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# The eta invariant and the real connective K-theory of the classifying space for quaternion groups

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# THE ETA INVARIANT AND THE REAL CONNECTIVE $K$-THEORY OF THE CLASSIFYING SPACE FOR QUATERNION GROUPS

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Abstract. We express the real connective $K$ theory groups $\tilde{k}o_{4k-1}(BQ_\ell)$ of the quaternion group $Q_\ell$ of order $\ell = 2^j \geq 8$ in terms of the representation theory of $Q_\ell$ by showing $\tilde{k}o_{4k-1}(BQ_\ell) = \tilde{K}Sp(S^{4k+3}/\tau Q_\ell)$ where $\tau$ is any fixed point free representation of $Q_\ell$ in $U(2k+2)$.
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## 1. Introduction

A compact Riemannian manifold $(M, g)$ is said to be a *spherical space form* if $(M, g)$ has constant sectional curvature $+1$. A finite group $G$ is said to be a *spherical space form group* if there exists a representation $\tau : G \to U(k)$ for $k \geq 2$ which is fixed point free - i.e. $\det(I - \tau(\xi)) \neq 0 \ \forall \ \xi \in G - \{1\}$. Let

$$M^{2k-1}(G, \tau) := S^{2k-1}/\tau(G)$$

be the associated spherical space form; $G$ is then the fundamental group of the manifold $M^{2k-1}(G, \tau)$. Every odd dimensional spherical space form arises in this manner; the only even dimensional spherical space forms are the sphere $S^{2k}$ and real projective space $\mathbb{RP}^{2k}$. The spherical space form groups all have periodic cohomology; conversely, any group with periodic cohomology acts without fixed points on some sphere, although not necessarily orthogonally. We refer to [18] for further details concerning spherical space form groups.

Any cyclic group is a spherical space form group since the group of $\ell^{th}$ roots of unity acts without fixed points by complex multiplication on the unit sphere $S^{2k-1}$ in $\mathbb{C}^k$. Let $\mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, \mathcal{I}, \mathcal{J}, \mathcal{K}\}$ be the quaternions, let $\ell = 2^j \geq 8$, and let $\xi := e^{4\pi\mathcal{I}/\ell} \in \mathbb{H}$ be a primitive $(\frac{\ell}{2})^{th}$ root of unity. The quaternion group $Q_\ell$ is the subgroup of $\mathbb{H}$ of order $\ell$ generated by $\xi$ and $\mathcal{J}$:

$$Q_\ell := \{1, \xi, ..., \xi^{\ell/2-1}, \mathcal{J}, \xi\mathcal{J}, ..., \xi^{\ell/2-1}\mathcal{J}\}. \tag{1.1}$$

Let $BG$ be the classifying space of a finite group and let $ko_*(BG)$ be the associated real connective $K$ theory groups; we refer to [2, 3, 7, 9, 14] for a further discussion of connective $K$ theory and related matters.

The $p$ Sylow subgroup of a spherical space form group $G$ is cyclic if $p$ is odd and either cyclic or a quaternion group $Q_\ell$ for $\ell = 2^j \geq 8$ if $p = 2$. This focuses attention on these two groups. We showed previously in [4] that:

**Theorem 1.1.** *Let $\mathbb{Z}_\ell$ be the cyclic group of order $\ell = 2^j > 1$. Let $k \geq 1$. Let $\tau : \mathbb{Z}_\ell \to U(2k+2)$ be a fixed point free representation. Then*

$$\tilde{k}o_{4k-1}(B\mathbb{Z}_\ell) = \tilde{K}Sp(M^{4k+3}(\mathbb{Z}_\ell, \tau)).$$

In this paper, we generalize Theorem 1.1 to the quaternion group:

*Key words and phrases.* quaternion spherical space form, eta invariant, symplectic K theory, real connective K theory.

**Theorem 1.2.** *Let $Q_\ell$ be the quaternion group of order $\ell = 2^j \geq 3$. Let $k \geq 1$. Let $\tau : Q_\ell \to U(2k+2)$ be a fixed point free representation. Then*

$$\tilde{ko}_{4k-1}(BQ_\ell) = \tilde{K}Sp(M^{4k+3}(Q_\ell, \tau)).$$

The quaternion (symplectic) $K$ theory groups $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$ are expressible in terms of the representation theory - see Theorem 4.1. Thus Theorem 1.2 expresses $\tilde{ko}_{4k-1}(BQ_\ell)$ in terms of representation theory. If $\ell = 8$, then these groups were determined previously [3, 5].

Here is a brief outline to this paper. In Section 2, we review some facts concerning the representation theory of $Q_\ell$ which we shall need. In Section 3, we review some results concerning the eta invariant. In Section 4, we use the eta invariant to study $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$. In Section 5, we use the eta invariant to study $\tilde{ko}(BQ_\ell)$ and complete the proof of Theorem 1.2.

The proof of Theorem 1.2 is quite a bit different from the proof of Theorem 1.1 given previously; the extension is not straightforward. This arises from the fact that unlike the classifying space $B\mathbb{Z}_\ell$, the 2 localization of $BQ_\ell$ is not irreducible. Let $SL_2(\mathbb{F}_q)$ be the group of $2 \times 2$ matrices of determinant 1 over the field $\mathbb{F}_q$ with $q$ elements where $q$ is odd. Then the 2-Sylow subgroup of $SL_2(\mathbb{F}_q)$ is $Q_\ell$ for $\ell = 2^j$ where $j$ is the power of 2 dividing $q^2 - 1$. There is a stable 2-local splitting of the classifying space $BQ_\ell$ in the form

$$BQ_\ell = BSL_2(\mathbb{F}_q) \vee \Sigma^{-1} BS^3/BN \vee \Sigma^{-1} BS^3/BN \tag{1.2}$$

where $N$ is the normalizer of a maximal torus in $S^3$ [16, 15]. It is necessary to find a corresponding splitting of $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$ that mirrors this decomposition; see Remark 5.2.

## 2. The Representation Theory of $Q_\ell$

We say that $f : Q_\ell \to \mathbb{C}$ is a *class function* if $f(xgx^{-1}) = f(g)$ for all $x, g \in Q_\ell$; let $\text{Class}\,(Q_\ell)$ be the Hilbert space of all class functions with the $L^2$ inner product

$$\langle f_1, f_2 \rangle = \ell^{-1} \textstyle\sum_{g \in Q_\ell} f_1(g) \bar{f}_2(g).$$

Let $\text{Irr}\,(Q_\ell)$ be a set of representatives for the equivalence classes of irreducible unitary representations of $Q_\ell$. The *orthogonality relations* show that $\{\text{Tr}\,(\sigma)\}_{\sigma \in \text{Irr}\,(Q_\ell)}$ is an orthonormal basis for $\text{Class}\,(Q_\ell)$, i.e. we may expand any class function:

$$f = \textstyle\sum_{\sigma \in \text{Irr}\,(Q_\ell)} \langle f, \text{Tr}\,(\sigma) \rangle \text{Tr}\,(\sigma).$$

The *unitary group representation ring* $RU(Q_\ell)$ and the *augmentation ideal* $RU_0(Q_\ell)$ are defined by:

$$\begin{aligned} RU(Q_\ell) &= \text{Span}_{\mathbb{Z}}\{\sigma\}_{\sigma \in \text{Irr}\,(Q_\ell)}, \text{ and} \\ RU_0(Q_\ell) &= \{\sigma \in RU(Q_\ell) : \dim \sigma = 0\}. \end{aligned}$$

We shall identify a representation with the class function defined by its trace henceforth; a class function $f$ has the form $f = \text{Tr}\,(\tau)$ for some $\tau \in RU(Q_\ell)$ if and only if $\langle f, \sigma \rangle \in \mathbb{Z}$ for all $\sigma \in \text{Irr}\,(Q_\ell)$.

Let $RSp(Q_\ell)$ and $RO(Q_\ell)$ be the $\mathbb{Z}$ vector spaces generated by equivalence classes of irreducible quaternion and real representations, respectively. Forgetting the symplectic structure and complexification of a real structure define natural inclusions $RSp(Q_\ell) \subset RU(Q_\ell)$ and $RO(Q_\ell) \subset RU(Q_\ell)$. We have:

$$\begin{aligned} &RO(Q_\ell) \cdot RO(Q_\ell) \subset RO(Q_\ell), \\ &RSp(Q_\ell) \cdot RSp(Q_\ell) \subset RO(Q_\ell), \\ &RO(Q_\ell) \cdot RSp(Q_\ell) \subset RSp(Q_\ell). \end{aligned} \tag{2.1}$$

The $\frac{\ell}{4}+3$ conjugacy classes of $Q_\ell$ have representatives:

$$\{1,\ \xi,\ ...,\ \xi^{\ell/4}=-1,\ \mathcal{J},\ \xi\mathcal{J}\}.$$

There are $\frac{\ell}{4}+3$ irreducible inequivalent complex representations of $Q_\ell$. Four of these representations are the 1 dimensional representations defined by:

$$\begin{array}{llll}\rho_0(\xi)=1, & \kappa_1(\xi)=-1, & \kappa_2(\xi)=1, & \kappa_3(\xi)=-1,\\ \rho_0(\mathcal{J})=1, & \kappa_1(\mathcal{J})=1, & \kappa_2(\mathcal{J})=-1, & \kappa_3(\mathcal{J})=-1.\end{array}$$

We define representations $\gamma_u: Q_\ell \to U(2)$ by setting:

$$\gamma_u(\xi) = \left(\begin{array}{cc}\xi^u & 0\\ 0 & \xi^{-u}\end{array}\right), \quad \gamma_u(\mathcal{J}) = \left(\begin{array}{cc}0 & (-1)^u\\ 1 & 0\end{array}\right).$$

The representations $\gamma_u$, $\gamma_{-u}$, and $\gamma_{u+\frac{\ell}{2}}$ are all equivalent. The representations $\gamma_u$ are irreducible and inequivalent for $1\le u\le \frac{\ell}{4}-1$; $\gamma_0$ is equivalent to $\rho_0+\kappa_2$ and $\gamma_{\frac{\ell}{4}}$ is equivalent to $\kappa_1+\kappa_3$. We have:

$$\mathrm{Irr}\,(Q_\ell)=\{\rho_0,\kappa_1,\kappa_2,\kappa_3,\gamma_1,...,\gamma_{\frac{\ell}{4}-1}\}.$$

If $\vec{s}=(s_1,...,s_k)$ is a $k$ tuple of odd integers, then

$$\gamma_{\vec{s}}:=\gamma_{s_1}\oplus...\oplus\gamma_{s_k}$$

is a fixed point free representation from $Q_\ell$ to $U(2k)$; conversely, every fixed point free representation of $Q_\ell$ is conjugate to such a representation. The associated spherical space forms are the *quaternion spherical space forms.*

The representations $\{\rho_0,\kappa_1,\kappa_2,\kappa_3\}$ are real, the representations $\gamma_{2i}$ are real, and the representations $\gamma_{2i+1}$ are quaternion. We have:

$$\begin{array}{ll}RO(Q_\ell) &= \mathrm{span}_{\mathbb{Z}}\{\ \rho_0,\ \kappa_1,\ \kappa_2,\ \kappa_3, 2\gamma_1,\ \gamma_2,...,2\gamma_{\ell/4-1}\},\\ RSp(Q_\ell) &= \mathrm{span}_{\mathbb{Z}}\{2\rho_0,2\kappa_1,2\kappa_2,2\kappa_3,\ \gamma_1,2\gamma_2,...,\ \gamma_{\ell/4-1}\}.\end{array}$$

We define:

$$\Theta_1(g) := \left\{\begin{array}{ll}\frac{\ell}{4} & \text{if } g=\pm\mathcal{I},\\ -2 & \text{if } g=\xi^{2i}\mathcal{J},\\ 0 & \text{otherwise},\end{array}\right.$$

(2.2)

$$\Theta_2(g) := \left\{\begin{array}{ll}\frac{\ell}{4} & \text{if } g=\pm\mathcal{I},\\ -2 & \text{if } g=\xi^{2i+1}\mathcal{J},\\ 0 & \text{otherwise}.\end{array}\right.$$

The two class functions $\Theta_i$ will be used to mirror in $RU(Q_\ell)$ the splitting of $BQ_\ell$ given in equation (1.2).

We identify virtual representations with the class functions they define henceforth. Let

$$\Delta := 2\rho_0-\gamma_1; \qquad \mathrm{Tr}\,(\Delta)=\det(I-\gamma_1).$$

**Lemma 2.1.**

(1) *We have* $\Theta_1\in RO_0(Q_\ell)$ *and* $\Theta_2\in RO_0(Q_\ell)$.
(2) *Let* $c_i:=\ell^{-1}\sum_{g\in Q_\ell-\{1\}}\Delta(g)^i$. *We have* $c_0=\frac{\ell-1}{\ell}$. *If* $i>0$, *then* $c_{2i}\in\mathbb{Z}$ *and* $c_{2i-1}\in 2\mathbb{Z}$.

**Proof:** We use equation (2.2) to compute:

$$\begin{array}{llll}\text{for any } \ell & \langle\Theta_1,\rho_0\rangle=0, & \langle\Theta_1,\gamma_{2i+1}\rangle=0, & \langle\Theta_1,\gamma_{2i}\rangle=(-1)^i,\\ & \langle\Theta_2,\rho_0\rangle=0, & \langle\Theta_2,\gamma_{2i+1}\rangle=0, & \langle\Theta_2,\gamma_{2i}\rangle=(-1)^i,\\ \text{for } \ell=8 & \langle\Theta_1,\kappa_1\rangle=-1, & \langle\Theta_1,\kappa_2\rangle=1, & \langle\Theta_1,\kappa_3\rangle=0,\\ & \langle\Theta_2,\kappa_1\rangle=0, & \langle\Theta_2,\kappa_2\rangle=1, & \langle\Theta_2,\kappa_3\rangle=-1,\\ \text{for } \ell>8 & \langle\Theta_1,\kappa_1\rangle=0, & \langle\Theta_1,\kappa_2\rangle=1, & \langle\Theta_1,\kappa_3\rangle=1,\\ & \langle\Theta_2,\kappa_1\rangle=1, & \langle\Theta_2,\kappa_2\rangle=1, & \langle\Theta_2,\kappa_3\rangle=0.\end{array}$$

We use equation (2.1) to complete the proof of assertion (1):

$$\Theta_1 = \begin{cases} \mathrm{Tr}\{\kappa_2 - \kappa_1\} & \text{if } \ell = 8, \\ \mathrm{Tr}\{\kappa_2 + \kappa_3 + \sum_{1\le i<\ell/8}(-1)^i\gamma_{2i}\} & \text{if } \ell \ge 16, \end{cases}$$
$$\Theta_2 = \begin{cases} \mathrm{Tr}\{\kappa_2 - \kappa_3\} & \text{if } \ell = 8, \\ \mathrm{Tr}\{\kappa_2 + \kappa_1 + \sum_{1\le i<\ell/8}(-1)^i\gamma_{2i}\} & \text{if } \ell \ge 16. \end{cases}$$

The first identity of assertion (2) is immediate. Let $r > 0$. As $\mathrm{Tr}\,(\Delta^r)(1) = 0$,

$$c_r = \ell^{-1} \sum_{g\in Q_\ell - \{1\}} \mathrm{Tr}\,(\Delta^r)(g) = \langle \Delta^r, \rho_0\rangle \in \mathbb{Z}.$$

If $r$ is odd, then $\gamma_1^r$ is quaternion so $\langle \gamma_1^r, \rho_0\rangle \in 2\mathbb{Z}$. Since $\Delta^r \equiv \gamma_1^r \bmod 2RU(Q_\ell)$, $\langle \Delta^r, \rho_0\rangle \in 2\mathbb{Z}$ if $r$ is odd. $\square$

## 3. The eta invariant, $K$ theory, and bordism

Let $V$ be a smooth complex vector bundle over a compact Riemannian manifold $M$. Let $V$ be equipped with a unitary (Hermitian) inner product. Let

$$P : C^\infty(V) \to C^\infty(V)$$

be a self-adjoint elliptic first order partial differential operator. Let $\{\lambda_i\}$ denote the eigenvalues of $P$ repeated according to multiplicity. Let

$$\eta(s, P) := \sum_i \mathrm{sign}\,(\lambda_i)|\lambda_i|^{-s}.$$

The series defining $\eta$ converges absolutely for $\Re(s) >> 0$ to define a holomorphic function of $s$. This function has a meromorphic extension to the entire complex plane with isolated simple poles. The value $s = 0$ is regular and one defines

$$\eta(P) := \tfrac{1}{2}\{\eta(s, P) + \dim(\ker P)\}|_{s=0}$$

as a measure of the spectral asymmetry of $P$; we refer to [11] for further details concerning this invariant which was first introduced by [1] and which plays an important role in the index theorem for manifolds with boundary.

We say that $P$ is *quaternion* if $V$ has a quaternion structure and if the action of $P$ commutes with this structure. We say that $P$ is *real* if $V$ is the complexification of an underlying real vector bundle and if $P$ is the complexification of an underlying real operator.

**Lemma 3.1.** *Let $M$ be a spin manifold of dimension $m$.*
1. *If $m \equiv 3, 4$ mod 8, then the Dirac operator is quaternion.*
2. *If $m \equiv 7, 8$ mod 8, then the Dirac operator is real.*

**Proof:** Let $\mathrm{Clif}\,(m)$ be the real Clifford algebra on $\mathbb{R}^m$. We have:

$$\begin{aligned} &\mathrm{Clif}\,(3) = \mathbb{H} \oplus \mathbb{H}, \\ &\mathrm{Clif}\,(4) = M_2(\mathbb{H}), \\ &\mathrm{Clif}\,(7) = M_8(\mathbb{R}) \oplus M_8(\mathbb{R}), \\ &\mathrm{Clif}\,(8) = M_{16}(\mathbb{R}), \text{ and} \\ &\mathrm{Clif}\,(m+8) = \mathrm{Clif}\,(m) \otimes_{\mathbb{R}} M_{16}(\mathbb{R}). \end{aligned}$$

Therefore, the fundamental spinor representation of $\mathrm{Clif}\,(m)$ is quaternion if we have $m \equiv 3, 4$ mod 8 and real if we have $m \equiv 7, 8$ mod 8. The Lemma now follows. $\square$

The following deformation result will be crucial to our investigations:

**Lemma 3.2.** *Let $P_u$ be a smooth 1 parameter family of self-adjoint first order elliptic partial differential operators on a compact manifold $M$.*
1. *The reduction mod $\mathbb{Z}$ of $\eta(P_u)$ is a smooth $\mathbb{R}/\mathbb{Z}$ valued function.*
2. *The variation $\partial_u \eta(P_u)$ is locally computable.*
3. *If the operators $P_u$ are quaternion, then the reduction mod $2\mathbb{Z}$ of $\eta(P_u)$ is a smooth $R/2\mathbb{Z}$ valued function.*

**Proof:** We sketch the proof briefly and refer to [11] Theorem 1.13.2 for further details. Since $\frac{1}{2}\mathrm{sign}\,(u)$ has an integer jump when $u=0$, $\eta(P_u)$ can have integer valued jumps at values of $u$ where $\dim(\ker(P_u))>0$. However, in $\mathbb{R}/\mathbb{Z}$, the jump disapears so the mod $\mathbb{Z}$ reduction of $\eta(P_u)$ is a smooth $\mathbb{R}/\mathbb{Z}$ valued function of $u$; one uses the pseudo-differential calculus to construct an approximate resolvant and to show that the variation $\partial_u\eta(P_u)$ is locally computable. Assertions (1) and (2) then follow. If $P_u$ is quaternion, then the eigenspaces of $P_u$ inherit quaternion structures. Thus $\dim(\ker P_u)$ is even so $\eta(P_u)$ has twice integer jumps as eigenvalues cross the origin. Consequently the reduction mod $2\mathbb{Z}$ of $\eta(P_u)$ is smooth and assertion (3) follows. □

Let $\tilde{M}$ be the universal cover of a connected manifold $M$ and let $\sigma$ be a representation of $\pi_1(M)$ in $U(k)$. The associated vector bundle is defined by:

$$\begin{aligned}&V^\sigma := \tilde{M}\times\mathbb{C}^k/\sim \text{ where we identify}\\&(\tilde{x},z)\sim(g\cdot\tilde{x},\sigma(g)\cdot z) \text{ for } g\in\pi_1(M),\ \tilde{x}\in\tilde{M}, \text{ and } z\in\mathbb{C}^k.\end{aligned}$$

The trivial connection on $\tilde{M}\times\mathbb{C}^k$ descends to define a flat connection on $V^\sigma$. The transition functions of $V^\sigma$ are locally constant; they are given by the representation $\sigma$. Thus the bundle $V^\sigma$ is said to be *locally flat*. Let $P: C^\infty(V)\to C^\infty(V)$ be a self-adjoint elliptic first order operator on $M$;

$$P^\sigma : C^\infty(V\otimes V^\sigma)\to C^\infty(V\otimes V^\sigma)$$

is a well defined operator which is locally isomorphic to $k$ copies of $P$. Define $\eta^\sigma(P):=\eta(P^\sigma)$; we extend by linearity to $\sigma\in RU(\pi_1(M))$.

This invariant is a homotopy invariant.

**Lemma 3.3.** *Let $P_u$ be a smooth 1 parameter family of elliptic first order self-adjoint partial differential operators over $M$.*

1. *If $\sigma\in RU_0(\pi_1(M))$, then the mod $\mathbb{Z}$ reduction of $\eta^\sigma(P_u)$ is independent of the parameter $u$.*
2. *If all the operators $P_u$ are quaternion and $\sigma\in RO_0(\pi_1(M))$ or if all the operators $P_u$ are real and $\sigma\in RSp_0(\pi_1(M))$, then the mod $2\mathbb{Z}$ reduction of $\eta(P_u,\sigma)$ is independent of the parameter $u$.*

**Proof:** If $\sigma$ is a representation of $\pi_1(M)$, then the mod $\mathbb{Z}$ reduction of $\eta^\sigma(P_u)$ is smooth a smooth function of $u$ by Lemma 3.2. Since $P_u^\sigma$ is locally isomorphic to $\dim\sigma$ copies of $P_u$ and since the variation is locally computable,

$$\partial_u\eta^\sigma(P_u)=\dim\sigma\cdot\partial_u\eta(P_u).$$

This formula continues to hold for virtual representations. In particular, if we have that $\sigma\in RU_0(\pi_1(M))$, then $\dim\sigma=0$ so $\partial_u\eta^\sigma(P_u)=0$; (1) follows.

If $P_u$ is quaternion and $\sigma$ is real or if $P_u$ is real and if $\sigma$ is quaternion, then $P_u^\sigma$ is quaternion and $\eta^\sigma(P_u)$ is a smooth $\mathbb{R}/2\mathbb{Z}$ valued function of $u$. The same argument shows that $\partial_u\eta^\sigma(P_u)=0$. □

We can use the eta invariant to construct invariants of $K$ theory. Let $P: C^\infty(V)\to C^\infty(V)$ be a first order self-adjoint elliptic partial differential operator with leading symbol $p$. Let $W$ be a unitary vector bundle over $M$. We use a partition of unity to construct a self-adjoint elliptic first order operator $P^W$ on $C^\infty(V\otimes W)$ with leading symbol $p\otimes\mathrm{id}$; this operator is not, of course, cannonically defined.

We can extend the invariant $\eta^\sigma$ to the the reduced unitary unitary and quaternion (symplectic) $K$ theory groups $\tilde{K}U$ and $\tilde{K}Sp$:

**Theorem 3.4.** *Let $P$ be an elliptic self-adjoint first order partial differential operator. Let $\sigma\in RU_0(\pi_1(M))$.*

(1) *The map* $W \to \eta^\sigma(P^W)$ *extends to a map* $\eta^\sigma_P : \tilde{K}U(M) \to \mathbb{R}/\mathbb{Z}$.
(2) *Suppose that* $P$ *and* $\sigma$ *are both real or that* $P$ *and* $\sigma$ *are both quaternion. The map* $W \to \eta^\sigma(P^W)$ *extends to a map*

$$\eta^\sigma_P : \tilde{K}Sp(M) \to \mathbb{R}/2\mathbb{Z}.$$

**Proof:** Let $P^W$ and $\tilde{P}^W$ be two first order self-adjoint partial differential operators on $C^\infty(V \otimes W)$ with leading symbol $p \otimes \mathrm{id}$. Set:

$$P_u := uP^W + (1-u)\tilde{P}^W.$$

This is a smooth 1 parameter family of first order self-adjoint partial differential operators. As the leading symbol of $P_u$ is $p \otimes \mathrm{id}$, the operators $P_u$ are elliptic. By Lemma 3.3, $\eta^\sigma(P_u) \in \mathbb{R}/\mathbb{Z}$ is independent of $u$. Consequently $\eta^\sigma_P(W) := \eta^\sigma(P^W) \in \mathbb{R}/\mathbb{Z}$ only depends on the isomorphism class of the bundle $W$. As the eta invariant is additive with respect to direct sums, we may extend $\eta^\sigma_P$ to $\tilde{K}U(M)$ as an $\mathbb{R}/\mathbb{Z}$ valued invariant. Let $W$ be quaternion. By Lemma 3.3, $\eta^\sigma(P_u) \in \mathbb{R}/2\mathbb{Z}$ is independent of $u$ if both $P$ and $\sigma$ are real or if both $P$ and $\sigma$ are quaternion and thus $\eta^\sigma$ extends to $\tilde{K}Sp$ as an $\mathbb{R}/2\mathbb{Z}$ valued invariant in this instance. □

We can use the Atiyah-Patodi-Singer index theorem [1] to see that the eta invariant also defines bordism invariants. Let $G$ be a finite group. A $G$ structure $f$ on a connected manifold $M$ is a representation $f$ from $\pi_1(M)$ to $G$. Equivalently, $f$ can also be regarded as a map from $M$ to the classifying space $BG$. We consider tuples $(M, g, s, f)$ where $(M, g)$ is a compact Riemannian manifold with a spin structure $s$ and a $G$ structure $f$. We introduce the bordism relation $[(M, g, s, f)] = 0$ if there exists a compact manifold $N$ with boundary $M$ so that the structures $(g, s, f)$ extend over $N$; this induces an equivalence relation and the equivariant bordism groups $\mathrm{MSpin}_m(BG)$ consists of bordism classes of these triples. Disjoint union defines the group structure.

Let $\mathrm{MSpin}_* := \mathrm{MSpin}_*(B\{1\})$ be defined by the trivial group. Cartesian product makes $\mathrm{MSpin}_*(BG)$ into an $\mathrm{MSpin}_*$ module. Let $\mathcal{F}$ be the forgetful homomorphism which forgets the $G$ structure $f$. The reduced bordism groups are then defined by:

$$\tilde{M}Spin_*(BG) := \ker(\mathcal{F}) : \mathrm{MSpin}_*(BG) \to \mathrm{MSpin}_*.$$

Since the eta invariant vanishes on $\mathrm{MSpin}_*$, we restrict henceforth to the reduced groups.

If $s$ is a spin structure on $(M, g)$, let $P_{(M,g,s)}$ be the associated Dirac operator. If $\sigma \in RU_0(G)$, then $f^*\sigma \in RU_0(\pi_1(M))$ and we may define:

$$\eta^\sigma(M, g, s, f) := \eta^{f^*\sigma}(P_{(M,g,s)}).$$

**Theorem 3.5.** *Let* $G$ *be a finite group. Assume either that* $m \equiv 3$ *mod* 8 *and that* $\sigma \in RO_0(G)$ *or that* $m \equiv 7$ *mod* 8 *and that* $\sigma \in RSp_0(G)$. *Then the map* $(M, g, s, f) \to \eta^\sigma(M, s, f)$ *extends to a map*

$$\eta^\sigma : \tilde{M}Spin_m(BG) \to \mathbb{R}/2\mathbb{Z}.$$

**Proof:** We sketch the proof and refer to [6] for further details. Suppose that $m \equiv 3$ mod 4 and that $[(M, g, s, f)] = 0$ in $\mathrm{MSpin}_m(BG)$. Then $M = dN$ where the spin and $G$ structures on $M$ extend over $N$. We may also extend the given Riemannian metric on $M$ to a Riemannian metric on $N$ which is product near the boundary.

Let $\sigma \in RU_0(G)$. The Dirac operator $P_{(M,g,s)}$ on $M$ is the tangential operator of the spin complex $Q_{(N,g,s)}$ on $N$. We twist these operators by taking coefficients in the locally flat virtual bundle $V^{f^*\sigma}$.

Let $\hat{A}(N, g, s)$ be the $A$-roof genus and let $ch(V^{f^*\sigma})$ be the Chern character. By the Atiyah-Patodi-Singer index theorem [1]:

$$\mathrm{index}\,(Q^{f^*\sigma}_{(N,g,s)}) = \textstyle\int_N \hat{A}(N, g, s) \wedge ch(V^{f^*\sigma}) + \eta(P^\sigma_{(M,g,s)}).$$

Since $V^{f^*\sigma}$ is a virtual bundle of virtual dimension 0 which admits a flat connection, the Chern character of $V^\sigma$ vanishes. Consequently:

$$\eta^\sigma(M,g,s,f)=\eta(P^{f^*\sigma}_{(M,g,s)})=\text{index}\,(Q^{f^*\sigma}_{N,g,s}).$$

The dimension of $N$ is $m+1$. We apply Lemma 3.1 to see that if $m\equiv 3$ mod 8 and if $\sigma$ is real or if $m\equiv 7$ mod 8 and if $\sigma$ is quaternion, then $Q^{f^*\sigma}_{(N,s,f)}$ is quaternion. Thus index $(Q^{f^*\sigma}_{(N,s,f)})\in 2\mathbb{Z}$ so $\eta^\sigma(M,g,s)$ vanishes as an $\mathbb{R}/2\mathbb{Z}$ valued invariant if $[(M,g,s,f)]=0$ in $\text{MSpin}_m(BG)$. $\square$

There is a geometric description of the real connective $K$ theory groups $\tilde{k}o_m(BG)$ in terms of the spin bordism groups. Let $\mathbb{HP}^2$ be the quaternionic projective plane. Let $\tilde T_m(BG)$ be the subgroup of $\tilde MSpin_m(BG)$ consisting of bordism classes $[(E,g,s,f)]$ where $E$ is the total space of a geometrical $\mathbb{HP}^2$ spin fibration and where the $G$ structure on $E$ is induced from a corresponding $G$ structure on the base. The following theorem is a special case of a more general result [17]:

**Theorem 3.6.** *Let $G$ be a finite group. There is a 2 local isomorphism between $\tilde{k}o_m(BG)$ and $\tilde MSpin_m(BG)/\tilde T_m(BG)$.*

We use Theorem 3.6 to draw the following consequence:

**Corollary 3.7.** *Assume either that $m\equiv 3$ mod 8 and $\sigma\in RO_0(Q_\ell)$ or that $m\equiv 7$ mod 8 and $\sigma\in RSp_0(Q_\ell)$. Then $\eta^\sigma$ extends to a map from $\tilde{k}o_m(BQ_\ell)$ to $\mathbb{Q}/2\mathbb{Z}$.*

**Proof:** If $[(E,s,f)]\in T_m(BQ_\ell)$, then $\eta^\sigma(P_{(E,g,s)})=0$; see [6] Lemma 4.3 or [13] Lemma 2.7.10 for details. Thus by Theorems 3.5 and Theorem 3.6, the eta invariant extends to $\tilde{k}o(BQ_\ell)$. By [6] Theorem 2.4, $\tilde{k}o_{4k-1}(BQ_\ell)$ is a finite 2 group. Thus it is not necessary to localize at the prime 2 and the eta invariant takes values in $\mathbb{Q}/2\mathbb{Z}$. $\square$

The eta invariant is combinatorially computable for spherical space forms. The following theorem follows from [8].

**Theorem 3.8.** *Let $\tau:G\to SU(2k)$ be fixed point free, let $P$ be the Dirac operator on $M^{4k-1}(G,\tau)$, and let $\sigma\in RU_0(G)$. Then*

$$\eta^\sigma(P)=\ell^{-1}\textstyle\sum_{g\in G-\{1\}}\text{Tr}\,(\sigma(g))\det(I-\tau(g))^{-1}.$$

## 4. The groups $\tilde KSp(M^{4\nu-1}(Q_\ell,\nu\cdot\gamma_1))$

Let $\Delta=\det(I-\gamma_1)\in RSp_0(Q_\ell)$. By equation (2.1):

$$\begin{array}{ll}\Delta^\nu RSp(Q_\ell)\subset RSp_0(Q_\ell) & \text{if }\nu\text{ is even},\\ \Delta^\nu RO(Q_\ell)\subset RSp_0(Q_\ell) & \text{if }\nu\text{ is odd}.\end{array}$$

The following Theorem is well known - see, for example [10, 12]:

**Theorem 4.1.** *Let $\tau:Q_\ell\to U(2\nu)$ be fixed point free. Then*

$$\tilde KSp(M^{4\nu-1}(Q_\ell,\tau))=\begin{cases}RSp_0(Q_\ell)/\Delta^\nu RSp(Q_\ell) & \textit{if }\nu\textit{ is even},\\ RSp_0(Q_\ell)/\Delta^\nu RO(Q_\ell) & \textit{if }\nu\textit{ is odd}.\end{cases}$$

By Theorem 4.1, the particular representation $\tau$ plays no role and we therefore set $\tau=\nu\cdot\gamma_1$. We use the eta invariant to study these groups. Let $\eta^\sigma_\nu(W)$ be the invariant described in Theorem 3.4 for the Dirac operator $P$ on $M^{4\nu-1}(Q_\ell,\nu\cdot\gamma_1)$. We define:

$$\vec\eta_\nu(W):=\begin{cases}(\eta_\nu^{\Theta_1},\eta_\nu^{\Theta_2},\eta_\nu^{2\Delta},\eta_\nu^{\Delta^2},...,\eta_\nu^{\Delta^{\nu-2}},\eta_\nu^{2\Delta^{\nu-1}})(W) & \text{if }\nu\text{ is even},\\ (\eta_\nu^{2\Theta_1},\eta_\nu^{2\Theta_2},\eta_\nu^{\Delta},\eta_\nu^{2\Delta^2},...,\eta_\nu^{\Delta^{\nu-2}},\eta_\nu^{2\Delta^{\nu-1}})(W) & \text{if }\nu\text{ is odd}.\end{cases}$$

**Lemma 4.2.** *Let $M:=M^{4\nu-1}(Q_\ell,\nu\cdot\gamma_1)$. Then*

$$\vec\eta_\nu:\tilde KSp(M)\to(\mathbb{Q}/2\mathbb{Z})^{\nu+1}.$$

**Proof:** We apply Lemma 3.1 and Theorem 3.4. We distinguish two cases:

(1) If $\nu$ is even, then $P$ is real. Thus $\eta_\nu^\sigma : \tilde{K}Sp(M) \to \mathbb{Q}/2\mathbb{Z}$ for real $\sigma$ and the Lemma follows as we have used the real representations $\{\Theta_1, \Theta_2, 2\Delta, \Delta^2, ..., \Delta^{\nu-2}, 2\Delta^{\nu-1}\}$ to define $\vec{\eta}_\nu$.

(2) If $\nu$ is is odd, then $P$ is quaternion. Thus $\eta_\nu^\sigma : \tilde{K}Sp(M) \to \mathbb{Q}/2\mathbb{Z}$ if $\sigma$ is quaternion and the Lemma follows as we have used the quaternion representations $\{2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, ..., \Delta^{\nu-2}, 2\Delta^{\nu-1}\}$ to define $\vec{\eta}_\nu$. □

Let $\varepsilon_{2i} = 2$ and $\varepsilon_{2i-1} = 1$; $\{2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, ..., \varepsilon_{\nu-1}\Delta^{\nu-1}\}$ are quaternion. In Lemma 2.1, we defined constants

$$c_i := \ell^{-1} \textstyle\sum_{g \in Q_\ell - \{1\}} \det(I - \gamma_1(g))^i.$$

Since $\Delta(g) = \det(I - \gamma_1(g))$, we use Theorem 3.8 to compute:

$$\eta_\nu^{\Delta^r}(\Delta^s) = \ell^{-1} \textstyle\sum_{g \in Q_\ell - \{1\}} \Delta(g)^{r+s} \Delta(g)^{-\nu} = c_{r+s-\nu}. \tag{4.1}$$

Since $\Theta_1$ and $\Theta_2$ are supported on the elements of order 4 in $Q_\ell$ and since $\Delta(g) = 2$ for such an element, we may use Theorem 3.8 and equation (2.2) to see:

$$\begin{aligned}
\eta_\nu^{\Delta^r}(\Theta_i) &= \eta_\nu^{\Theta_i}(\Delta^r) = \ell^{-1} \textstyle\sum_{g \in Q_\ell - \{1\}} 2^r \mathrm{Tr}\,(\Theta_i(g)) 2^{-\nu} \\
&= \ell^{-1} 2^{r-\nu} \textstyle\sum_{g \in Q_\ell - \{1\}} \mathrm{Tr}\,(\Theta_i(g)) = 0, \\
\eta_\nu^{\Theta_1}(\Theta_1) &= \eta_\nu^{\Theta_2}(\Theta_2) = \ell^{-1} 2^{-\nu} \textstyle\sum_{g \in Q_\ell - \{1\}} \mathrm{Tr}\,(\Theta_1(g))^2 \\
&= \ell^{-1} 2^{-\nu} \{2 \cdot \tfrac{\ell^2}{16} + 4 \cdot \tfrac{\ell}{4}\}, \\
\eta_\nu^{\Theta_1}(\Theta_2) &= \eta_\nu^{\Theta_2}(\Theta_1) = \ell^{-1} 2^{-\nu} \textstyle\sum_{g \in Q_\ell - \{1\}} \mathrm{Tr}\,(\Theta_1(g)) \mathrm{Tr}\,(\Theta_2(g)) \\
&= \ell^{-1} 2^{-\nu} \{2 \cdot \tfrac{\ell^2}{16}\}.
\end{aligned} \tag{4.2}$$

We have $\ell = 2^j$. We use equation (4.1), equation (4.2), and Lemma 2.1 to see:

$$\vec{\eta}_\nu \begin{pmatrix} 2\Theta_1 \\ 2\Theta_2 \\ \Delta \\ 2\Delta^2 \\ ... \\ \varepsilon_{\nu-1}\Delta^{\nu-1} \end{pmatrix} = \begin{pmatrix} A_\nu & 0 \\ 0 & B_\nu \end{pmatrix} \in M_{\nu+1}(\mathbb{Q}/2\mathbb{Z})$$

where $A$ is the $2 \times 2$ matrix given by

$$A_\nu = 2^{1-\nu} \begin{pmatrix} 2^{j-3}+1 & 2^{j-3} \\ 2^{j-3} & 2^{j-3}+1 \end{pmatrix} \text{ if } \nu \text{ is even}$$

$$A_\nu = 2^{2-\nu} \begin{pmatrix} 2^{j-3}+1 & 2^{j-3} \\ 2^{j-3} & 2^{j-3}+1 \end{pmatrix} \text{ if } \nu \text{ is odd}$$

and where $B$ is the $\nu - 1 \times \nu - 1$ matrix given by:

$$B_\nu = \begin{pmatrix}
2c_{2-\nu} & c_{3-\nu} & 2c_{4-\nu} & ... & 2c_{-2} & c_{-1} & 2c_0 \\
4c_{3-\nu} & 2c_{4-\nu} & 4c_{5-\nu} & ... & 4c_{-1} & 2c_0 & 0 \\
2c_{4-\nu} & c_{5-\nu} & 2c_{6-\nu} & ... & 2c_0 & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... \\
2c_{-2} & c_{-1} & 2c_0 & ... & 0 & 0 & 0 \\
4c_{-1} & 2c_0 & 0 & ... & 0 & 0 & 0 \\
2c_0 & 0 & 0 & ... & 0 & 0 & 0
\end{pmatrix} \text{ if } \nu \text{ is even}$$

$$B_\nu = \begin{pmatrix} c_{2-\nu} & 2c_{3-\nu} & c_{4-\nu} & \dots & 2c_{-2} & c_{-1} & 2c_0 \\ 2c_{3-\nu} & 4c_{4-\nu} & 2c_{5-\nu} & \dots & 4c_{-1} & 2c_0 & 0 \\ c_{4-\nu} & 2c_{5-\nu} & c_{6-\nu} & \dots & 2c_0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2c_{-2} & 4c_{-1} & 2c_0 & \dots & 0 & 0 & 0 \\ c_{-1} & 2c_0 & 0 & \dots & 0 & 0 & 0 \\ 2c_0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \text{ if } \nu \text{ is odd.}$$

**Theorem 4.3.** *Let $\mathcal{B}_\nu$ be the subgroup of $(\mathbb{Q}/2\mathbb{Z})^{\nu-1}$ spanned by the rows of the matrix $B_\nu$ defined above. Let $M = M^{4\nu-1}(Q_\ell, \nu\cdot\gamma_1)$. Then*

$$\tilde{K}Sp(M) = \begin{cases} \mathbb{Z}_{2^\nu} \oplus \mathbb{Z}_{2^\nu} \oplus \mathcal{B}_\nu & \text{if } \nu \text{ is even,} \\ \mathbb{Z}_{2^{\nu-1}} \oplus \mathbb{Z}_{2^{\nu-1}} \oplus \mathcal{B}_\nu & \text{if } \nu \text{ is odd.} \end{cases}$$

**Proof:** Let $\mathcal{K}_\nu$ be the subspace of $\tilde{K}Sp(M)$ spanned by the virtual vector bundles defined by $\{2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, ..., \varepsilon_{\nu-1}\Delta^{\nu-1}\}$. It is then immediate from the definition and from the form of the matrix $A_\nu$ that

$$\vec{\eta}_\nu(\mathcal{K}_\nu) = \begin{cases} \mathbb{Z}_{2^\nu} \oplus \mathbb{Z}_{2^\nu} \oplus \mathcal{B}_\nu & \text{if } \nu \text{ is even,} \\ \mathbb{Z}_{2^{\nu-1}} \oplus \mathbb{Z}_{2^{\nu-1}} \oplus \mathcal{B}_\nu & \text{if } \nu \text{ is odd.} \end{cases} \tag{4.3}$$

We use Lemma 2.1 to see $c_0 = \frac{\ell-1}{\ell}$. Thus $2c_0$ is an element of order $\ell$ in $\mathbb{Q}/2\mathbb{Z}$. We use the diagonal nature of matrix $B_\nu$ to see that:

$$|\vec{\eta}_\nu(\mathcal{K}_\nu)| \geq \begin{cases} 4^\nu \ell^{\nu-1} & \text{if } \nu \text{ is even,} \\ 4^{\nu-1}\ell^{\nu-1} & \text{if } \nu \text{ is odd.} \end{cases} \tag{4.4}$$

The $E_2$ term in the Atiyah-Hirezbruch spectral sequence for the $K$ theory groups $\tilde{K}Sp^*(M)$ is

$$\oplus_{u+v=w}\tilde{H}^u(M; \mathrm{KSp}^v(pt)).$$

We take $w = 0$ and study the reduced groups to obtain the estimate:

$$|\tilde{K}Sp(M)| \leq |\oplus_{u+v=0}\tilde{H}^u(M; \mathrm{KSp}^v(pt))|. \tag{4.5}$$

We have that:

$$\begin{array}{rclcl} \mathrm{KSp}^v(pt) & = & \mathbb{Z} & \text{if } v \equiv 0, 4 \bmod 8, \\ \mathrm{KSp}^v(pt) & = & \mathbb{Z}_2 & \text{if } v \equiv -5, -6 \bmod 8, \\ \mathrm{KSp}^v(pt) & = & 0 & \text{otherwise,} \\ \tilde{H}^u(M;\mathbb{Z}) & = & \mathbb{Z}_\ell & \text{if } u \equiv 0, 4 \bmod 8,\ u < 4\nu - 1, \\ \tilde{H}^u(M;\mathbb{Z}_2) & = & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } u \equiv 1, 2, 5, 6 \bmod 8,\ u \leq 4\nu - 1. \end{array} \tag{4.6}$$

Equations (4.5) and (4.6) then imply:

$$|\tilde{K}Sp(M)| \leq \begin{cases} 4^\nu \ell^{\nu-1} & \text{if } \nu \text{ is even} \\ 4^{\nu-1}\ell^{\nu-1} & \text{if } \nu \text{ is odd.} \end{cases} \tag{4.7}$$

Thus equations (4.4) and (4.7) show $|\tilde{K}Sp(M)| \leq |\vec{\eta}_\nu(\mathcal{K}_\nu)|$. As the opposite inequality is immediate, we have

$$\vec{\eta}_\nu(\mathcal{K}_\nu) = \mathcal{K}_\nu = \tilde{K}Sp(M).$$

The Theorem now follows from equation (4.3). □

## 5. The groups $\tilde{k}o_{4k-1}(BQ_\ell)$

Let $x = (M, g, s, f)$ where $s$ is a spin structure and $f$ is a $G$ structure on a compact Riemannian manifold $(M, g)$ of dimension $4k - 1$. Let $\eta^\sigma(x)$ be the eta invariant of the associated Dirac operator with coefficients in $f^*\sigma$. We reverse the parities of the invariant defined in the previous section to define:

$$\vec{\eta}_k(x) := \begin{cases} (\eta^{2\Theta_1}(x), \eta^{2\Theta_2}(x), \eta^{\Delta}(x), \eta^{2\Delta^2}(x) ..., \eta^{2\Delta^k}(x)) & (k \text{ even}) \\ (\eta^{\ \Theta_1}(x), \eta^{\ \Theta_2}(x), \eta^{2\Delta}(x), \eta^{\Delta^2}(x) ..., \eta^{2\Delta^k}(x)) & (k \text{ odd}). \end{cases}$$

We have used real representations if $k$ is odd and quaternion representations if $k$ is even. Therefore, by Corollary 3.7, $\vec{\eta}_k$ extends to:

$$\vec{\eta}_k : \tilde{ko}_{4k-1}(BG) \to (\mathbb{Q}/2\mathbb{Z})^{k+2}.$$

The group $Q_\ell$ has 3 non-conjugate elements of order 4: $\{\mathcal{I}, \mathcal{J}, \xi\mathcal{J}\}$ which generate the 3 non-conjugate subgroups $\{\langle\mathcal{I}\rangle, \langle\mathcal{J}\rangle, \langle\xi\mathcal{J}\rangle\}$ of order 4. The representation $\gamma_1$ restricts to a fixed point free representation of any subgroup of $Q_\ell$. We define the following spherical space forms:

$$\begin{array}{ll} M_Q^{4k-1} := M^{4k-1}(Q_\ell, k\gamma_1), & M_{\mathcal{I}}^{4k-1} := M^{4k-1}(\langle\mathcal{I}\rangle, k\gamma_1) \\ M_{\mathcal{J}}^{4k-1} := M^{4k-1}(\langle\mathcal{J}\rangle, k\gamma_1), & M_{\xi\mathcal{J}}^{4k-1} := M^{4k-1}(\langle\xi\mathcal{J}\rangle, k\gamma_1). \end{array}$$

Give the lens spaces $M_g^{4k-1}$ the $Q_\ell$ structure induced by the natural inclusion $\langle g\rangle \subset Q_\ell$. We project into the reduced group $\tilde{MSpin}_{4k-1}(Q_\ell)$; this does not affect the eta invariant as $\eta^\sigma(\mathrm{MSpin}_*(pt)) = 0$. Let $i > 0$. By Theorem 3.8:

$$(\eta^{\Theta_1}, \eta^{\Theta_2}, \eta^{\Delta^i})(M_{\mathcal{I}}^{4k-1} - M_{\mathcal{J}}^{4k-1}) = \left\{ \begin{array}{ll} 2^{-k}(2,\ 1,\ 0) & \text{if } \ell = 8, \\ 2^{-k}(1,\ 0,\ 0) & \text{if } \ell > 8, \end{array} \right.$$

$$(\eta^{\Theta_1}, \eta^{\Theta_2}, \eta^{\Delta^i})(M_{\mathcal{I}}^{4k-1} - M_{\xi\mathcal{J}}^{4k-1}) = \left\{ \begin{array}{ll} 2^{-k}(1,\ 2,\ 0) & \text{if } \ell = 8, \\ 2^{-k}(0,\ 1,\ 0) & \text{if } \ell > 8, \end{array} \right.$$

$$(\eta^{\Theta_1}, \eta^{\Theta_2}, \eta^{\Delta^i})(M_Q^{4k-1}) = (0,\ 0,\ c_{i-k}) \qquad \text{any } \ell.$$

Let $K^4$ be a spin manifold with $\hat{A}(K^4) = 2$ and let $B^8$ be a spin manifold with $\hat{A}(B^8) = 1$. Let $Z^{8k-4} := K^4 \times B^{8k-8}$ and $Z^{8k} = (B^8)^k$. Standard product formulas [10] then show

$$\eta^\sigma(M^{4k-1} \times Z^{4j}) = \eta^\sigma(M^{4k-1})\hat{A}(Z^{4j}) = \left\{ \begin{array}{ll} 2\eta^\sigma(M^{4k-1}) & \text{if } j \text{ is odd}, \\ \eta^\sigma(M^{4k-1}) & \text{if } j \text{ is even}. \end{array} \right.$$

Let $B_\nu$ and $\mathcal{B}_\nu$ be as defined in Section 4. There is a dimension shift involved as we must set $\nu = k+1$. We use the same arguments as those given previously to see

$$\vec{\eta}_k \left( \begin{array}{l} M_{\mathcal{I}}^{4k-1} - M_{\mathcal{J}}^{4k-1} \\ M_{\mathcal{I}}^{4k-1} - M_{\xi\mathcal{J}}^{4k-1} \\ M_Q^{4k-1} \\ M_Q^{4k-5} \times Z^4 \\ \dots \\ M_Q^3 \times Z^{4k-4} \end{array} \right) = \left( \begin{array}{ll} C_k & 0 \\ 0 & B_{k+1} \end{array} \right) \in M_{k+2}(\mathbb{Q}/2\mathbb{Z})$$

where $C_k$ is the $2 \times 2$ matrix given by

$$C_k = \left\{ \begin{array}{ll} 2^{1-k} \left( \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right) & \text{if } \ell = 8 \text{ and } k \text{ is even}, \\ 2^{1-k} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) & \text{if } \ell > 8 \text{ and } k \text{ is even}, \\ 2^{-k} \left( \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right) & \text{if } \ell = 8 \text{ and } k \text{ is odd}, \\ 2^{-k} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) & \text{if } \ell > 8 \text{ and } k \text{ is odd}, \end{array} \right.$$

Theorem 1.2 will follow from Theorem 4.3 and from the following:

**Theorem 5.1.** *We have*

$$\tilde{ko}_{4k-1}(BQ_\ell) = \left\{ \begin{array}{ll} \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k} \oplus \mathcal{B}_{k+1} & \text{if } k \text{ is even}, \\ \mathbb{Z}_{2^{k+1}} \oplus \mathbb{Z}_{2^{k+1}} \oplus \mathcal{B}_{k+1} & \text{if } k \text{ is odd}. \end{array} \right.$$

**Proof:** We use the same argument used to prove Theorem 4.3. Let

$$\begin{aligned}\mathcal{L}_k := \operatorname{Span}_{\mathbb{Z}}\{&M_{\mathcal{I}}^{4k-1}-M_{\mathcal{J}}^{4k-1}, M_{\mathcal{I}}^{4k-1}-M_{\xi\mathcal{J}}^{4k-1}, M_Q^{4k-1},\\ &M_Q^{4k-5}\times Z^4,...,M_Q^3\times Z^{4k-4}\}\subset \tilde{ko}_{4k-1}(BQ_\ell).\end{aligned}$$

We then have that

$$\vec{\eta}_k(\mathcal{L}_k)=\begin{cases}\mathbb{Z}_{2^k}\oplus\mathbb{Z}_{2^k}\oplus\mathcal{B}_{k+1} & \text{if } k \text{ is even,}\\ \mathbb{Z}_{2^{k+1}}\oplus\mathbb{Z}_{2^{k+1}}\oplus\mathcal{B}_{k+1} & \text{if } k \text{ is odd.}\end{cases}$$

By Lemma 2.1 we have $c_0=\frac{\ell-1}{\ell}$ and thus $2c_0$ is an element of order $\ell$ in $\mathbb{Q}/2\mathbb{Z}$. We use the diagonal nature of the matrix $B_{k+1}$ to see that:

$$|\vec{\eta}_k(\mathcal{L}_k)|\ge\begin{cases}4^k\ell^k & \text{if } k \text{ is even,}\\ 4^{k+1}\ell^k & \text{if } k \text{ is odd.}\end{cases}$$

We use [6] Theorem 2.4 see:

$$|\tilde{ko}_{4k-1}(BQ_\ell)|=\begin{cases}4^k\ell^k & \text{if } k \text{ is even.}\\ 4^{k+1}\ell^k & \text{if } k \text{ is odd.}\end{cases}\qquad\square$$

**Remark 5.2.** Let $n\ge 0$. One has [3] that:

$$\tilde{ko}_{8n+\varepsilon}(\Sigma^{-1}BS^3/BN)=\begin{cases}\mathbb{Z}_2 & \text{if } \varepsilon=1,2,\\ \mathbb{Z}_{2^{2n+2}} & \text{if } \varepsilon=3,7,\\ 0 & \text{if } \varepsilon=4,5,6,8,\end{cases}$$

We may use equation (1.2) to decompose:

$$\begin{aligned}\tilde{ko}_*(BQ_\ell) &= \tilde{ko}_*(\Sigma^{-1}BS^3/BN)\oplus\tilde{ko}_*(\Sigma^{-1}BS^3/BN)\\ &\oplus \tilde{ko}_*(BSL_2(\mathbb{F}_q)).\end{aligned}$$

This is the decomposition given in Theorems 4.3 and 5.1:

$$\begin{aligned}\mathcal{A}_k &= \tilde{ko}_{4k-1}(\Sigma^{-1}BS^3/BN)\oplus\tilde{ko}_{4k-1}(\Sigma^{-1}BS^3/BN)\\ &= \operatorname{Span}\{[V^{\Theta_1}],[V^{\Theta_2}]\}\subset\tilde{KSp}(M^{4k+3}(Q_\ell,\tau))\\ &= \operatorname{Span}\{[M_{\mathcal{I}}^{4k-1}-M_{\mathcal{J}}^{4k-1}],[M_{\mathcal{I}}^{4k-1}-M_{\xi\mathcal{J}}^{4k-1}]\}\subset\tilde{ko}_{4k-1}(BQ_\ell),\\ \mathcal{B}_k &= \tilde{ko}_{4k-1}(BSL_2(\mathbb{F}_q))\\ &= \operatorname{Span}_{\mathbb{Z}}\{[V^{\varepsilon_j\Delta^j}]\}\subset\tilde{KSp}(M^{4k+3}(Q_\ell,\tau))\\ &= \operatorname{Span}_{\mathbb{Z}}\{[M_Q^{4k-1-4\mu}\times Z^{4\mu}]\}\subset\tilde{ko}_{4k-1}(BQ_\ell).\end{aligned}$$

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