viewpoint of geometry and arithmetic. In the body of the Introduction we expose the background and the motivation of the study, and formulate our main results.

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1.1. **Motivation and Background.** In W. Thurston's classification of surface diffeomorphisms [Th88], he studied flat Riemannian metrics with conical singularities. In a particular setting, these give rise to translation surfaces. See the survey [MT01] for more material on this aspect of translation surfaces and [Tr86] for the metrics with general cone singularities.

From the classical complex analysis perspective, translation surfaces arise from Riemann surfaces. Integrating a holomorphic 1-form on a Riemann surface, we obtain local coordinates whose transition functions are translations. Zeros of the form yield the cone points of the translation structure. Namely, a zero of multiplicity $m-1$ yields a cone point with angle $2m\pi$. See, e.g., [MT01, Tr86], and [Wrd98] for details.

In the subject of mathematical billiards there is a well known construction that replaces a *rational polygon*, P, by an associated translation surface, $\mathcal{S} = \mathcal{S}(P)$. The relation between P and $\mathcal{S}$ is such that the billiard ball orbits in P unfold into the geodesics in $\mathcal{S}$, reducing the billiard flow in P to the geodesic flow in $\mathcal{S}$. A. Katok and A. Zemlyakov [KZ75] used this construction to obtain new results on billiards in polygons. For this reason, $\mathcal{S}(P)$ is often called "the Katok-Zemlyakov surface". In fact, it is a classical geometric construct. For instance, when P is a square, $\mathcal{S}$ is the square translation torus, obtained from four copies of P. See [Gut84] and the references there on the geometry of this classical construct.

Classifying surface diffeomorphisms, Thurston focused upon *affine diffeomorphisms* of (half-)translation surfaces. The differential of a general diffeomorphism of any surface $\mathcal{S}$ is a linear operator on the tangent space at each regular point of $\mathcal{S}$. Using the atlas of a translation surface $\mathcal{S}$, the differential of an area preserving diffeomorphism at each point, $s \in \mathcal{S} \setminus C(\mathcal{S})$, can be expressed as an element of $\mathrm{SL}(2,\mathbb{R})$.

The *affine diffeomorphisms* preserve the translation structure. They form a group, $\mathrm{Aff}(\mathcal{S})$. Any affine diffeomorphism has constant differential. Assigning to $g \in \mathrm{Aff}(\mathcal{S})$ its differential, we obtain the *differential homomorphism* $D : \mathrm{Aff}(\mathcal{S}) \to \mathrm{SL}(2,\mathbb{R})$. We denote the range of this homomorphism by $\Gamma(\mathcal{S}) \subset \mathrm{SL}(2,\mathbb{R})$. W. Veech was the first to relate $\Gamma(\mathcal{S})$ to the geometry and dynamics of the geodesic flow of $\mathcal{S}$. See [Vch89]. It is thus customary to call $\Gamma(\mathcal{S})$ the *Veech group* of a translation surface. In [Vch89] Veech proved, in particular, that the differential homomorphism has a finite kernel, and that $\Gamma(\mathcal{S})$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. That is, $\Gamma(\mathcal{S})$ is a Fuchsian group. [1]

[1] More precisely, the flat structures considered by Veech and many other authors correspond to *half-translation surfaces* [GJ00]. These are induced by the quadratic differentials on a Riemann surface; the translation surfaces which we treat here are induced by those quadratic differentials which are squares of holomorphic 1-forms. All of our results extend mutatis mutandis to the half-translation surfaces. This follows, for example, from the construction in [GJ00] that associates with a half-translation surface a 2-sheeted covering by a translation surface. See also [HM79] and [Vch84].

A Fuchsian group, Γ, is a *lattice* if the quotient $\mathrm{SL}(2,\mathbb{R})/\Gamma$ has finite Haar volume. We say that $\mathcal{S}$ has the *lattice property*, or, for brevity, that $\mathcal{S}$ is a *lattice surface* if $\Gamma(\mathcal{S})$ is a lattice. It is necessarily nonuniform [Vch89] (equivalently: $\mathrm{SL}(2,\mathbb{R})/\Gamma$ is noncompact). For instance, the standard square torus has the lattice property — its Veech group is $\mathrm{SL}(2,\mathbb{Z})$. Recall that nonuniform lattices have many parabolic elements. We say that a diffeomorphism $g \in \mathrm{Aff}(\mathcal{S})$ is parabolic if $Dg \in \mathrm{SL}(2,\mathbb{R})$ is parabolic.

A nonuniform Fuchsian group, Γ, is *arithmetic* if it is commensurable (in the wide sense) to $\mathrm{SL}(2,\mathbb{Z})$. That is, Γ is arithmetic if it admits a finite index subgroup which is $\mathrm{SL}(2,\mathbb{R})$-conjugate to a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. A translation surface is *arithmetic* if its Veech group is arithmetic. An arithmetic group is a lattice, thus arithmetic surfaces satisfy the lattice property. They were investigated already in [Gut84]. In [Vch89, Vch92] Veech gave the first examples of nonarithmetic lattice surfaces. He also showed that the geodesics on lattice surfaces, just as the geodesics of a flat torus, have especially simple properties. Every geodesic is either closed (i.e., is a periodic geodesic or a saddle connection), or it is uniformly distributed on the surface. This is known as the *Veech dichotomy*. Veech's results led to further investigations of lattice translation surfaces. See [GJ96], [Vo96], [EG97], [Wrd98], [KS00], [GJ00], [HS00], [HS01].

1.2. **The Setting and Main Results.** There are two major branches to the study of translation surfaces. One is the study of the general or, at least, the generic translation surface. See, for instance, [EM01, KMS86] and the survey [MT01]. The other major branch is the study of special translation surfaces, e.g., those satisfying the Veech dichotomy. This approach naturally subdivides into two: The purely geometric one [Vo96, Gut00, GM01] and the algebro-geometric one [Wrd98, GJ96, KS00, GJ00, HS00, HS01]. The present work is of the latter type. We elaborate on this below. See the body of the paper for the definitions and explanations of those concepts which we only briefly mention here.

Recall the standard classification of elements of $\mathrm{SL}(2,\mathbb{R}) \setminus \{\pm I\}$ as *parabolic, elliptic*, or *hyperbolic*. See, for instance, [B83]. By convention, we consider the trivial elements $\pm I \in \mathrm{SL}(2,\mathbb{R})$ elliptic. Let $\phi \in \mathrm{Aff}(\mathcal{S})$ be arbitrary. We say that ϕ is a *parabolic, elliptic*, or a *hyperbolic* diffeomorphism, if its differential, $D\phi \in \mathrm{SL}(2,\mathbb{R})$, is parabolic, elliptic, or hyperbolic respectively.

A typical translation surface has no nontrivial affine symmetries. Our translation surfaces, $\mathcal{S}$, are exceptional in the sense that they have large groups of affine diffeomorphisms. We will concentrate on subgroups "generated" by the *parabolic directions*. A direction, θ, is parabolic for $\mathcal{S}$ if

(i) Every geodesic in this direction is either periodic or a saddle connection;
(ii) The moduli of the cylinders in $\mathcal{S}$, formed by the geodesics in this direction are commensurate.

With any parabolic direction, we associate a parabolic diffeomorphism, $\phi_\theta \in \mathrm{Aff}(\mathcal{S})$. See § 2.1 for elaboration of this and what follows. The restriction of ϕ_θ to a cylinder in the direction θ is a power of the Dehn twist of that cylinder. This allows one to identify ϕ_θ with its differential, a parabolic element of $\Gamma(\mathcal{S})$. Slightly abusing notation, we will regard ϕ_θ as an element in $\mathrm{Aff}(\mathcal{S})$ and in $\Gamma(\mathcal{S})$. If $g \in \mathrm{Aff}(\mathcal{S})$ is parabolic, then there exist $m, n \in \mathbb{N}$, and a parabolic direction θ such that $g^m = \phi_\theta^n$. We say that θ is the direction of the parabolic diffeomorphism g. If α, β are distinct arbitrary directions, then either $\alpha = -\beta$ or α and β are transversal. We will say that α, β is a *pair of (transversal) parabolic directions* to mean that α and β are parabolic for $\mathcal{S}$, and $\alpha \neq \pm\beta$.

Definition 1. A translation surface $\mathcal{S}$ is a *prelattice surface* if it has a pair of parabolic directions. Equivalently, $\mathcal{S}$ is a prelattice surface if the Veech group $\Gamma(\mathcal{S})$ contains non-commuting parabolic elements. Equivalently, $\mathcal{S}$ is a prelattice surface if the group $\mathrm{Aff}(\mathcal{S})$ contains parabolic diffeomorphisms with distinct directions.

In the seminal work [Vch89] Veech briefly considered surfaces satisfying Definition 1. See §9 of [Vch89]. He asked if any such surface is, in fact, a lattice surface. The class of prelattice surfaces is one of the principal subjects of this work. In particular, we answer Veech's question in the negative — see Corollary 4 below.

The *periodic points*, with respect to various subgroups of affine diffeomorphisms, play a major role in our paper. Let $\mathcal{S}$ be a translation surface and let $G \subset \mathrm{Aff}(\mathcal{S})$ be an infinite subgroup. A point of $\mathcal{S}$ is *G-periodic* if its G-orbit is finite. When $G = \mathrm{Aff}(\mathcal{S})$, we simply speak of the periodic points of $\mathcal{S}$. This concept is especially meaningful when $\mathcal{S}$ is a prelattice surface. Several characterizations of arithmetic translation surfaces [GJ96, GJ00] imply that the set of periodic points of an arithmetic surface is countable and dense. For instance, periodic points of the standard square torus $\mathbb{R}^2/\mathbb{Z}^2$ are its rational points, i.e., the set $\mathbb{Q}^2/\mathbb{Z}^2$. We extend the concept of *rational points* to prelattice surfaces. We prove that the sets of periodic and rational points of $\mathcal{S}$ coincide if and only if $\mathcal{S}$ is an arithmetic surface. Otherwise, the set of periodic points is finite. See Theorem 4 and Corollary 2 below. For the moment we formulate a result in this direction that does not use the concept of rational points.

Theorem 1. *Let $\mathcal{S}$ be a prelattice translation surface. Then the following dichotomy holds:*

(i) *The surface $\mathcal{S}$ is arithmetic and its periodic points form a dense countable subset;*
(ii) *The surface $\mathcal{S}$ is nonarithmetic and its set of periodic points is finite.*

We will need the following notation and terminology. If α, β is a pair of parabolic directions on $\mathcal{S}$, we denote by $\mathrm{Aff}_{\alpha,\beta}(\mathcal{S}) \subset \mathrm{Aff}(\mathcal{S})$ the subgroup generated by the parabolic affine diffeomorphisms ϕ_α and ϕ_β. We will denote by $\Gamma_{\alpha,\beta}(\mathcal{S}) \subset \Gamma(\mathcal{S})$ its (isomorphic) image in the Veech group. Subgroups of this form will be called *basic*. More generally, any subgroup of affine diffeomorphisms will be called basic if it is commensurable with $\mathrm{Aff}_{\alpha,\beta}(\mathcal{S})$ for some pair of parabolic directions. We apply the term *basic* to subgroups of Veech groups in the corresponding manner. A group which contains a basic subgroup, is a *prelattice group*. Thus, $\mathcal{S}$ is a prelattice translation surface if $\mathrm{Aff}(\mathcal{S})$, or equivalently $\Gamma(\mathcal{S})$, is a prelattice group. The following result is an extension of Theorem 1.

Theorem 2. *Let $\mathcal{S}$ be a translation surface, and let $G \subset Aff(\mathcal{S})$ be a prelattice subgroup. If the set of G-periodic points of $\mathcal{S}$ is infinite, then $\mathcal{S}$ is an arithmetic translation surface.*

Theorems 1 and 2, and further related results will follow from the quantitative Theorem 11 in § 3. Theorem 11 gives an upper bound on the number of periodic points that a nonarithmetic translation surface can have.

Recall that an action of a group on a compact is *minimal* if there are no nontrivial closed invariant subsets. As a byproduct of our analysis of diffeomorphisms of a prelattice surface, we show that the action of its affine group on the surface is "nearly" minimal.

Theorem 3. *Let $\mathcal{S}$ be a prelattice translation surface. Then the only closed infinite subset of $\mathcal{S}$ invariant under $Aff(\mathcal{S})$ is $\mathcal{S}$ itself.*

To formulate our next result, we recall the notion of a punctured translation surface.

Definition 2. Let $\mathcal{S}$ be a translation surface. Let $C(\mathcal{S}) \subset \mathcal{S}$ be its set of cone points. Let $s_1, \dots, s_p \in \mathcal{S} \setminus C(\mathcal{S})$ be any regular points. *Puncturing* $\mathcal{S}$ at $s_1, \dots, s_p$ we create a formally new translation surface $(\mathcal{S}; s_1, \dots, s_p)$. Its set of cone points is $C(\mathcal{S}) \cup \{s_1, \dots, s_p\}$. The group of affine diffeomorphisms of $(\mathcal{S}; s_1, \dots, s_p)$ is the subgroup of $\mathrm{Aff}(\mathcal{S})$ consisting of the diffeomorphisms that preserve $C(\mathcal{S}) \cup \{s_1, \dots, s_p\}$. The Veech group of $(\mathcal{S}; s_1, \dots, s_p)$ is the corresponding subgroup of $\Gamma(\mathcal{S})$. We denote it by $\Gamma(\mathcal{S}; s_1, \dots, s_p)$. We will say that $(\mathcal{S}; s_1, \dots, s_p)$ is the surface $\mathcal{S}$ punctured at the points $s_1, \dots, s_p$.

The operation of puncturing a translation surface naturally arises in the context of affine coverings. See [GJ00, Gut00, HS00, HS01]. Note that the puncturing of a surface does not change the dynamics of its geodesic flow. However, it may drastically change the Veech group, and the counting functions associated with the translation surface. See [Gut00] and [HS00]. Our next result characterizes nonarithmetic lattice surfaces in terms of their puncturings. In order to formulate the result, we need to introduce the notion of *rational points* of a prelattice translation surface. We do this briefly, postponing a formal definition until § 5. See Definition 5 there.

Let $\mathcal{S}$ be any translation surface, let γ be a periodic oriented geodesic in $\mathcal{S}$, and let α be the direction of γ. Then γ is contained in a unique maximal flat cylinder,

$\mathcal{C} \subset \mathcal{S}$. This cylinder is a disjoint union of closed geodesics, γ_t, parallel to γ. The *Dehn twist* $T_{\mathcal{C}}$ induces a rotation by t on γ_t, where $0 \le t < 1$. Let $\gamma = \gamma_\rho$. It might seem natural to call $\rho = \rho(\gamma)$ the rotation number of the geodesic γ. However, our rotation number is slightly different, and we will define it under the additional assumption that α is a parabolic direction for $\mathcal{S}$. Then $\mathcal{S}$ decomposes as a union of the cylinders $\mathcal{C}_i$, $1 \le i \le k(\alpha)$. The Dehn twists T_i of the cylinders $\mathcal{C}_i$ define, after raising them to appropriate powers, a parabolic diffeomorphism ϕ_α of $\mathcal{S}$. The mapping ϕ_α induces a rotation on every geodesic in direction α, in particular on γ. The amount, $0 \le r(\gamma) < 1$, of this rotation is the *rotation number* of γ. See § 2.1 for an elaboration of this discussion and for further details.

If θ is a parabolic direction, then every geodesic γ in direction θ is either periodic or a saddle connection. In both cases, we say that γ is a *closed geodesic*. Furthermore, we say that a *geodesic is parabolic* to mean that it belongs to a parabolic direction. Note that on lattice translation surfaces every closed geodesic is parabolic. See [Vch89] and [GJ00]. Thus, our rotation number, $0 \le r(\gamma) < 1$, is defined for parabolic geodesics in $\mathcal{S}$. Such a *geodesic is rational* if $r(\gamma) \in \mathbb{Q}$.

Now let $\mathcal{S}$ be a prelattice surface. We say that $s \in \mathcal{S}$ is a *rational point* of $\mathcal{S}$ if it is an intersection point of two transversal rational geodesics.

Theorem 4. *Let $\mathcal{S}$ be a prelattice translation surface. Let $\mathcal{S}_{\mathbb{Q}} \subset \mathcal{S}$ be the set of rational points and let $P(\mathcal{S})$ be the set of periodic points.*

(a) *We have the inclusion $P(\mathcal{S}) \subset \mathcal{S}_{\mathbb{Q}}$, and the set $\mathcal{S}_{\mathbb{Q}}$ is countable and dense.*
(b) *The equality $P(\mathcal{S}) = \mathcal{S}_{\mathbb{Q}}$ holds if and only if $\mathcal{S}$ is arithmetic.*
(c) *Let $s \in \mathcal{S}$ be arbitrary. Then the marked surface $(\mathcal{S}; s)$ is a prelattice surface if and only if $s \in \mathcal{S}_{\mathbb{Q}}$.*

Applying this result to lattice translation surfaces, we obtain a classification of their points.

Corollary 1. *Let $\mathcal{S}$ be a lattice translation surface. Then the points of $\mathcal{S}$ satisfy the following trichotomy.*

(i) *We have $s \in P(\mathcal{S})$ if and only if $(\mathcal{S}; s)$ is a lattice surface.*
(ii) *We have $s \in \mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$ if and only if $(\mathcal{S}; s)$ is prelattice, but not a lattice surface.*
(iii) *We have $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}$ if and only if the Veech group of $(\mathcal{S}; s)$ is not a prelattice.*

In addition, Theorem 4 yields two new characterizations of arithmetic translation surfaces.

Corollary 2. *Let $\mathcal{S}$ be a prelattice translation surface. Then $\mathcal{S}$ is an arithmetic translation surface if and only if its set of rational points coincides with its set of periodic points.*

Corollary 3. *Let $\mathcal{S}$ be a lattice translation surface. Then $\mathcal{S}$ is arithmetic if and only if the following dichotomy holds:*

For any $s \in \mathcal{S}$ the marked surface $(\mathcal{S}; s)$ is either a (necessarily arithmetic) lattice surface, or the Veech group of $(\mathcal{S}; s)$ is is not a prelattice.

Remark 1. Fuchsian groups can be classified in terms of their action on the extended (i. e., including the absolute) hyperbolic plane. A Fuchsian group, Γ, is *elementary* if one of its orbits in the extended hyperbolic plane is finite. If Γ is a prelattice, then it has at least two distinct cyclic subgroups of parabolic elements. Hence, elementary groups are not prelattices.

Any nonelementary Fuchsian group contains a hyperbolic element. See say [B83]. Recall further that a hyperbolic and a parabolic element of a Fuchsian group cannot fix the same point of the absolute. Thus, the conjugate of the parabolic by the hyperbolic element has a distinct fixed point from that of the original parabolic. We conclude that a nonelementary Fuchsian group which is not a prelattice has no parabolic elements.

Let $\mathcal{S}$ be the Katok–Zemlyakov translation surface of a rational polygon. R. Kenyon and J. Smillie pointed out in [KS00] that if $\Gamma(\mathcal{S})$ has a parabolic element, then it also has a hyperbolic element. Hence, if $\Gamma(\mathcal{S})$ contains any parabolic element, then $\mathcal{S}$ is a prelattice translation surface.

Our further results deal with the mappings of translation surfaces that are compatible with their structures. These are the (branched) *affine coverings*. See [GJ00] and § 2.2 for the background on this material. Those coverings which are compatible with the respective cone sets are the so-called *balanced coverings* [Gut00]. See Definition 4. Balanced coverings are particularly interesting because the groups of affine diffeomorphisms (and hence the Veech groups) behave naturally under them. In particular, the lattice property is preserved under balanced coverings. See [GJ00, HS00, HS01]. Let $p : \mathcal{R} \to \mathcal{S}$ be an affine covering. Its differential, Dp, is a matrix, $Dp \in \mathrm{GL}(2, \mathbb{R})$. *Translation coverings* are the affine coverings such that $Dp = 1$. The group $\mathrm{GL}(2, \mathbb{R})$ acts on translation surfaces, by way of composition with coordinate functions. Let $\mathcal{S} \to g \cdot \mathcal{S}$ denote the action. Two translation surfaces $\mathcal{S}, \mathcal{S}'$ are *equivalent* if $\mathcal{S}' = g \cdot \mathcal{S}$, with $g \in \mathrm{SL}(2, \mathbb{R})$. Should g instead be in $\mathrm{GL}(2, \mathbb{R})$, then we say that $\mathcal{S}$ and $\mathcal{S}'$ are *equivalent in the extended sense.*

Again let $p : \mathcal{R} \to \mathcal{S}$ be an affine covering. Replacing either surface by an equivalent one (possibly in the extended sense), we obtain a translation covering $p' : \mathcal{R}' \to \mathcal{S}'$. If p is balanced, then p' is balanced, as well. This observation is very useful in studying the affine coverings of translation surfaces [GJ96, Vo96, GJ00].

Recall that two Fuchsian groups Γ, Γ' are *commensurable* if $\Gamma \cap \Gamma'$ is of finite index in both. They are *commensurable in the wide sense* if there is some $g \in \mathrm{SL}(2, \mathbb{R})$ for which $\Gamma \cap g\Gamma' g^{-1}$ is of finite index in both Γ and $g\Gamma' g^{-1}$. See § 2.2 and the references there for more on this material. Now we can state our next result.

When we speak of coverings, we always mean that they are nontrivial, i.e. the covering surface is connected, and the degree of the covering is greater than one.

Theorem 5. *A. Let $\mathcal{S}$ be a prelattice translation surface, and let $s \in \mathcal{S}$. Let $p : \mathcal{R} \to (S; s)$ be a balanced affine covering. Then the following trichotomy holds.*

(i) *The surface $\mathcal{R}$ satisfies the prelattice condition, and the groups $\Gamma(\mathcal{R}), \Gamma(\mathcal{S})$ are commensurable in the wide sense if and only if $s \in P(\mathcal{S})$.*
(ii) *The surface $\mathcal{R}$ does satisfy the prelattice condition, but the group $\Gamma(\mathcal{R})$ is commensurable in the wide sense to a prelattice subgroup of infinite index in $\Gamma(\mathcal{S})$ if and only if $s \in \mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$.*
(iii) *The surface $\mathcal{R}$ does not satisfy the prelattice condition if and only if $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}$.*

B. Let the assumptions be as above, except that now p is a balanced translation covering. Then the conclusions are also as above, but now the groups in question are commensurable (in the "narrow" sense).

In order to turn Theorem 5 into a source of examples of translation surfaces whose Veech groups have desired properties, we need a theorem that guarantees the existence of balanced translation coverings with prescribed branch points. This is our next result.

Theorem 6. *Let $\mathcal{S}$ be any translation surface. Let $s \in \mathcal{S} \setminus C(\mathcal{S})$. Then there exist balanced translation coverings $p : \mathcal{R} \to (\mathcal{S}; s)$.*

Combining Theorem 6 with Theorem 5, we obtain in § 5 several corollaries about Veech groups, as well as specific examples. For the moment we formulate two immediate consequences.

Corollary 4. *There exist prelattice, nonlattice translation surfaces.*

Corollary 5. *Let $\mathcal{S}$ be a nonarithmetic lattice translation surface. Let α, β be a pair of parabolic directions for $\mathcal{S}$. Then there exists a nonlattice, prelattice translation surface $\mathcal{R}$, and a translation covering $p : \mathcal{R} \to \mathcal{S}$ such that α and β are parabolic directions for $\mathcal{S}$.*

In order to formulate our next result, we introduce a definition.

Definition 3. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a Fuchsian group. We say that Γ is *realizable as a Veech group*, if there exists a translation surface $\mathcal{S}$ such that $\Gamma = \Gamma(\mathcal{S})$. Similarly, Γ is *nearly realizable* if it is commensurable with some $\Gamma(\mathcal{S})$.

Note that if Γ is a lattice (resp. prelattice, but not a lattice), realizable as Veech group, then the translation surface "realizing" Γ is a lattice (resp. prelattice, but nonlattice) surface. See [HS00, HS01] for more material on the realization of Fuchsian groups.

Theorem 7. *Let $\mathcal{S}$ be a lattice translation surface. Then $\mathcal{S}$ is nonarithmetic if and only if $\Gamma = \Gamma(\mathcal{S})$ has a subgroup, Γ', of infinite index, such that Γ' is a prelattice and Γ' is nearly realizable as a Veech group.*

Theorem 7 yields, in particular, the following "non-realization" result.

Corollary 6. *Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group. Suppose that Γ is commensurable in the wide sense to a prelattice in $SL(2, \mathbb{Z})$. Then Γ is nearly realizable as a Veech group if and only if it is a lattice.*

Recall that a Riemann surface is *hyperelliptic* if the Riemann sphere is the quotient of this surface by a holomorphic involution. This involution is called the *hyperelliptic involution*. The *Weierstrass* points of a hyperelliptic Riemann surface are exactly the fixed points of the hyperelliptic involution. A holomorphic 1-form is hyperelliptic if it is anti-invariant under the hyperelliptic involution.

Since a translation surface is given by a Riemann surface with a holomorphic 1-form, we can speak of *hyperelliptic translation surfaces*. See § 5.3 and Definition 7 below. The notion of *Weierstrass points of a translation surface* is also well defined. As Veech [Vch89] showed, his initial examples of nonarithmetic lattice surfaces are hyperelliptic translation surfaces. Their Veech groups are either generated by elliptic elements or else can be generated by a elliptic element and a parabolic element, [Vch89]. The following theorems indicate that there are simple relations between the periodic points and the Weierstrass points of a hyperelliptic translation surface.

Theorem 8. *Let $\mathcal{S}$ be a hyperelliptic translation surface. Suppose that the group Aff($\mathcal{S}$) is generated by elliptic elements. Then the set of Weierstrass points is a subset of the set of periodic points of $\mathcal{S}$.*

For the general hyperelliptic translation surface $\mathcal{S}$ the two sets do not coincide. For instance, if $\mathcal{S}$ is arithmetic, then the set of its periodic points is infinite, while the set of Weierstrass points is always finite. However, the two sets are equal for some translation surfaces.

Theorem 9. *There exist hyperelliptic translation surfaces whose sets of Weierstrass points and of periodic points coincide.*

1.3. **Organization of Paper.** The exposition in the body of the paper is organized as follows. In § 2 we treat background material and miscellaneous preliminaries. We recommend that the reader consult the corresponding sections of [GJ96, GJ00, HS00, HS01] for further details. Section 3 is the backbone of the paper. There we formulate and prove several technical results about the structure of translation surfaces and their parabolic diffeomorphisms. Some of these results are of independent interest. However, the main purpose of § 3 is to formulate and to prepare a proof of Theorem 11. This is a quantitative result, which implies the qualitative Theorems 1 and 2. Section 4 contains a few more auxiliary propositions (see § 4.1) and a proof of Theorem 11. See § 4.2 for the proof. Theorem 1 and Theorem 2 follow immediately. The proof of Theorem 3 is more involved, and we give it in the final subsection of § 4. See § 4.3. In § 5 we prove a few more of our claims and present explicit examples. In § 5.1 we expand the material on the rational points of prelattice surfaces. In §§ 5.2 and 5.3 we prove the remaining claims of § 1.2, except for Theorems 8 and 9. § 5.4 contains a few applications and examples, illustrating our results. In § 6 we prove the remaining claims of § 1.2 and give more examples concerning the material of Theorems 8 and 9.

2. Background and Preliminaries

2.1. **Parabolic Diffeomorphisms of a Translation Surface.** We recall some of the main concepts of the subject. We refer the reader to [GJ00] and to the survey [MT01] for elaborations on this material. A closed translation surface, $\mathcal{S}$, has a finite set, $C(\mathcal{S})$, of cone points. The points in $\mathcal{S} \setminus C(\mathcal{S})$ are called regular. Every nonzero tangent vector to $\mathcal{S}$, based at a regular point, has a *direction.* For any direction, $\theta \in [0, 2\pi)$, the unit tangent vectors in direction θ form a vector field, V_θ, with singularities at the cone points. Integral curves of V_θ are the *geodesics on $\mathcal{S}$ in direction θ.* We parametrize the geodesics by arclength. If $\gamma(t)$ is a geodesic, defined for $-\infty < t < \infty$, and $\gamma(t+\ell) = \gamma(t)$, $\gamma(t+\ell/n) \neq \gamma(t)$ for $n \in \mathbb{N}$, then γ is a (prime) *periodic geodesic* of length ℓ. If $\gamma(t)$ is a geodesic, defined for $0 \le t \le \ell$, where $\gamma(0), \gamma(\ell) \in C(\mathcal{S})$, and the interior points of γ are regular, then γ is a *saddle connection of length ℓ.* We use the name *closed geodesic* to designate both periodic geodesics and saddle connections.

The only closed translation surfaces without cone points are the flat tori. To unify our treatment, we make a convention that the origin of a flat torus is a cone point. See [GJ00]. Moving any periodic geodesic parallel to itself, we obtain a maximal flat cylinder, $\mathcal{C} \subset \mathcal{S}$. These are the elementary building blocks of $\mathcal{S}$. The flat cylinder, $\mathcal{C}(\ell, w)$, of *length* ℓ and *width* w, is obtained by identifying the two vertical sides of the rectangle $\mathcal{R}(\ell, w) = \{(x, y), 0 \le x \le \ell, 0 \le y \le w\}$. All $\mathcal{C}(\ell, w)$ are affinely equivalent. However, the modulus $\mu = w/\ell = \mu(\mathcal{C})$ is the conformal invariant. Vice versa, the interior of any (maximal) cylinder $\mathcal{C}$ in $\mathcal{S}$ is isometric to $\mathrm{Int}(\mathcal{C}(\ell, w))$, where $\ell = \ell(\mathcal{C})$ and $w = w(\mathcal{C})$ are respectively the length and the width of $\mathcal{C}$. The curves L_y, $y = \text{const.}$, for $0 \le y \le w$, are the closed geodesics in $\mathcal{C}(\ell, w)$. If $\mathcal{C} \subset \mathcal{S}$ is a cylinder of length ℓ and width w, then L_y, $0 < y < w$ are the periodic geodesics in $\mathcal{S}$ of length ℓ and direction $\theta = \theta(\mathcal{C})$. The curves L_0 and L_w in $\mathcal{S}$ are the unions of saddle connections in the same direction.

The group of affine diffeomorphisms of $\mathcal{C} = \mathcal{C}(\ell, w)$ is generated by the Dehn twist $T = T_{\mathcal{C}}$. In the coordinates above we have $T_{\mathcal{C}} : (s, t) \mapsto (s + t\ell/w \mod \ell\mathbb{Z}, t)$. The points in the boundary, $\partial\mathcal{C}(\ell, w)$, are fixed under T. Hence the formula above defines the Dehn twist for any cylinder, $\mathcal{C} \subset \mathcal{S}$, of length ℓ and width

w. A direction θ is *periodic* for $\mathcal{S}$, if every geodesic in direction θ is closed. A periodic direction defines a decomposition of $\mathcal{S}$ as a finite union of cylinders $\mathcal{C}_i$, $1 \le i \le k(\theta)$. Let w_i, ℓ_i, μ_i be the respective parameters, and let $T_i : \mathcal{C}_i \to \mathcal{C}_i$ be the respective Dehn twists. Then there exist $N_i \in \mathbb{N}$ such that the powers $T_i^{N_i}$, $1 \le i \le k(\theta)$ fit together, yielding an affine diffeomorphism $\phi_\theta : \mathcal{S} \to \mathcal{S}$ if and only if the moduli μ_i are commensurable. Then θ is a *parabolic direction*. The smallest positive $\mu = \mu(\theta)$ such that $\mu = N_i\mu_i, 1 \le i \le k(\theta)$, is the *modulus of the parabolic direction* θ. The corresponding diffeomorphism $\phi_\theta \in \mathrm{Aff}(\mathcal{S})$ is uniquely defined. We call ϕ_θ the *principal parabolic diffeomorphism* corresponding to θ. We use the same notation for its differential, which belongs to the group $\Gamma(\mathcal{S})$. In appropriate coordinates ϕ_θ is given by the parabolic upper triangular 2×2 matrix with $\mu(\theta)$ in the corner.

As opposed to the generic translation surface, lattice surfaces have many parabolic directions. See [Vch89].

2.2. **Affine Equivalence and Coverings.** There is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on the space of translation surfaces. It is especially easy to describe in terms of the coordinate charts, see [Vch84], [Vch86], [GJ00]. If $\mathcal{S}$ is a translation surface, and $g \in \mathrm{SL}(2,\mathbb{R})$, we denote by $g \cdot \mathcal{S}$ the new translation surface. The translation surfaces $\mathcal{S}$ and $g \cdot \mathcal{S}$ are *affinely equivalent*; one has $\Gamma(g \cdot \mathcal{S}) = g\Gamma(\mathcal{S})g^{-1}$. Therefore, this action preserves properties such as arithmeticity, the prelattice property and the lattice property. In particular, if α, β is a pair of parabolic directions for $\mathcal{S}$, and $g \in \mathrm{SL}(2,\mathbb{R})$, then $g \cdot \alpha, g \cdot \beta$ is the corresponding pair of parabolic directions for $g \cdot \mathcal{S}$. In view of these remarks, the statements formulated in § 1.2 are either invariant or equivariant under the affine equivalence of translation surfaces. The preceding remarks apply as well if $g \in \mathrm{GL}(2,\mathbb{R})$, i.e., the two translation surfaces are equivalent in the extended sense.

We use this observation for two purposes: 1) To *normalize* a pair of parabolic directions; 2) To replace an *affine covering* by a *translation covering*. The former is immediate from one of the preceding remarks. Namely, let $\mathcal{S}$ be a translation surface, and let α, β be a pair of parabolic directions for $\mathcal{S}$. Replacing $\mathcal{S}$ by an affinely equivalent surface, if need be, we assume without loss of generality that α and β are the coordinate directions. Since the change of sign of a direction does not interfere with our considerations, we can assume that α is the positive x-direction and β the positive y-direction.

Affine coverings form a natural class of mappings of translation surfaces. They are easy to define in terms of the coordinate charts of the two surfaces, see say [GJ00] for this. We list a few relevant properties of affine coverings. Let $p : \mathcal{X} \to \mathcal{Y}$ be one such cover. Then p defines a (surjective) branched covering of the corresponding closed topological surfaces. Furthermore, p is an affine mapping outside of the cone sets $C(\mathcal{X}), C(\mathcal{Y})$. Hence, its differential $Dp(x)$ is a constant matrix, $g \in \mathrm{GL}(2,\mathbb{R})$. Translation coverings are the affine coverings whose differential is the identity matrix. Replacing either $\mathcal{X}$ or $\mathcal{Y}$ by an affinely equivalent surface (in general, in the extended sense), we can assume that $p : \mathcal{X} \to \mathcal{Y}$ is a translation covering. This device has been widely used in the literature [GJ96, Vo96, GJ00, HS01]. We freely apply it here, again as the statements on coverings in § 1.2 are either invariant or equivariant under the extended affine equivalence. We elaborate on this as needed in the relevant proofs.

Definition 4. Let $p : \mathcal{X} \to \mathcal{Y}$ be an affine covering of translation surfaces. Then p is *balanced* if $p(C(\mathcal{X})) = C(\mathcal{Y})$ and $p^{-1}(C(\mathcal{Y})) = C(\mathcal{X})$.

We need the following result. It was proved independently by E. Gutkin and C. Judge [GJ96, GJ00] and by Ya. Vorobets [Vo96].

Theorem 10. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a balanced affine covering of translation surfaces. Then the groups $\Gamma(\mathcal{X})$ and $\Gamma(\mathcal{Y})$ are commensurable in the wide sense. If, besides, p is a translation covering, then $\Gamma(\mathcal{X})$ and $\Gamma(\mathcal{Y})$ are commensurable.*

3. Periodic Points of Translation Surfaces

Let $\mathcal{C}$ be a flat cylinder, and let $T : \mathcal{C} \to \mathcal{C}$ be the Dehn twist. A point $z \in \mathcal{C}$ is periodic if $T^n z = z$, for some $n > 0$. The smallest such n is the period of z.

Since affine equivalence respects sets of periodic points, we restrict the computations below to the standard cylinder $\mathcal{C} = \mathcal{C}(1,1)$. It is straightforward to extend our formulas to the arbitrary $\mathcal{C}(\ell, w)$. Thus consider $T : (x,y) \mapsto (x+y \mod 1, y)$. The circles $L_y = \{y = \text{const}\} \subset \mathcal{C}$ are the closed geodesics of $\mathcal{C}$. The restriction of T to L_y is the rotation by y. Hence, a point $z \in \mathcal{C}$ is periodic if and only if $z \in L_y$, where y is rational. Moreover, the set of points of period n is the union of $L_{k/n}$, with k and n relatively prime. Thus, we have $\phi(n)$ closed geodesics consisting of the points of period n, where ϕ is Euler's totient function.

The number of geodesics in $\mathcal{C}$, consisting of the points of period at most n is $\Phi(n) := \sum_{m=1}^{n} \phi(m)$. As n tends to infinity, $\Phi(n) = (3/\pi^2) \cdot n^2 + O(n \log n)$. See say Theorem 330 of [HW38].

We consider the subgroups of affine diffeomorphisms of $\mathcal{C}$, generated by powers of T. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the set of rational rotation numbers with denominator at most n. Thus, $\mathcal{F}_n := \{ (k,l) \in \mathbb{N}^2 \mid \gcd(k,l) = 1, k < l \le n \}$, and $|\mathcal{F}_n| = \Phi(n) \le n^2$. The map of the unit interval to itself, $x \mapsto \{Nx\}$, is N-to-1 and sends $\mathcal{F}_n$ to itself. In particular, the points of period at most n under T^N lie on $N\Phi(n)$ closed geodesics in $\mathcal{C}$.

We now apply the material above to an affine torus, as a model case. Again for simplicity, we give the computations for the standard torus $\mathbb{T}$, leaving the general case to the reader. The group of affine diffeomorphisms of $\mathbb{T}$ is $SL(2,\mathbb{Z})$. It is generated by the horizontal and the vertical Dehn twists, T_h and T_v respectively. We have $T_h : (x,y) \mapsto (x+y \mod 1, y)$ and $T_v : (x,y) \mapsto (x, y+x \mod 1)$.

The points $(x, y) \in \mathbb{T}$ which are periodic with respect to $SL(2, \mathbb{Z})$ are the rational points $(x, y) \in \mathbb{Q}^2/\mathbb{Z}^2$. The set of points which are periodic of period at most n under T_v and of period at most m under T_h is the intersection of the horizontal and vertical closed geodesics that we have just considered. The cardinality of this set is asymptotic to $(9/\pi^4) \cdot m^2 n^2$, as $m, n \to \infty$.

Let θ be a parabolic direction on a translation surface $\mathcal{S}$. We use the preceding material and the terminology of § 1.2.

In particular, we speak of *rational closed geodesics*, their *periods* and their *rotation numbers*.

Note that the periodic points of period n under the restriction of ϕ_θ to the cylinder $\mathcal{C}_i$ lie on $N_i\phi(n)$ rational geodesics of $\mathcal{C}_i$. The set of rotation numbers of these geodesics is $\mathcal{F}_n$.

We now state our crucial quantitative result.

Theorem 11. *Let $\mathcal{S}$ be a translation surface. Suppose that $\mathcal{S}$ has a pair of parabolic directions, α and β. Let $Aff_{\alpha,\beta}(\mathcal{S}) \subset Aff(\mathcal{S})$ be the group generated by the parabolic affine diffeomorphisms ϕ_α and ϕ_β. Then there exist positive integers M and N, depending only on the ratios of the parameters of the two decompositions, so that the following statements hold.*

(i) *If $\mathcal{S}$ has more than M periodic points with respect to $Aff_{\alpha,\beta}(\mathcal{S})$, then $\mathcal{S}$ is arithmetic.*

(ii) *If $\mathcal{S}$ has an $Aff_{\alpha,\beta}(\mathcal{S})$-periodic point of period greater than N, then $\mathcal{S}$ is arithmetic.*

Theorem 11 follows from several technical lemmas and propositions — some of these being of independent interest — about translation surfaces which satisfy the assumptions of Theorem 11. Note that these are the prelattice surfaces. By the remarks in § 2.2, we assume without loss of generality that the two parabolic directions are the coordinate directions. We use notational labels v and h referring to the *vertical* and the *horizontal* directions respectively. From now until further notice the standing assumption is that both coordinate directions are parabolic for our translation surface. A *rectangle* in $\mathcal{S}$ is a connected component of the intersection $\mathcal{C}_i^h \cap \mathcal{C}_j^v$. The interior of any rectangle is isometric to the Euclidean rectangle $(0, w_j^v) \times (0, w_i^h)$. Let $\mu_{i,j}$ be the number of rectangles formed by this intersection. We denote the rectangles by $\mathcal{R}_{i,j}^l, 1 \le l \le \mu_{i,j}$. The (essentially disjoint) decomposition

$$\mathcal{S} = \cup_{i=1}^{k(h)} \cup_{j=1}^{k(v)} \cup_{l=1}^{\mu_{i,j}} \mathcal{R}_{i,j}^l \tag{1}$$

implies

$$\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} w_i^h w_j^v = \text{Area}(\mathcal{S})\,.$$

Lemma 1. *For $1 \leq i \leq k(h)$ (resp. $1 \leq j \leq k(v)$) let H_i (resp. V_j) be a finite set of closed geodesics in $\mathcal{C}_i^h$ (resp. $\mathcal{C}_j^v$). Then*

$$|(\cup_{i=1}^{k(h)} H_i) \cap (\cup_{j=1}^{k(v)} V_j)| = \sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} |H_i||V_j|. \tag{2}$$

Proof. The intersection of a longitude in $\mathcal{C}_i^h$ with a longitude in $\mathcal{C}_j^v$ consists of $\mu_{i,j}$ points. ∎

To simplify notation, we denote the subgroups of $\text{Aff}(\mathcal{S})$ generated by the diffeomorphisms ϕ_h and ϕ_v by A and B, respectively. We denote by $\langle A, B\rangle$ the subgroup generated by A and B. Later on, while dealing with any pair of parabolic directions on $\mathcal{S}$, say α and β, we will use the same conventions. We will call $\langle A, B\rangle$ the *basic subgroups* of $\text{Aff}(\mathcal{S})$.

If f and g are functions of natural argument, we use the notation $f \leq\sim g$ to indicate that $f(n) \leq g(n)$ for n sufficiently large. As usual, $f \sim g$ means that the ratio $f(n)/g(n)$ converges to one as n goes to infinity.

The proposition below is immediate from Lemma 1 and the preceding remarks about periodic points.

Proposition 1. *For any subgroup $G \subset Aff(\mathcal{S})$ let $P^G \subset \mathcal{S}$ be the set of G-periodic points. Denote by $P_n^G \subset P^G$ the subset of points of periods at most n. Then*

(i) *For any m and n we have*

$$|P_m^A \cap P_n^B| = \Phi(m)\Phi(n) \sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} N_i^h N_j^v \,. \tag{3}$$

(ii) *We have*

$$|P_m^A \cap P_n^B| \sim \frac{9}{\pi^4} \left(\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} N_i^h N_j^v \right) m^2 n^2. \tag{4}$$

Corollary 7. *We have the asymptotic inequality*

$$|P_n^{\langle A,B\rangle}| \leq\sim \frac{9}{\pi^4} \left(\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} m_{i,j} N_i^h N_j^v \right) n^4 \,. \tag{5}$$

Proof. Use equation (4) and the inclusion $P_n^{\langle A,B\rangle} \subset P_n^A \cap P_n^B$. ∎

We now formulate a few immediate consequences of the propositions above.

Corollary 8. *Let $\mathcal{S}$ be a prelattice translation surface. Let $G \subset Aff(\mathcal{S})$ be any prelattice subgroup. Then*

(i) *The sets* P_n^G *are finite.*

(ii) *The cardinality* $|P_n^G|$ *grows at most quartically in* n*, as* n *tends to infinity.*

(iii) *The set* P^G *is infinite if and only if it contains periodic points of arbitrarily large periods.*

We will need a few technical lemmas.

Lemma 2. *There exist constants* c_0 *and* n_0*, depending only on the parameters of the pair of parabolic decompositions of* $\mathcal{S}$*, such that the following holds:*

Any finite orbit of $\langle A, B\rangle$ *of cardinality* $n > n_0$ *contains points of periods at least* $c_0\sqrt[4]{n}$ *with respect to each of* A *and* B*.*

Proof. We choose $c_0 > 0$ so that

$$c_0^4 = (\frac{3}{\pi^2} + 1)^{-2} \left(\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} N_i^h N_j^v \right)^{-1} .$$

By equations (4) and (5), there exists $m_0 \in \mathbb{N}$ such that for $m > m_0$ one has

$$|P_m^{\langle A,B\rangle}| < c_0^{-1} m^4. \tag{6}$$

Rewriting this inequality as $m^4 > c_0^4\,|P_m^{\langle A,B\rangle}|$ and setting $n_0 = c_0 m_0^4$, we obtain the claim. ∎

If x_i^α, $1 \le i \le k(\alpha)$ are any relevant parameters of the cylinders of a parabolic direction α, we use the notation $x_{\min}^\alpha$ and $x_{\max}^\alpha$ for the smallest and the biggest among them.

Lemma 3. *There is* $m_0 \in \mathbb{N}$*, depending only on the parameters of the horizontal and vertical decompositions of* $\mathcal{S}$*, such that the following holds:*

If any finite $\langle A, B\rangle$*-orbit contains a point of* A*-period* $m > m_0$*, then the* A*-orbit of this point contains a point of* B*-period at least*

$$\sqrt{\frac{2(m\frac{w_{\min}^v}{\ell_{\max}^h} - 1)}{N_{\max}^v}}.$$

Proof. Suppose that $\mathcal{O}$ is a finite $\langle A, B\rangle$-orbit, and $s \in \mathcal{O}$ is of A-period m. We assume, without loss of generality, that $s \in \mathcal{C}_1^h$, and let $L \subset \mathcal{C}_1^h$ be the closed geodesic containing s. It intersects at least one vertical cylinder. Again, we can assume that L intersects $\mathcal{C}_1^v$. Let $\mathcal{R} \subset \mathcal{C}_1^h \cap \mathcal{C}_1^v$ be one of the rectangles.

The distance between consecutive points of $A\cdot s$ is ℓ_1^h/m. Hence the number of points of the orbit $A\cdot s$ in the interval $L\cap R$ is at least $\lfloor w_1^v/(\ell_1^h/m)\rfloor \geq (mw_1^v/\ell_1^h)-1$. The interval $L\cap\mathcal{R}$ intersects each closed geodesic of $\mathcal{C}_1^v$ exactly once. Hence $\{A\cdot s\}\cap\mathcal{R}$ intersects at least $(mw_1^v/\ell_1^h)-1$ distinct closed geodesics of $\mathcal{C}_1^v$.

Let $X\subset[0,1]\cap\mathbb{Q}$ be the set of rotation numbers of these geodesics with respect to the basic Dehn twist of $\mathcal{C}_1^v$. Recall that the closed geodesics in a cylinder are parametrized by their rotation numbers. Set $N:=N_1^v$ and $Y:=\{\ \{Nx\}\,|\,x\in X\,\}$. Then Y is the set of rotation numbers of these geodesics with respect to the diffeomorphism ϕ_v of $\mathcal{S}$. Let n be the smallest positive integer such that $Y\subset\mathcal{F}_n$. Then n is the largest B-period of the geodesics in question. Using that $|Y|\geq|X|/N$ and the obvious upper bound for $|\mathcal{F}_n|$, we have

$$\frac{m\frac{w^v_{\min}}{l^h_{\max}}-1}{N^v_{\max}}<\frac{n^2}{2}. \tag{7}$$

Suppose that $m>m_0=l^h_{\max}/w^v_{\min}$. Then the left hand side of equation (7) is positive, and we obtain the claim. ∎

The following two lemmas put the statements obtained in the course of the proofs of Lemma 2 and Lemma 3 into a more suitable form. The proofs are straightforward, and we leave them to the reader.

Lemma 4. *There exist $c_1>0$ and $n_0\in\mathbb{N}$ depending only on the parameters of the two decompositions of $\mathcal{S}$, and such that the following holds:*

Let $n>n_0$, and let $\mathcal{O}\subset\mathcal{S}$ be an $\langle A,B\rangle$-periodic orbit of cardinality at least c_1n^8. Then $\mathcal{O}$ contains a point, s, enjoying the following properties:

(i) *The A-period of s is at least n;*
(ii) *Every vertical cylinder which intersects nontrivially the horizontal cylinder containing s contains a point of $B\cdot\{A\cdot s\}$, whose B-period is at least n.*

These statements remain true under interchange of A and B.

Lemma 5. *There exist $c_2,c_3>0$ and $n_0\in\mathbb{N}$ so that the following holds:*

(i) *Let $n\geq n_0$, and let $\mathcal{O}\subset\mathcal{S}$ be a finite $\langle A,B\rangle$-orbit of cardinality greater than c_2n^4. Then $\mathcal{O}$ contains a point of A-period at least n, and a point of B-period at least n.*
(ii) *Suppose that an $\langle A,B\rangle$-periodic orbit $\mathcal{O}$ contains an A-periodic point, s, of period at least c_3n^2 with $n\geq n_0$. Then every vertical cylinder which intersects nontrivially the horizontal cylinder containing s contains a point of $A\cdot s$, whose B-period is greater than or equal to n.*

These statements remain true under the interchange of A and B.

The proposition below is the main technical result about the prelattice surfaces.

Proposition 2. *There exist $c_4 > 0$, $n_0 \in \mathbb{N}$ and $d \in \mathbb{N}$, depending only on the parameters of the two decompositions, so that the following holds:*

Let $\mathcal{O} \subset \mathcal{S}$ be a finite $\langle A, B\rangle$-orbit of cardinality greater than $c_4 n^{2^{d+2}}$ with $n \geq n_0$. Then in every horizontal (resp. vertical) cylinder there is a point of $\mathcal{O}$ whose A-period (resp. B-period) is at least n.

Proof. We only sketch the proof, leaving the details to the reader. In particular, we will pretend that in the lemmas above the constants c_i are equal to one and that all the thresholds n_0 are the same. The latter can always be achieved by taking the biggest threshold of them all. The former can be arranged by (for instance) increasing the exponents in the lemmas by an arbitrarily small, but positive amount, and raising the threshold. By the first claim of Lemma 5, there is a horizontal cylinder, $\mathcal{C}_1^h$, such that $\mathcal{O} \cap \mathcal{C}_1^h$ contains a finite A-orbit of cardinality at least n^{2^d}. Then every vertical cylinder intersecting $\mathcal{C}_1^h$ contains a B-periodic point of $\mathcal{O}$, whose period is greater than or equal to $n^{2^{d-1}}$. See the second claim of Lemma 5. If the union of these vertical cylinders with $\mathcal{C}_1^h$ covers $\mathcal{S}$, we are done. Otherwise, we continue the inductive argument. On each consecutive step of the argument we lose a factor of 2 in the exponent 2^k. Since $\mathcal{S}$ is connected, after a finite number of steps we exhaust the surface. Thus, we take d to be the number of steps in this process. ■

4. Large Periodic Orbits Imply Arithmeticity

We continue with our standing assumption, as well as the preceding conventions. We begin this section with a few more technical propositions.

4.1. **Commensurability of Parameters.**

Lemma 6. *Let $\mathcal{C}_i^v$ and $\mathcal{C}_j^h$ be two cylinders such that $\mathcal{C}_i^h \cap \mathcal{C}_j^v \neq \emptyset$. Let $\mathcal{R} \subset \mathcal{C}_i^v \cap \mathcal{C}_j^h$ be one of the rectangles in the intersection. Suppose that two distinct points of $\mathcal{R}$ lie in the same A-orbit and in a finite $\langle A, B\rangle$-orbit. Then $w_j^v/\ell_i^h \in \mathbb{Q}$.*

Proof. We denote by (x, y) the natural coordinates in $\mathcal{R}$. Then $0 \leq x \leq w_j^v$, $0 \leq y \leq w_i^h$. Let $s = (x, y)$ and $s' = (x', y')$ be the two points in question. By assumption, there is $0 \neq n \in \mathbb{Z}$ so that

$$x' = x + n\frac{y}{w_i^h}\ell_i^h, \quad y' = y\,. \tag{8}$$

Since s is A-periodic, $\frac{y}{w_i^h} \in \mathbb{Q}$. On the other hand, since s and s' are both B-periodic, they belong to rational closed geodesics in $\mathcal{C}_j^v$. Thus, both x/w_j^v and

x'/w^v_j are rational numbers. Hence

$$\frac{x'-x}{w^v_j} = n\frac{y}{w^h_i}\frac{\ell^h_i}{w^v_j} \in \mathbb{Q}. \tag{9}$$

Since, as we already noted, $\frac{y}{w^h_i} \in \mathbb{Q}$, we obtain the claim. ■

Remark 2. Interchange of A and B in assumptions of the preceding Lemma yields the conclusion $w^h_i/\ell^v_j \in \mathbb{Q}$.

The following technical proposition is crucial. It is also of independent interest.

Proposition 3. *Let the notation be as in Proposition 2. Set*

$$m = m(A,B) = \max\{\frac{\ell^h_{\max}}{w^v_{\min}}, \frac{\ell^v_{\max}}{w^h_{\min}}\}. \tag{10}$$

Suppose that $\mathcal{S}$ has an $\langle A,B\rangle$-periodic point of period greater than or equal to $c_4m^{2^{d+2}}$. Then

(i) *All numbers w^v_j/ℓ^h_i and w^h_i/ℓ^v_j are rational;*

(ii) *The lengths ℓ^h_i, $1 \le i \le k(h)$, are commensurate, and the lengths ℓ^v_j, $1 \le j \le k(v)$, are commensurate, as well.*

(iii) *The widths w^h_i, $1 \le i \le k(h)$, are commensurate, and the widths w^v_j, $1 \le j \le k(v)$, are commensurate, as well.*

Proof. Let $\mathcal{O}$ be the $\langle A,B\rangle$-orbit in question. By Proposition 2, every horizontal (resp. vertical) cylinder contains a point of $\mathcal{O}$ of A-period (resp. B-period) greater than m. In view of equation (10), every rectangle $\mathcal{R} \subset \mathcal{C}^h_i \cap \mathcal{C}^v_j$ contains (at least) two points, s and s' of $\mathcal{O}$, such that $s' = \phi_h \cdot s$ (resp. $s' = \phi_v \cdot s$). Lemma 6 and Remark 2 imply our first claim.

Suppose that $\mathcal{C}^h_i$ and $\mathcal{C}^h_{i'}$ intersect the same vertical cylinder, $\mathcal{C}^v_j$. We have already proved that w^v_j/ℓ^h_i and $w^v_j/\ell^h_{i'}$ are rational. Thus ℓ^h_i and $\ell^h_{i'}$ are commensurate. In view of the connectedness of $\mathcal{S}$, for any pair $\mathcal{C}, \mathcal{C}'$ of horizontal cylinders, there is a sequence $\mathcal{C}_1, \dots, \mathcal{C}_k$ of horizontal cylinders such that $\mathcal{C} = \mathcal{C}^h_1$, $\mathcal{C}' = \mathcal{C}^h_k$, and every two consecutive cylinders of the sequence intersect some common vertical cylinder. Thus $\ell(\mathcal{C})/\ell(\mathcal{C}')$ is rational. The same argument works for vertical cylinders, proving our second claim. The proof of the last claim is essentially identical, and we leave it to the reader. ■

At this point we drop our standing assumption. Until further notice, we will explicitly formulate all of our assumptions. The following proposition is of independent interest.

Proposition 4. *Let $\mathcal{S}$ be a translation surface. Let α and β be two transversal parabolic directions. Let w_i^α, $1 \le i \le k(\alpha)$, and w_j^β, $1 \le j \le k(\beta)$, be the widths of the respective cylinders. Suppose that the numbers w_i^α are all commensurate, and the numbers w_j^β are commensurate, as well. Then $\mathcal{S}$ is an arithmetic translation surface.*

Proof. Replacing $\mathcal{S}$ by an affinely equivalent surface, we assume without loss of generality that α and β are the coordinate directions. In what follows we use h for α and v for β.

Changing the translation structure $\mathcal{S}$ by a diagonal transformation, if need be, we ensure that all the widths w_i^h and w_i^v are rational. Applying a homothety, we make them integral. Now we use the relations

$$\ell_j^v = \sum_{i=1}^{k(h)} \mu_{i,j} w_i^h \ , \quad \ell_i^h = \sum_{j=1}^{k(v)} \mu_{i,j} w_j^v . \tag{11}$$

Thus, all the lengths ℓ_i^h, ℓ_j^v are integral. Since of the parameters of the coordinate decompositions of $\mathcal{S}$ are integral, by Theorem 5.5 of [GJ00], the translation surface is arithmetic. ∎

In some sense, Proposition 4 is a special case of a more general statement: A translation surface whose parameters are commensurate is arithmetic. See, e.g., [GJ00].

4.2. **Proofs of Theorems 1, 2, 11.** Now we are almost ready to prove the claims formulated in § 1.2. First, we prove the main quantitative theorem.

Proof of Theorem 11. We prove the second claim first. Let $m = m(\alpha, \beta)$, as given by equation (10). By Proposition 3, if $N \ge m(\alpha, \beta)$, then the assumptions of Proposition 4 are satisfied. Hence, $\mathcal{S}$ is arithmetic.

To prove the first claim, we note that, by Corollary 7, the existence of M periodic points implies the existence of a periodic point of period at least $N = \text{const}\sqrt[4]{M}$. This holds only for M greater than a certain threshold, depending on the data, which also determines the constant in question. Therefore, if $\mathcal{S}$ satisfies the assumption of claim (i), then it satisfies the assumption of claim (ii), as well. Since that claim is already proven, we are done. ∎

Note that the hypothesis of the second claim of Theorem 11 implies the hypothesis of the first claim, with $M = N$. This observation and the preceding argument allow us to reformulate Theorem 11 as follows.

Corollary 9. *Let $\mathcal{S}$ be a prelattice translation surface. Then there exists $n \in \mathbb{N}$, determined from the parameters of any pair of transversal parabolic directions in $\mathcal{S}$, so that the following holds:*

If $\mathcal{S}$ either has at least n periodic points, or has a periodic point of period at least n, then $\mathcal{S}$ is arithmetic.

Proof of Theorem 2. If $H \subset G \subset \mathrm{Aff}(\mathcal{S})$ is a tower of subgroups, then $P \subset P^G \subset P^H$ for the respective sets of periodic points. The claim hence follows directly from Theorem 11. ■

Proof of Theorem 1. We have already proved that a nonarithmetic (pre)lattice translation surface necessarily has a finite number of periodic points. Now let $\mathcal{S}$ be an arithmetic translation surface. Replacing $\mathcal{S}$ by an equivalent translation surface, if need be, we can assume that we have $p : \mathcal{S} \to \mathbb{T}$, a balanced translation covering of the standard torus, $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$. We have $\mathrm{Aff}(\mathbb{T}) = \mathrm{SL}(2, \mathbb{Z})$, and the group $\mathrm{Aff}(\mathcal{S})$ is commensurable with this. See [GJ96, GJ00].

The set $\mathbb{Q}^2/\mathbb{Z}^2 := \mathbb{T}_\mathbb{Q} \subset \mathbb{T}$ of rational points is dense in $\mathbb{T}$. But $\mathbb{T}_\mathbb{Q}$ is the set of $\mathrm{SL}(2, \mathbb{Z})$-periodic points in $\mathbb{T}$. The set of periodic points in $\mathcal{S}$ satisfies $P(\mathcal{S}) = p^{-1}(\mathbb{T}_\mathbb{Q})$. As the preimage of a dense set, $P(\mathcal{S})$ is dense in $\mathcal{S}$. ■

4.3. **Proof of Theorem 3.** The claim is invariant under the affine equivalence of translation surfaces. Hence, it suffices to prove the claim under the convention introduced in § 3: The coordinate directions are parabolic. We use the pertinent notation as well.

Let $X \subset \mathcal{S}$ be an infinite closed $\langle A, B\rangle$-invariant subset. Suppose that X contains a coordinate closed geodesic, L. We can assume without loss of generality that L is vertical. Let $\mathcal{R}$ be one of the rectangles intersecting L. The set of ϕ_h-rotation numbers of the points in the vertical interval $\mathcal{R} \cap L \subset X$ is the interval $(0, 1)$. For every point $z \in \mathcal{R} \cap L$ of irrational rotation number, the ϕ_h-orbit of z is dense in the horizontal geodesic containing z. Since X is closed, all of this geodesic belongs to X. Since the set of irrational numbers is dense in $(0, 1)$, we conclude that all of the horizontal cylinder containing $\mathcal{R} \cap L$ belongs to X. Since $\mathcal{R}$ was chosen arbitrarily, we see that X contains the union, X_1, of the horizontal cylinders intersecting L. Replacing L by a horizontal closed geodesic in X_1, we conclude that X contains the union, X_2, of the vertical cylinders intersecting X_1. This inductive process produces an increasing tower $L \subset X_1 \subset X_2 \subset \cdots \subset X$. By construction, either $X_{i+1} \setminus X_i$ contains at least one coordinate cylinder, or $X_i = \mathcal{S}$. Since the number of these cylinders is finite, we have $X = \mathcal{S}$.

It remains to prove that X contains a coordinate closed geodesic. Let $\mathcal{R}$ be a coordinate rectangle, and let $z = (x, y) \in \mathcal{R}$ be an arbitrary point. Denote by $r_h(z)$ and $r_v(z)$ the ϕ_h and ϕ_v rotation numbers respectively. Note that r_h is (essentially) a linear function of y alone; similarly for r_v with respect to x. Since X is infinite, there is at least one $\mathcal{R}$ such that the set $X \cap \mathcal{R}$ is infinite. Denote by $R_h(X)$ and $R_v(X)$ the sets of horizontal and vertical rotation numbers of the points in $X \cap \mathcal{R}$. Since $X \cap \mathcal{R}$ is closed, both $R_h(X)$ and $R_v(X)$ are closed subsets of $[0, 1]$.

If $R_h(X) \cup R_v(X)$ contains an irrational number, then there exists a closed (vertical, without loss of generality) geodesic, L, with an irrational rotation number, containing a point of X. Then, by minimality of irrational rotations, $L \subset X$. Assume from now, and until the end of the proof, that $R_h(X) \cup R_v(X) \subset \mathbb{Q}$. There are two possibilities: the set $R_h(X) \cup R_v(X)$ is either infinite or finite.

Suppose first that both sets of rotation numbers are finite. Then there is a closed (horizontal, without loss of generality) geodesic, L, with a rational rotation number which contains infinitely many points of X. Since $X \cap L \cap \mathcal{R}$ is infinite, we have infinitely many vertical rotation numbers, contrary to the assumption.

It remains to consider the possibility when $R_h(X) \cup R_v(X)$ is infinite. Assume, without loss of generality, that $|R_h(X)| = \infty$. Let $r \in \mathbb{Q}$ be an accumulation point of $R_h(X)$. Then there is an infinite sequence of points $z_n \in X \cap \mathcal{R}$ converging to $z \in X \cap \mathcal{R}$, and $r = r_h(z)$. Set $r_h(z_n) = p_n/q_n$. Since $p_n/q_n \to r$, as $n \to \infty$, the sequence of the denominators q_n is unbounded. Let L_n (resp. L) be the horizontal closed geodesic containing z_n (resp. z). The distance between the consecutive points of the orbit $A \cdot z_n \subset L_n$ is of the order of q_n^{-1}. Since L_n converges to L, we conclude that L consists of accumulation points of X. But X is closed. Thus, $L \subset X$, which concludes our proof. ■

5. Prelattice Surfaces

5.1. **Rational Points.** Let $\mathcal{S}$ be a prelattice translation surface, and let α, β be a pair of parabolic directions. We assume without loss of generality that α, β correspond to the positive orientation on $\mathcal{S}$. In the discussion that follows we work with an arbitrary pair α, β and use an explicit affine equivalence to reduce it to the coordinate pair x, y. Let $\mathcal{R} \subset \mathcal{C}_i^h \cap \mathcal{C}_j^v$ be one of the *parallelograms* $\mathcal{R}_{i,j}^l$ of the associated decomposition. See equation (1). We change the affine structure of $\mathcal{S}$ by the unique $g \in \mathrm{SL}(2, \mathbb{R})$ which sends α and β to the coordinate directions. Let x, y be the Euclidean coordinates such that the interior of $\mathcal{R}$ is parametrized by $(0 < x < w_v, 0 < y < w_h)$. In view of possible identifications on the boundary, $\mathcal{R}$ itself may not be isometric to the Euclidean rectangle $[0, w_v] \times [0, w_h]$. However, there is a unique mapping $f_{\mathcal{R}} : [0, w_v] \times [0, w_h] \to \mathcal{R}$, inducing an isometry of $(0, w_v) \times (0, w_h)$ onto $\mathrm{Int}(\mathcal{R})$.

Reversing the affine equivalence above, we return to the original directions α, β. This construction yields a unique affine mapping $f_{\mathcal{R}} : [0, w_v] \times [0, w_h] \to \mathcal{R}$, which is onto, preserves orientation and area, and is an affine isomorphism of $(0, w_v) \times (0, w_h)$ and $\mathrm{Int}(\mathcal{R})$.

Definition 5. Let $\mathcal{S}$ be a translation surface, and let α, β be a pair of parabolic directions. Let $z \in \mathcal{S}$ be an arbitrary point, let $\mathcal{R}$ be a parallelogram of the decomposition equation (1), containing z, and let $f_{\mathcal{R}} : [0, w_v] \times [0, w_h] \to \mathcal{R}$ be the corresponding affine mapping. Then z is *rational with respect to the pair* α, β if $z = f_{\mathcal{R}}(x, y)$, where $x/w_v, y/w_h \in \mathbb{Q}$. A point $z \in \mathcal{S}$ is *rational*, if there is a pair of parabolic directions such that z is rational with respect to it.

We use the name *irrational* for all points that are not rational in the sense of Definition 5. If $\mathcal{R}$ is a parallelogram of the decomposition equation (1), we denote by $\mathcal{R}_{\mathbb{Q}}$ the set of its rational points. We use the notation $\mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ for the set of rational points with respect to the pair α, β, and $\mathcal{S}_{\mathbb{Q}}$ for the set of rational points of $\mathcal{S}$. Note that the concepts of rational and irrational points applies only to prelattice surfaces. The following proposition justifies it. Its proof is straightforward, and we leave it to the reader.

Proposition 5. *Let $\mathcal{S}$ be a prelattice translation surface, and let α, β be a pair of parabolic directions for $\mathcal{S}$. Let $s \in \mathcal{S} \setminus C(\mathcal{S})$ be an arbitrary point. Then the following statements are equivalent.*

(i) *The point s is rational with respect to α, β.*
(ii) *The directions α, β are parabolic for the punctured surface $(\mathcal{S}; s)$.*
(iii) *The point s is periodic with respect to each of ϕ_α and ϕ_β.*
(iv) *The point s is an intersection point of two rational geodesics, with directions α and β respectively.*

If $\mathcal{S}$ is a lattice translation surface, then the qualifier "parabolic" in Definition 5 and in Proposition 5 may be replaced with the formally weaker "periodic".

5.2. **Proofs of Theorems 4, 5, and Corollaries 1, 2, and 3.** Now we turn to the proofs of the relevant claims of § 1.2.

Proof of part (a) of Theorem 4. We first show that the set of rational points of $\mathcal{S}$ is countable and dense. It is the union of $\mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ over all pairs of parabolic directions. Each set $\mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ is countable and dense in $\mathcal{S}$. It remains to show that the set of parabolic directions of $\mathcal{S}$ is (at most) countable. A parabolic direction is, in particular, periodic. It is known since [KZ75] that the set of periodic directions of a translation surface is at most countable. See [Gut96] for further developments.

Let $s \in P(\mathcal{S})$. Then s is periodic with respect to every $\mathrm{Aff}_{\alpha,\beta}(\mathcal{S}) \subset \mathrm{Aff}(\mathcal{S})$, hence

$$s \in \cap_{\alpha,\beta} \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta} \subset \cup_{\alpha,\beta} \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta} = \mathcal{S}_{\mathbb{Q}}.$$

This proves the inclusion $P(\mathcal{S}) \subset \mathcal{S}_{\mathbb{Q}}$. ∎

Proofs of part (b) of Theorem 4 and Corollary 2 . If $\mathcal{S}$ is arithmetic, then $\mathcal{S}_{\mathbb{Q}} = \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ for every pair α, β. See the proof of Theorem 1. But, $P(\mathcal{S}) = \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ for every α, β. Hence $P(\mathcal{S}) = \mathcal{S}_{\mathbb{Q}}$. On the other hand, if $P(\mathcal{S}) = \mathcal{S}_{\mathbb{Q}}$ holds, then in particular, the set $P(\mathcal{S})$ is infinite. By Theorem 1 or by Theorem 2, $\mathcal{S}$ is arithmetic. This proves Corollary 2 and part (b) of Theorem 4. ∎

Proof of part (c) of Theorem 4 . Proposition 5 immediately gives that $(\mathcal{S}, s)$ is a prelattice surface for $s \in \mathcal{S}_{\mathbb{Q}}$

The Veech group of a translation surface is not a prelattice if and only if the surface does not have a pair of parabolic directions. The set of parabolic directions for $(\mathcal{S}; s)$ is a subset of the set of parabolic directions for $\mathcal{S}$. By Proposition 5, $\Gamma(\mathcal{S}; s)$ is not a prelattice if and only if $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ for all parabolic pairs α, β. But, this is equivalent to the statement that $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}$. ∎

Proof of Corollary 1 . The first claim of Corollary 1 is in the literature [GJ96, GJ00, HS00]. The second claim follows from the first and Theorem 4. The third claim is a special case of the part (c) in Theorem 4. ∎

Proof of Corollary 3. By Corollary 1, $\mathcal{S}$ is arithmetic if and only if the second option in the trichotomy of Corollary 1 is empty. Thus, Corollary 1 implies Corollary 3. ∎

Proof of Theorem 5. If $p : \mathcal{R} \to (S; s)$ is a balanced affine covering, we replace $\mathcal{R}$ by an equivalent surface to make it a balanced translation covering. Thus, it suffices to prove the theorem under this assumption. By [GJ00], the groups $\Gamma(\mathcal{R})$ and $\Gamma((S; s))$ are commensurable. This reduces all but one claim of Theorem 5 to the statements of Theorem 4. It remains to analyze the group $\Gamma((S; s))$ when $s \in \mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$.

Let α, β be a pair of parabolic directions for $\mathcal{S}$ such that $s \in \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$. Then the "local isotropy group" $\Gamma_s^{\alpha,\beta} \subset \Gamma((S; s))$ is a prelattice. Therefore $\Gamma((S; s)) \subset \Gamma(\mathcal{S})$ is a prelattice as well. But, since the orbit $\mathrm{Aff}(\mathcal{S}) \cdot s$ is infinite, $\Gamma((S; s))$ has infinite index in $\Gamma(\mathcal{S})$. This completes the analysis of possible cases. ∎

5.3. **Proofs of Theorems 6, 7, and Corollaries 4, 5, and 6.** A translation surface, $\mathcal{S}$, can be viewed as a Riemann surface, S, with a holomorphic 1-form, ω. The cone points, $s \in C(\mathcal{S})$, are the zeros of ω. See [Vch84], [HM79], [GJ00], [Gut96] or [MT01] for details. Let $p : R \to S$ be a branched covering of Riemann surfaces, and let α be the pull-back of ω. Then the pair (R, α) determines a translation surface, $\mathcal{R}$, and p gives a translation covering, $p : \mathcal{R} \to \mathcal{S}$. The cone set $C(\mathcal{R})$ is the union of $p^{-1}(C(\mathcal{S}))$ and the set of the ramification points of the covering $p : R \to S$.

Proof of Theorem 6. Suppose now that $\mathcal{S}$ is a translation surface, and let $s \in \mathcal{S} \setminus C(\mathcal{S})$. Let (S, ω) be the corresponding Riemann surface and the holomorphic 1-form. By the remark above, we have a one-to-one correspondence between the balanced translation coverings $p : \mathcal{R} \to (\mathcal{S}; s)$ and the coverings of Riemann surfaces $p : R \to S$, satisfying the following conditions: 1) The points in $S \setminus \{C(\mathcal{S}) \cup \{s\}\}$ do not belong to the branch locus of p; 2) Every point in $p^{-1}(s) \subset R$ is a ramification point of p. For instance, if s is the only branch point of p in $S \setminus C(\mathcal{S})$, and $|p^{-1}(s)| = 1$, the conditions above are satisfied. There are coverings of arbitrarily high degree which satisfy the two conditions [FK80]. ∎

Proofs of Corollaries 4, 5. Now let $\mathcal{S}$ be a nonarithmetic lattice surface. See [Vch89] for simple examples. By Theorems 1 and 4, the set $\mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$ is countable and dense, hence nonempty. Let $s \in \mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$ be any point. Let $p : \mathcal{R} \to (S; s)$ be any balanced translation covering. By Theorem 5, $\mathcal{R}$ is a prelattice, nonlattice translation surface. This proves Corollary 4. To obtain Corollary 5, we use the preceding argument, but choose any point $s \in \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta} \setminus P(\mathcal{S})$. ∎

Proofs of Theorem 7 and Corollary 6. Let $\mathcal{S}$ be a nonarithmetic lattice surface. Let α, β be a pair of parabolic directions. Let s be as in the proof of Corollary 5. Set $\Gamma' = \Gamma((S; s))$. By Theorem 5, the prelattice group Γ' is of infinite index in $\Gamma(\mathcal{S})$. Thus, Γ' is not a lattice. Let $p : \mathcal{R} \to (S; s)$ be as in Theorem 6. By Theorem 10, the groups $\Gamma(\mathcal{R})$ and Γ' are commensurable. Thus, $\mathcal{R}$ provides a near realization of Γ'.

Let now $\mathcal{S}$ be arithmetic, and let $\Gamma' \subset \Gamma(\mathcal{S})$ be a prelattice subgroup of infinite index. Suppose that Γ' is nearly realizable as a Veech group. Then there exists a translation surface $\mathcal{R}$ and a finite index subgroup $\Gamma'' \subset \Gamma'$, such that Γ'' is of finite index in $\Gamma(\mathcal{R})$. Let $g \in \Gamma''$ be a hyperbolic element. Since Γ is arithmetic, the trace of g is a rational number. Therefore, by [KS00] (Theorem 28), the holonomy field of $\mathcal{R}$ is $\mathbb{Q}$. (See [KS00] for the notion of the holonomy field of a translation surface.) Then, by Theorem 5.5 of [GJ00], $\mathcal{R}$ is arithmetic. On the other hand, $\Gamma(\mathcal{R})$ has infinite covolume. This contradiction yields Theorem 7. Finally, Corollary 6 is a special case of Theorem 7. ∎

5.4. **Examples and Applications.** In this subsection we illustrate and augment the preceding material. We also give an application to polygonal billiards. The example below provides a family of prelattice subgroups of $\mathrm{SL}(2, \mathbb{Z})$, none of which can be (even nearly) realized as Veech groups.

Example 1. For $m, n \in \mathbb{N}$, let $G_{m,n} \subset \mathrm{SL}(2, \mathbb{Z})$ be the group generated by the parabolic matrices $\mu = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\nu = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$. Our family consists of $G_{m,n}$ for which $mn > 4$.

Note that each $G_{m,n}$ satisfies the following condition: For each hyperbolic element $g \in G_{m,n}$ the fields $\mathbb{Q}(|\mathrm{tr}(g^2)|)$ and $\mathbb{Q}(|\mathrm{tr}(g)|)$ coincide. This condition is necessary for the realizability of a Fuchsian group as a Veech group, see [HS01].

For any m, n with $mn > 4$, let $\Omega_{m,n}$ be the domain in the upper half-plane, which is bounded by the two vertical lines $x = \pm m/2$ and by the two half-circles with the endpoints at $x = -2/n, x = 0$ and $x = 0, x = 2/n$ respectively. Then $\Omega_{m,n}$ is a fundamental domain for $G_{m,n}$, see say [B83]. But $\Omega_{m,n}$ is of infinite hyperbolic area. By Corollary 6, $G_{m,n}$ is not realizable as a Veech group if $mn > 4$. Moreover, since any group commensurable to $G_{m,n}$ will also be of infinite covolume, $G_{m,n}$ is not nearly realizable as a Veech group.

For $mn \leq 4$ the group $G_{m,n}$ is of finite index in $\mathrm{SL}(2,\mathbb{Z})$, and hence is nearly realizable.

There is a well known connection between *rational polygons* and translation surfaces. For the reader's convenience, we outline it here. See the surveys [Gut96] and [MT01] for more information. A Euclidean polygon, P, is rational if the angles between its sides are of the form $m\pi/n$. Let $N = N(P)$ be the common denominator of these rational numbers. There is a canonical translation surface, $\mathcal{S} = \mathcal{S}(P)$. Combinatorially, $\mathcal{S}$ is made from $2N$ copies of P glued along their boundaries in a canonical fashion. The relation between P and $\mathcal{S}$ is such that the billiard ball orbits in P unfold into the geodesics in $\mathcal{S}$, reducing the billiard flow in P to the geodesic flow in $\mathcal{S}$. This observation was used by A. Katok and A. Zemlyakov in [KZ75] to prove the topological transitivity of the billiard in the typical (i.e. , irrational) polygon. For this reason, the translation surface $\mathcal{S}(P)$ is often called "the Katok-Zemlyakov surface" of a rational polygon. In fact, it is a classical geometric construct. See [Gut84] and the references there.

Definition 6. Let P be a rational polygon, and let $\mathcal{S}$ be the corresponding translation surface. We say that P is a *lattice polygon* (resp. a *prelattice polygon*) if $\mathcal{S}$ is a lattice (resp. a prelattice) translation surface.

The simplest lattice polygon is the square. Its translation surface is the standard torus. Gutkin [Gut84] introduced and investigated a class of rational polygons P that naturally generalize this example: The surface $\mathcal{S}(P)$ is arithmetic. We call these the *arithmetic polygons*. Veech in [Vch89] gave the first examples of *nonarithmetic lattice polygons*. These are the right triangles whose smallest angle is π/n, if $n \neq 4, 6$.

Let $p, q, r \in \mathbb{N}$ be relatively prime. We denote by $T(p, q, r)$ the Euclidean triangle with angles $p\pi/(p+q+r)$, $q\pi/(p+q+r)$, $r\pi/(p+q+r)$. In this notation, the right triangle above is $T(2, n-2, n)$ if n is odd and $T(1, m-1, m)$ if $n = 2m$.

One of the remarkable properties of lattice polygons concerns the asymptotics of the number of *periodic billiard orbits*. These occur in bands of parallel orbits of the same (physical) length. Let $f_P(x)$ be the number of the periodic bands of length at most x in P. Veech proved in [Vch89] that if P is a lattice polygon, then

$$f_P(x) \sim c(P)x^2, \text{ as } x \to \infty. \tag{12}$$

We denote by $\Gamma(P)$ the Veech group of the lattice translation surface $\mathcal{S}(P)$. We call it the *Veech group of the polygon P*. Using harmonic analysis and explicit computations, Veech calculated the *quadratic constants* $c(T(2, n-2, n))$ and $c(T(1, m-1, m))$. See [GJ00] for an elementary approach to the *quadratic asymptotics* for the number of periodic billiard orbits expressed in equation (12).

There are several other papers concerning lattice polygons in the literature [Vch92, Vo96, Wrd98, KS00]. They provide, in particular, many examples of rational polygons that satisfy or do not satisfy the lattice condition. The results of [KS00] and [Pu01] yield a complete description of lattice acute triangles. Below we give an explicit example of a prelattice but nonlattice triangle.

Example 2. Set $T_1 = T(2,3,5)$ and $T_2 = T(3,3,4)$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the corresponding translation surfaces, and let Γ_1 and Γ_2 be the respective Veech groups. We prove that T_2 is a nonlattice, prelattice triangle.

The triangle T_1 has angles $\pi/2, \pi/5, 3\pi/10$. It belongs to the family of lattice triangles treated in [Vch89]. The surface $\mathcal{S}_1$ is obtained by glueing along the sides two copies of the regular pentagon. Their vertices are glued into a single point, the cone set $C(\mathcal{S}_1)$. The isosceles triangle T_2 has angles $2\pi/5, 3\pi/10, 3\pi/10$. It is the "doubling" of T_1 along one of its sides. Accordingly, there is a canonical two-to-one translation covering $p : \mathcal{S}_2 \to \mathcal{S}_1$.

Let o_1, o_2 be the centers of the two pentagons. The covering above is not balanced, but it defines a balanced covering $p : \mathcal{S}_2 \to (S_1; o_1, o_2)$. Hence, Γ_2 is commensurable with $\Gamma((S_1; o_1, o_2))$. See Proposition 4 and related material in [HS00].

By Proposition 3 of [HS00], the group Γ_2 is not a lattice. Thus, T_2 is not a lattice triangle. Therefore, the points o_1, o_2 are not periodic points of the lattice surface $\mathcal{S}_1$. Choose any two diagonals of the regular pentagon. Their directions, α and β, are parabolic [Vch89]. Since o_1, o_2 are intersection points of the saddle connections of a pair of parabolic directions, they are rational points of $\mathcal{S}_1$. Hence, $\Gamma((S_1; o_1, o_2)) \subset \Gamma_1$ is a prelattice.

The group Γ_2, being commensurable with $\Gamma((S_1; o_1, o_2))$, is a prelattice as well. Thus, T_2 is a prelattice triangle.

6. Weierstrass Points versus Periodic Points

Definition 7. Let $\mathcal{S}$ be a translation surface. Let (S, ω) be the corresponding Riemann surface with a holomorphic 1-form. We say that $\mathcal{S}$ is a *hyperelliptic translation surface* if S is a hyperelliptic Riemann surface, and ω is anti-invariant under the hyperelliptic involution of S.

6.1. **Proof of Theorem 8.** We begin with a general Lemma which will be useful later on.

Lemma 7. *Let $\mathcal{S}$ be an arbitrary translation surface. Let $\phi \in Aff(\mathcal{S})$ be an elliptic element. Then there is an affinely equivalent translation surface $\mathcal{T}$ such that the induced diffeomorphism $\psi \in Aff(\mathcal{T})$ is an isometry.*

Proof. Let α be the 1-form on $\mathcal{S}$ giving the translation structure and the metric. By assumption, either $D\phi = \pm 1$, or $D\phi$ is an elliptic element of $\mathrm{SL}(2, \mathbb{R})$. In the former case, ϕ itself is an isometry. Suppose now that $D\phi$ is elliptic. Then $D\phi$ fixes a unique point in the upper half-plane. Let $g \in \mathrm{SL}(2, \mathbb{R})$ be an element that sends that point to i. Set $\mathcal{T} = g \cdot \mathcal{S}$, and let β be the corresponding 1-form on $\mathcal{T}$. The induced diffeomorphism $\psi \in \mathrm{Aff}(\mathcal{T})$ satisfies $D\psi = g \cdot D\phi \cdot g^{-1}$. Since $D\psi(i) = i$, it belongs to the group $\mathrm{SO}(2)$. Let θ be the rotation angle of $D\psi$. Then $\psi(\beta) = e^{i\theta}\beta$. Thus, ψ is an isometry of $\mathcal{T}$. ∎

Proof of Theorem 8. Let $\mathcal{S}$ be a translation surface and let $\mathcal{T} = g \cdot \mathcal{S}$ be an equivalent translation surface. Denote by S and T the corresponding Riemann surfaces, and let $W(S), W(T)$ be the respective sets of Weierstrass points. We regard $W(S)$ and $W(T)$ as subsets of the underlying "physical surface", M. In general there is no relation between the sets $W(S), W(T) \subset M$.

Let now $\mathcal{S}$ be a hyperelliptic translation surface. Then $\mathcal{S}$ is obtained by identifying the opposite sides of a centrally symmetric polygon, P, whose centerpoint is the origin o. The points of $W(S)$ are represented by o, the midpoints of the sides of P, and (possibly) its vertices. In particular, $W(S)$ contains the cone points of $\mathcal{S}$, which, if any, come from the vertices. All of this is indicated in [Vch93b].

Let $\mathcal{T} = g \cdot \mathcal{S}$ be an equivalent translation surface. Set $Q = g \cdot P$. The polygon Q is centrally symmetric. But $\mathcal{T}$ is obtained by identifying the opposite sides of Q. Thus, $\mathcal{T}$ is a hyperelliptic translation surface, as well. In view of the preceding remarks, $g \in \mathrm{SL}(2, \mathbb{R})$ induces a bijection of $W(S)$ and $W(T)$.

Let now $\phi \in \mathrm{Aff}(\mathcal{S})$ be an elliptic diffeomorphism. Let $\mathcal{T} = g \cdot \mathcal{S}$ be such that the diffeomorphism $\psi = g \cdot \phi \cdot g^{-1} \in \mathrm{Aff}(\mathcal{T})$ is an isometry. It exists, by Lemma 7. Since ψ is a conformal automorphism of the Riemann surface T, it preserves the set $W(T)$. See say [FK80]. By remarks above, ϕ preserves $W(S)$.

We have now shown that $\mathrm{Aff}(\mathcal{S})$ preserves the set of Weierstrass points of $\mathcal{S}$. Since this set is finite, we obtain the claim. ∎

The following result is an immediate corollary of the preceding argument. In combination with Theorem 8, it shows that the Weierstrass points of the surfaces studied by Veech [Vch89] are, in fact, periodic.

Corollary 10. *Let $\mathcal{S}$ be a hyperelliptic translation surface. Suppose that the group Aff($\mathcal{S}$) admits a generating set consisting of elliptic elements, and of parabolic elements which preserve the set $W(S)$. Then the Weierstrass points of $\mathcal{S}$ are periodic.*

6.2. **Examples that prove Theorem 9.** Below we discuss a few examples that led to the present work. In particular, we show that these hyperelliptic translation surfaces admit their Weierstrass points as periodic points, thus proving Theorem 9.

Example 3: A Gnomon. Let P be the cross of translation $\lambda = (1+\sqrt{5})/2$, discussed in Lemma 2 of [HS01]. See also Figure 1 there. It yields a hyperelliptic translation surface $\mathcal{S}$, of genus 2. The six Weierstrass points of $\mathcal{S}$ come from the center, the exterior corners of the cross — these are identified to a single cone point — and from the midpoints of the opposite sides (identified in pairs). We will show that $\mathcal{S}$ has the property stated in Theorem 9.

By a cut-translate-and-paste operation, we put P into the form of an "L". The L-shaped polygons are often called "gnomons". In Figure 1 below the Weierstrass points of P are marked. The cone point is black, the remaining five points are marked by open circles: $O, A, \ldots, D$. The invariants of the vertical cylinders are: $w_1^v = 1$, $\ell_1^v = \lambda$, $w_2^v = \lambda - 1$ and $\ell_2^v = 1$. We use two coordinate systems in the unit square U inside P: The standard x, y-coordinates with the origin at the center of the square and the x', y'-coordinates, obtained by the rotation of these by $-\pi/4$. Note that any periodic point which is in U must have rational x, y-coordinates.

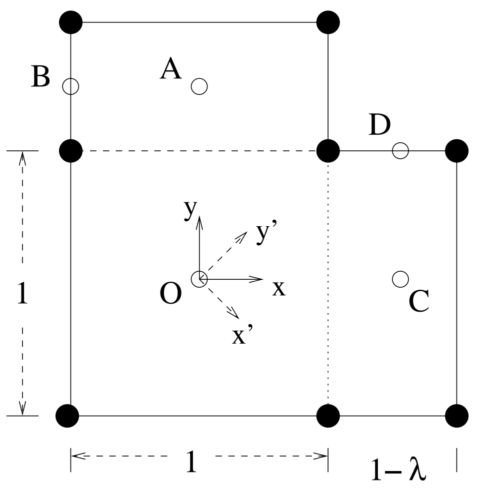

Figure 1. The gnomon, $\lambda = (1+\sqrt{5})/2$.

The two cylinders in the direction $\pi/4$ have the following parameters: $w_1^{\pi/4} = (\lambda - 1)/\sqrt{2}$, $\ell_1^{\pi/4} = (\lambda+1)\sqrt{2}$ and $w_2^{\pi/4} = (2-\lambda)/\sqrt{2}$, $\ell_2^{\pi/4} = \lambda\sqrt{2}$. The two cylinders in the direction $3\pi/4$ have the same parameters. See Figure 2 below.

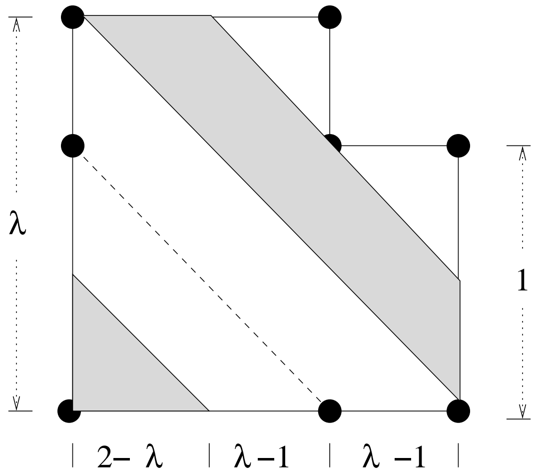

Figure 2. Cylinders for $\theta = 3\pi/4$.

Intersecting U with the cylinders of width $w = (\lambda - 1)/\sqrt{2}$ for both directions deletes from U the four corners. They are the images of the bottom left hand corner shaded in gray in Figure 2 under the standard symmetries of the square. Denote the intersection by I. Let $s = (x, y) \in I$ be a periodic point. We have $(x, y) = (\,(x' + y')/\sqrt{2},\, (-x' + y')/\sqrt{2}\,)$, hence $\sqrt{2}y' = x - y \in \mathbb{Q}$. Since s is periodic, y' is a rational multiple of the width of the cylinders meeting this region. Thus, $\sqrt{2}y' \in \mathbb{Q} \cap (\lambda - 1)\mathbb{Q}$, implying $y' = 0$. Analogous argument yields $x' = 0$.

Suppose now that $s = (x, y) \in U \setminus I$. By symmetry, it suffices to assume that s belongs to the bottom left corner of U. As before, x' is a rational multiple of the cylinder width, implying $x' = 0$. Analogous rationality considerations imply that $y' \in \sqrt{2}\mathbb{Q} \cap \frac{2-\lambda}{\sqrt{2}}\mathbb{Q}$, yielding $y' = 0$. Hence, the set $U \setminus I$ contains no periodic points. The only periodic point in U is its center point.

We will use the horizontal and vertical parabolic diffeomorphisms to push any periodic point s into U. Suppose that s is in the interior of the vertical cylinder of width 1 and length λ. Since the intersection of U with each closed geodesic is longer than half of its length, the orbit of s meets the unit square. Therefore, the only periodic points in the interior of this cylinder are the center of U and its image under the vertical parabolic. The horizontal Dehn twist maps the boundary of the vertical cylinder into the interior of the horizontal cylinder. By symmetry, we obtain the claim.

Example 4: An Octagon. Let P be the regular octagon, inscribed in the unit circle. Identifying the opposite sides of P by translations, we obtain a translation surface, $\mathcal{S}$. It is hyperelliptic, and has genus 2. By results of Veech [Vch89], $\mathcal{S}$ is a nonarithmetic lattice surface. See also [AH00] for more on this and related surfaces. As for the gnomon of the preceding example, the six Weierstrass points of $\mathcal{S}$ come from the center of P, the midpoints of the edges, and the vertices. We will show that $\mathcal{S}$ has the property stated in Theorem 9.

The coordinate directions form a parabolic pair. Each of them yields two cylinders. Rotating coordinates by $\pi/8$, we obtain another pair of parabolic directions, with two cylinders in each. In the notation of Figure 3, we have $w_1 = \sqrt{2}/2$, $w_2 = (2-\sqrt{2})/2$, and $w_{1'} = 2\sin\pi/8$, $w_{2'} = \cos\pi/8 - \sin\pi/8$.

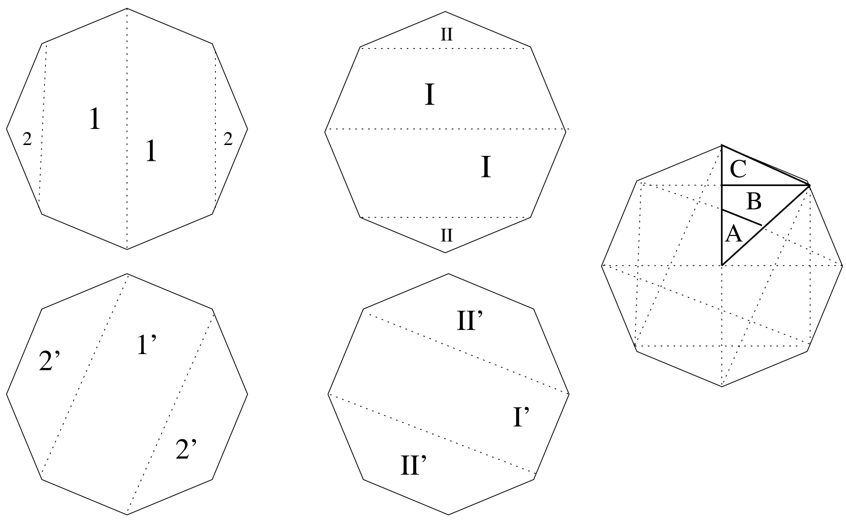

Figure 3. Cylinders for several directions; induced partition of triangle.

We will show that a periodic point in P is either its center, a midpoint of an edge, or a vertex. By symmetry, it suffices to show this for points of the triangle with vertices $0, e^{i\pi/4}, i$. Denote by A, B and C the parts of this triangle, obtained by intersecting it with the cylinders above. See Figure 3. The triangle A contains points in the cylinders labeled 1, I, $1'$ and I'. The quadrangle B contains points in the cylinders 1, I, $1'$ and II'. The triangle C contains points in the cylinders 1, II, $1'$ and II'.

We denote by x, y the standard coordinates centered at $z = 0$, and by x', y' for the same coordinates rotated by $-\pi/8$.

Let $s = (x, y) = (x', y') \in A$ be a periodic point. Then $x, y \in \sqrt{2}\mathbb{Q}$ and $x', y' \in (\sin\pi/8)\mathbb{Q}$. The relation between the two coordinate systems yields

$$x = x'\cos\pi/8 + y'\sin\pi/8,\ y = -x'\sin\pi/8 + y'\cos\pi/8. \tag{13}$$

From $x' = \frac{p}{q}\sin\pi/8$ and the double angle equation, we obtain $x'\cos\pi/8 \in \sqrt{2}\mathbb{Q}$. Hence $y'\sin\pi/8 \in \sqrt{2}\mathbb{Q}$. But $y' = \frac{u}{v}\sin\pi/8$, yielding $x' = y' = 0$. Thus, s is the center of P.

Suppose now that $s = (x, y) = (x', y') \in B \cup C$ is a periodic point. Then $x \in \sqrt{2}\mathbb{Q}$, $x' \in (\sin\pi/8)\mathbb{Q}$, and $y' \in \sin\pi/8 + (\cos\pi/8 - \sin\pi/8)\mathbb{Q}$. (Here,

$\lambda + \mu\mathbb{Q}$ denotes the set of reals of the form $\lambda + \mu a/b$ with rational a/b.) Exactly as above, we obtain $x' \cos \pi/8 \in \sqrt{2}\mathbb{Q}$. Therefore, equation (13) allows us to conclude that $y' \sin \pi/8 \in \sqrt{2}\mathbb{Q}$. Let $y' = \sin \pi/8 + (\cos \pi/8 - \sin \pi/8)\frac{u}{v}$. We have $(2-\sqrt{2})/4 + [\sqrt{2}/2 - (2-\sqrt{2})/4]\frac{u}{v} \in \sqrt{2}\mathbb{Q}$. It is easily seen that this membership implies that $\frac{u}{v}$ must equal 1. Therefore, we find $y' = \cos \pi/8$. Hence, s belongs to the edge of P. See Figure 3. The Dehn twist of the cylinder $1'$ sends the midpoint of the edge to the center and fixes the vertices. Other points of the edge go to the interior points of P. Those points are not periodic, by our preceding argument.

We have shown that the periodic points of $\mathcal{S}$ belong to the set of Weierstrass points. By the results of [Vch93b] or [AH00], the group $\mathrm{Aff}(\mathcal{S})$ satisfies the assumptions of Corollary 10. Thus, the opposite inclusion holds, as well. This proves our claim.

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