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by

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DERIVATION OF THE NONLINEAR BENDING-TORSION THEORY FOR INEXTENSIBLE RODS BY Γ-CONVERGENCE

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1 Introduction

A fundamental problem in nonlinear elasticity is to understand the relation between three-dimensional theory and lower dimensional theories for domains which are thin in one or more dimensions. The derivation of such theories has a long history with contributions from many authors (we refer to S.S. Antman [1, 2] for a survey about one-dimensional models and a discussion of the history of the subject; see also [7], [11]). The derivations are usually based on some a priori assumptions leading to a variety of lower dimensional theories which are often not consistent with each other.

The starting point of our rigorous approach is the elastic energy

$$E^{(h)}(v) := \int_{\Omega_h} W(\nabla v(z))\, dz \tag{1.1}$$

of a deformation $v \in W^{1,2}(\Omega_h;\mathbb{R}^3)$, where $\Omega_h := (0,L)\times hS$ and S is an open subset of $\mathbb{R}^2$. Heuristically one expects that energies $E^{(h)}$ of order h^2 correspond to stretching and shearing deformations, leading to a *string theory,* while energies of order h^4 correspond to bending flexures and torsions keeping the domain unextended, leading to a *rod theory.* The elastic theory for strings has been rigorously justified by E. Acerbi, G. Buttazzo, D. Percivale in [3] by means of Γ-convergence (see [5] for a comprehensive introduction to Γ-convergence). In this paper we rigorously derive the bending and torsion theory for rods through Γ-convergence. A very different approach to the rod equations, based on centre manifold theory, was pursued by A. Mielke (see [8]). He fixes the cross section and considers the limit $L \to \infty$. For $\Omega = \mathbb{R}\times S$ he shows that all solutions whose strain is uniformly sufficiently small must lie on a 12-dimensional centre manifold and that the equation on the centre manifold is given by the Timoshenko beam equations. For related results in the context of linear elasticity see [4], [10].

To state our result it is convenient to introduce in (1.1) the following change of variables:

$$z_1 = x_1, \quad z_2 = hx_2, \quad z_3 = hx_3,$$

and to rescale deformations according to $y(x) := v(z(x))$, so that y belongs to $W^{1,2}(\Omega;\mathbb{R}^3)$, where $\Omega := (0,L)\times S$. We will use the notation

$$\nabla_h y := \left(y_{,1} \,\Big|\, \frac{1}{h} y_{,2} \,\Big|\, \frac{1}{h} y_{,3} \right),$$

so that

$$\frac{1}{h^2} E^{(h)}(v) = I^{(h)}(y) := \int_\Omega W(\nabla_h y(x))\, dx.$$

We assume that the stored energy function $W : \mathbb{M}^{3\times 3} \to [0,+\infty]$ satisfies the following assumptions:

i) $W \in C^0(\mathbb{M}^{3\times 3})$, W is of class C^2 in a neighbourhood of $\mathrm{SO}(3)$;

ii) W is frame-indifferent, i.e., $W(F) = W(RF)$ for every $F \in \mathbb{M}^{3\times 3}$ and $R \in \mathrm{SO}(3)$;

iii) $W(F) \geq C\,\mathrm{dist}^2(F, \mathrm{SO}(3))$, $W(F) = 0$ if $F \in \mathrm{SO}(3)$.

In Theorem 2.1 we show that for any sequence $(y^{(h)})$ such that

$$\limsup_{h\to 0} \frac{1}{h^2} I^{(h)}(y^{(h)}) < +\infty,$$

there exists a subsequence such that $\nabla_h y^{(h)} \to R$ strongly in $L^2(\Omega)$, where $R = (y_{,1}\,|\,d_2\,|\,d_3)$ and (y, d_2, d_3) belongs to the class

$$\begin{aligned}\mathcal{A} := \{(y, d_2, d_3) \in W^{2,2}(\Omega;\mathbb{R}^3)\times W^{1,2}(\Omega;\mathbb{R}^3)\times W^{1,2}(\Omega;\mathbb{R}^3) : \\ y, d_2, d_3 \text{ do not depend on } x_2, x_3,\ |y_{,1}| = |d_2| = |d_3| = 1, \\ y_{,1}\cdot d_2 = y_{,1}\cdot d_3 = d_2\cdot d_3 = 0\}.\end{aligned}$$

In our main theorem (Theorem 3.1) we identify the Γ-limit of the sequence of functionals $(\frac{1}{h^2} I^{(h)})$ with respect to the weak (and strong) topology of $W^{1,2}$. The limiting one-dimensional energy depends on (y, d_2, d_3) and is of the form

$$I(y, d_2, d_3) := \begin{cases} \frac{1}{2}\int_0^L Q_2(R^T R_{,1})\, dx_1 & \text{if } (y, d_2, d_3) \in \mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases}$$

where, as above, $R := (y_{,1}\,|\,d_2\,|\,d_3)$, while Q_2 is a quadratic form defined through a suitable minimization procedure involving the quadratic form of linearized elasticity $Q_3(G) := \frac{\partial^2 W}{\partial F^2}(Id)(G,G)$ (see (3.1)). The limiting energy is thus finite only on isometric deformations of $(0,L)$ and is a quadratic form in the entries of the matrix $R^T R_{,1}$. Note that, when $y \in \mathcal{A}$, $R^T R_{,1}$ is skew-symmetric. For $k = 2, 3$

we have $(R^T R_{,1})_{1k} = -(R^T R_{,1})_{k1} = y_{,1} \cdot d_{k,1}$, and this is related to curvature (and therefore, to bending effects), while $(R^T R_{,1})_{23} = -(R^T R_{,1})_{32} = d_2 \cdot d_{3,1}$ is related to torsion.

The key ingredient in the proofs is a geometric rigidity result, proved by G. Friesecke, R.D. James, and S. Müller in [6], which guarantees that low energy maps are close to a rigid motion (see Theorem 2.2) and provides the compactness result of Theorem 2.1.

In the last section of the paper we deal with a refined version of the Γ-convergence result: we let the functional $I^{(h)}$ depend explicitly on some additional variables, as the averaged deformation gradient and the rescaled nonlinear strain. We obtain as Γ-limit a one-dimensional functional with a richer structure and an additional term related to stretching and shearing effects. This term may play a role if we consider for instance deformations of Ω_h, whose energy is of order h^4, but which are only approximately isometries on the boundary.

After this work was finished, we have learnt that similar results have been obtained independently by O. Pantz [9].

2 Compactness

In the sequel S is a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary and $\Omega := (0, L) \times S$. We denote the variables in S by x_2, x_3 and we assume that $\mathcal{L}^2(S) = 1$ and

$$\int_S x_2 x_3 \, dx_2 dx_3 = \int_S x_2 \, dx_2 dx_3 = \int_S x_3 \, dx_2 dx_3 = 0. \tag{2.1}$$

Theorem 2.1 *Let $(y^{(h)})$ be a sequence in $W^{1,2}(\Omega; \mathbb{R}^3)$ such that*

$$\limsup_{h \to 0} \frac{1}{h^2} \int_\Omega \operatorname{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3)) \, dx < +\infty. \tag{2.2}$$

Then, there exists a subsequence (not relabelled) such that

$$\nabla_h y^{(h)} \to (y_{,1} \,|\, d_2 \,|\, d_3) \quad \text{in } L^2(\Omega), \tag{2.3}$$

where $y \in W^{2,2}(\Omega; \mathbb{R}^3)$, $d_2, d_3 \in W^{1,2}(\Omega; \mathbb{R}^3)$. Moreover, $(y_{,1} \,|\, d_2 \,|\, d_3) \in \mathrm{SO}(3)$ a.e. and is independent of x_2, x_3.

The key ingredient in the proof is the following rigidity result, proved by G. Friesecke, R.D. James, and S. Müller in [6].

Theorem 2.2 *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $v \in W^{1,2}(U; \mathbb{R}^n)$ there is an associated rotation $R \in \mathrm{SO}(n)$ such that*

$$\|\nabla v - R\|_{L^2(U)} \leq C(U) \|\operatorname{dist}(\nabla v, \mathrm{SO}(n))\|_{L^2(U)}. \tag{2.4}$$

Proof of Theorem 2.1. – The argument follows closely the proof of Theorem 4.1 in [6]. We include the details for the convenience of the reader. For every $h > 0$ let $k_h \in \mathbb{N}$ be such that $h \le L/k_h < 2h$, and let

$$I_{a,k_h} := \left(a, a + \frac{L}{k_h}\right), \quad a \in [0, L) \cap \frac{L}{k_h}\mathbb{N}. \tag{2.5}$$

By applying Theorem 2.2 to the function $v^{(h)}(z) := y^{(h)}(z_1, \frac{z_2}{h}, \frac{z_3}{h})$ restricted to the set $(a, a+2h) \times S_h$ (when $a < L - L/k_h$; to the set $(L-2h, L) \times S_h$, otherwise), we have that there exists a piecewise constant map $R^{(h)} : [0, L] \to \mathrm{SO}(3)$ such that

$$\int_{I_{a,k_h} \times S} |\nabla_h y^{(h)} - R^{(h)}|^2 dx \le C \int_{(a,a+2h)\times S} \mathrm{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3))\, dx, \tag{2.6}$$

(when $a = L - L/k_h$ just replace the interval $(a, a + 2h)$ by $(L - 2h, L)$ in the second integral above). Summing over a, we obtain

$$\int_\Omega |\nabla_h y^{(h)} - R^{(h)}|^2 dx \le C \int_\Omega \mathrm{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3))\, dx \le Ch^2. \tag{2.7}$$

Let now $a \in [0, L) \cap \frac{L}{k_h}\mathbb{N}$ be such that $(a, a + 4h) \subset (0, L)$ and let $b = a + L/k_h$. Then, using the fact that I_{a,k_h}, I_{b,k_h} are contained in $(a, a + 4h)$, the estimate (2.7), and its analog for the set $(a, a + 4h)\times S$, we have

$$\frac{L}{k_h}|R^{(h)}(a) - R^{(h)}(b)|^2 \le C \int_{(a,a+4h)\times S} \mathrm{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3))\, dx.$$

Since $R^{(h)}$ is piecewise constant, the inequality above can be rewritten as

$$\int_{I_{a,k_h}} |R^{(h)}(x_1) - R^{(h)}(x_1 + L/k_h)|^2 dx_1 \le C \int_{(a,a+4h)\times S} \mathrm{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3))\, dx.$$

Hence for every $0 \le \xi \le L/k_h$,

$$\int_{I_{a,k_h}} |R^{(h)}(x_1+\xi) - R^{(h)}(x_1)|^2 dx_1 \le C \int_{(a,a+4h)\times S} \mathrm{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3))\, dx. \tag{2.8}$$

In the same way one can show that for every a such that $(a - 2h, a + 2h) \subset (0, L)$ and for every $L/k_h \le \xi \le 0$,

$$\int_{I_{a,k_h}} |R^{(h)}(x_1 + \xi) - R^{(h)}(x_1)|^2 dx_1 \le C \int_{(a-2h,a+2h)\times S} \mathrm{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3))\, dx. \tag{2.9}$$

Now let I' be an open interval compactly contained in $(0,L)$ and let $\xi \in \mathbb{R}$ satisfy $|\xi| \leq \operatorname{dist}(I', \{0,L\})$. Then iterative applications of the estimates (2.8) and (2.9) yield

$$\int_{I'} |R^{(h)}(x_1+\xi) - R^{(h)}(x_1)|^2 dx_1$$
$$\leq C\left(\frac{|\xi|}{h}+1\right)^2 \int_\Omega \operatorname{dist}^2(\nabla_h y^{(h)}, \mathrm{SO}(3))\, dx \quad \leq C(|\xi|+h)^2. \tag{2.10}$$

Using the Fréchet-Kolmogorov criterion, one can deduce from this estimate that for any sequence $h_j \to 0$ there exists a subsequence of $(R^{(h_j)})$ strongly converging in $L^2(I')$ to some $\overline{R} \in L^2(I')$ with $\overline{R}(x_1) \in \mathrm{SO}(3)$ for a.e. $x_1 \in I'$.

From the bound (2.2) it follows that, up to subsequences, $(\nabla_{h_j} y^{(h_j)})$ converges weakly in $L^2(\Omega)$ to $(y_{,1}\,|\,d_2\,|\,d_3)$. By (2.7) we have that $R^{(h_j)} - \nabla_{h_j} y^{(h_j)} \to 0$ strongly in $L^2(\Omega)$, so that $(y_{,1}\,|\,d_2\,|\,d_3) = \overline{R}$ a.e. on $I' \times S$. In particular, $(y_{,1}\,|\,d_2\,|\,d_3)$ depends only on x_1 and belongs to $\mathrm{SO}(3)$ for a.e. $x_1 \in I'$. Since I' is an arbitrary compact interval contained in $(0,L)$, the properties above hold in the whole $(0,L)$. Since $\operatorname{dist}(\nabla_{h_j} y^{(h_j)}, \mathrm{SO}(3))$ tends to 0 in $L^2(\Omega)$, we have that $|\nabla_{h_j} y^{(h_j)}|^2 \to 3 = |\overline{R}|^2$ in $L^1(\Omega)$, so that $\|\nabla_{h_j} y^{(h_j)}\|_{L^2(\Omega)}$ converges to $\|\overline{R}\|_{L^2(\Omega)}$, which together with weak convergence in $L^2(\Omega)$ implies strong convergence in $L^2(\Omega)$. Hence, by (2.7) the sequence $(R^{(h)})$ is in fact converging to $\overline{R}$ strongly in $L^2(0,L)$.

Finally, passing to the limit in (2.10) as $h \to 0$, we obtain

$$\int_{I'} \frac{|(y_{,1}\,|\,d_2\,|\,d_3)(x_1+\xi) - (y_{,1}\,|\,d_2\,|\,d_3)(x_1)|^2}{|\xi|^2} dx_1 \leq C,$$

which implies $(y_{,1}\,|\,d_2\,|\,d_3) \in W^{1,2}(I'; \mathbb{M}^{3\times 3})$. Since C is independent of I', we actually have that $(y_{,1}\,|\,d_2\,|\,d_3) \in W^{1,2}((0,L); \mathbb{M}^{3\times 3})$. □

3 Γ-convergence

Theorem 3.1 *As $h \to 0$, the functionals $\frac{1}{h^2} I^{(h)}$ are Γ-convergent to the functional I given below, in the following sense:*

(i) (liminf inequality) *for every sequence of positive (h_j) converging to 0 and for every sequence $(y^{(h_j)}) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $y^{(h_j)} \to y$ in $W^{1,2}$ and $(\frac{1}{h_j} y^{(h_j)}_{,2}, \frac{1}{h_j} y^{(h_j)}_{,3}) \to (d_2, d_3)$ in L^2,*

$$I(y, d_2, d_3) \leq \liminf_{j\to\infty} \frac{1}{h_j^2} I^{(h_j)}(y^{(h_j)});$$

(ii) (limsup inequality) *for every sequence of positive (h_j) converging to 0 and for every $y \in W^{1,2}(\Omega;\mathbb{R}^3)$, $d_2, d_3 \in L^2(\Omega;\mathbb{R}^3)$ there exists a sequence $(y^{(h_j)}) \subset$*

$W^{1,2}(\Omega;\mathbb{R}^3)$ such that $y^{(h_j)} \to y$ in $W^{1,2}$, $(\frac{1}{h_j}y^{(h_j)}_{,2}, \frac{1}{h_j}y^{(h_j)}_{,3}) \to (d_2, d_3)$ in L^2, and

$$\limsup_{j\to\infty} \frac{1}{h_j^2} I^{(h_j)}(y^{(h_j)}) = I(y, d_2, d_3).$$

The limit functional is defined as

$$I(y, d_2, d_3) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L Q_2(R^T R_{,1})\, dx_1 & \textit{if } (y, d_2, d_3) \in \mathcal{A}, \\ +\infty & \textit{otherwise,} \end{cases}$$

where $R := (y_{,1}\,|\,d_2\,|\,d_3)$, while the class $\mathcal{A}$ is given by

$$\begin{aligned} \mathcal{A} := \{(y, d_2, d_3) \in W^{2,2}(\Omega;\mathbb{R}^3)\times W^{1,2}(\Omega;\mathbb{R}^3)\times W^{1,2}(\Omega;\mathbb{R}^3) :& \\ y, d_2, d_3 \text{ do not depend on } x_2, x_3,\ |y_{,1}| = |d_2| = |d_3| = 1,& \\ y_{,1}\cdot d_2 = y_{,1}\cdot d_3 = d_2\cdot d_3 = 0\}.& \end{aligned}$$

The quadratic form $Q_2 : \mathbb{M}^{3\times3}_{skew} \to [0, +\infty)$ is defined as

$$Q_2(A) := \min_{\alpha\in W^{1,2}(S;\mathbb{R}^3)} \int_S Q_3\left(A\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \middle|\, \alpha_{,2} \,\middle|\, \alpha_{,3} \right) dx_2 dx_3, \qquad (3.1)$$

while

$$Q_3(G) = \frac{\partial^2 W}{\partial F^2}(Id)(G, G)$$

is twice the quadratic form of linearized elasticity.

Remark 3.2 The result of the theorem remains valid if we replace the strong convergence in $W^{1,2}$ and L^2 of the sequences $(y^{(h)})$ and $(\frac{1}{h}y^{(h)}_{,2}, \frac{1}{h}y^{(h)}_{,3})$ by the weak convergence in the same spaces, as shown in the proof.

Remark 3.3 Notice that when $y \in \mathcal{A}$, the matrix R belongs to SO(3), so that $R^T R_{,1}$ is skew-symmetric.

Remark 3.4 (Euler-Lagrange equations) By standard arguments one can prove that the minimum problem in (3.1) has a solution; indeed, it is easy to show that the minimum can be equivalently computed on the class of functions

$$V := \left\{ \alpha \in W^{1,2}(S;\mathbb{R}^3) : \int_S \alpha\, dx_2 dx_3 = \int_S \nabla\alpha\, dx_2 dx_3 = 0 \right\},$$

where the fact that Q_3 is strictly positive definite on symmetric matrices is enough to guarantee compactness with respect to the the weak topology of $W^{1,2}$. Moreover, the functional to minimize is lower semicontinuous with respect to this topology. The strict convexity of Q_3 on symmetric matrices ensures also that the minimizer is unique in V.

In order to derive the Euler-Lagrange equations associated to the minimum problem, it is convenient to introduce some notation. Given a matrix $G \in \mathbb{M}^{3\times 3}_{sym}$, we denote its entries as follows:

$$G = \begin{pmatrix} g_1 & g_2 & g_3 \\ g_2 & g_4 & g_5 \\ g_3 & g_5 & g_6 \end{pmatrix},$$

and we write the quadratic form Q_3 in the following way:

$$Q_3(G) = \sum_{i,j\in\{1,4,6\}} \frac{1}{2} q_{ij} g_i g_j + \sum_{i,j\in\{2,3,5\}} 2 q_{ij} g_i g_j + \sum_{i\in\{1,4,6\}} \sum_{j\in\{2,3,5\}} 2 q_{ij} g_i g_j.$$

Note that the matrix $Q := (q_{ij})_{i,j=1,\dots,6}$ is positive definite. If M is a matrix in $\mathbb{M}^{n\times m}$, we denote by $M^{j_1 j_2}_{i_1 i_2}$ the (2×2)-submatrix of M given by the i_1, i_2-th rows and the j_1, j_2-th columns of M. Using this notation one can show that the minimizer $\alpha \in V$ of the problem (3.1) must satisfy the following Euler-Lagrange equations:

$$\begin{cases} \operatorname{div}(Q^{23}_{23}\nabla\alpha_1 + Q^{45}_{23}\nabla\alpha_2 + Q^{56}_{23}\nabla\alpha_3) = -a_{12}q_{12} - a_{13}q_{13} \\ \operatorname{div}(Q^{23}_{45}\nabla\alpha_1 + Q^{45}_{45}\nabla\alpha_2 + Q^{56}_{45}\nabla\alpha_3) = -a_{12}q_{14} - a_{13}q_{15} + a_{23}(q_{34} - q_{25}) \text{ in } S, \\ \operatorname{div}(Q^{23}_{56}\nabla\alpha_1 + Q^{45}_{56}\nabla\alpha_2 + Q^{56}_{56}\nabla\alpha_3) = -a_{12}q_{15} - a_{13}q_{16} + a_{23}(q_{35} - q_{26}) \end{cases} \tag{3.2}$$

with the following boundary conditions:

$$\begin{cases} (Q^{23}_{23}\nabla\alpha_1 + Q^{45}_{23}\nabla\alpha_2 + Q^{56}_{23}\nabla\alpha_3)\cdot\nu = -n_{23}\cdot\nu \\ (Q^{23}_{45}\nabla\alpha_1 + Q^{45}_{45}\nabla\alpha_2 + Q^{56}_{45}\nabla\alpha_3)\cdot\nu = -n_{45}\cdot\nu \quad \text{on } \partial S, \\ (Q^{23}_{56}\nabla\alpha_1 + Q^{45}_{56}\nabla\alpha_2 + Q^{56}_{56}\nabla\alpha_3)\cdot\nu = -n_{56}\cdot\nu \end{cases} \tag{3.3}$$

where we have set $n_{ij}(x_2, x_3) := (a_{12}x_2 + a_{13}x_3)(q_{1i}, q_{1j}) + a_{23}\, Q^{23}_{ij}(x_3, -x_2)$. It is clear that any solution α to (3.2)-(3.3) depends linearly on the entries (a_{ij}) of A. Hence Q_2 is in fact a quadratic form of A.

The formulae (3.2) and (3.3) simplify considerably if W is isotropic or if Ω has circular cross section (see Remarks 3.5 and 3.6 below).

Proof of Theorem 3.1. – (i) Let (h_j) be a positive sequence converging to 0 and let $(y^{(h_j)})$ be a sequence in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that $(y^{(h_j)}) \rightharpoonup y$ in $W^{1,2}$, $(\frac{1}{h_j}y^{(h_j)}_{,2}, \frac{1}{h_j}y^{(h_j)}_{,3}) \rightharpoonup (d_2, d_3)$ in L^2, and

$$\liminf_{j\to\infty} \frac{1}{h_j^2} I^{(h_j)}(y^{(h_j)}) < +\infty.$$

Passing to a subsequence if needed, we can assume that $\lim_{j\to\infty} \frac{1}{h_j^2} I^{(h_j)}(y^{(h_j)})$ exists and equals $\liminf_{j\to\infty} \frac{1}{h_j^2} I^{(h_j)}(y^{(h_j)})$.

By the proof of Theorem 2.1 we can costruct a piecewise constant approximation $R^{(h_j)} : (0, L) \to \mathrm{SO}(3)$ satisfying (2.7). We consider the function $G^{(h_j)} : \Omega \to \mathbb{M}^{3\times 3}$ defined as

$$G^{(h_j)}(x) := \frac{R^{(h_j)}(x_1)^T \nabla_{h_j} y^{(h_j)}(x) - Id}{h_j}. \tag{3.4}$$

It follows from (2.7) that the L^2-norm of $G^{(h_j)}$ in Ω is bounded; therefore, up to subsequences, there exists $G \in L^2(\Omega; \mathbb{M}^{3\times 3})$ such that

$$G^{(h_j)} \rightharpoonup G \quad \text{in } L^2(\Omega). \tag{3.5}$$

By expanding W around the identity and by using the frame-indifference of W one can show that

$$\liminf_{j\to\infty} \frac{1}{h_j^2} \int_\Omega W(\nabla_{h_j} y^{(h_j)})\, dx \geq \frac{1}{2} \int_\Omega Q_3(G)\, dx \tag{3.6}$$

(see the analogous argument in the proof of Theorem 6.1-(i) in [6]).

Now the main point is to identify G in terms of y, d_2, d_3. Let $G_1^{(h_j)}$ and G_1 denote the first column of $G^{(h_j)}$ and G, respectively, and consider the difference quotients in the x_k-direction with $k = 2, 3$:

$$\begin{aligned} H_k^{(h_j)}(x) &:= \frac{G_1^{(h_j)}(x + te_k) - G_1^{(h_j)}(x)}{t} \\ &= R^{(h_j)}(x_1)^T \frac{y_{,1}^{(h_j)}(x + te_k) - y_{,1}^{(h_j)}(x)}{th_j}. \end{aligned}$$

Let S' be a compact subset of S, let t be such that $|t| < \mathrm{dist}(S', \partial S)$, and let $\Omega' := (0, L) \times S'$. From (3.5) it follows that $H_k^{(h_j)} \rightharpoonup H_k$ in $L^2(\Omega')$, where

$$H_k(x) := \frac{G_1(x + te_k) - G_1(x)}{t}.$$

From the proof of Theorem 2.1 we know that $(R^{(h_j)})$ converges in $L^2(\Omega)$ to $R = (y_{,1} \,|\, d_2 \,|\, d_3)$; therefore,

$$\frac{y_{,1}^{(h_j)}(x + te_k) - y_{,1}^{(h_j)}(x)}{th_j} = R^{(h_j)} H_k^{(h_j)} \rightharpoonup RH_k \quad \text{in } L^2(\Omega'). \tag{3.7}$$

Note that the left-hand side can be rewritten as follows:

$$\frac{y_{,1}^{(h_j)}(x + te_k) - y_{,1}^{(h_j)}(x)}{th_j} = \partial_{x_1} \left(\frac{1}{t} \int_0^t \frac{1}{h_j} y_{,k}^{(h_j)}(x + se_k)\, ds \right). \tag{3.8}$$

By Theorem 2.1 we have that $\frac{1}{h_j} y^{(h_j)}_{,k}$ converges strongly in $L^2(\Omega)$ to d_k, hence the average $\frac{1}{t}\int_0^t \frac{1}{h_j} y^{(h_j)}_{,k}(\cdot + se_k)\, ds$ converges strongly in $L^2(\Omega')$ to $\frac{1}{t}\int_0^t d_k(\cdot + se_k)\, ds$, which is equal to d_k, since d_k does not depend on x_k. By (3.8) we obtain

$$\frac{y^{(h_j)}_{,1}(x + te_k) - y^{(h_j)}_{,1}(x)}{th_j} \rightharpoonup d_{k,1} \quad \text{in } W^{-1,2}(\Omega'). \tag{3.9}$$

Combining (3.7) and (3.9) we have $H_k = R^T d_{k,1}$. In particular, H_k is independent of x_2, x_3 and hence

$$G_1(x) = G_1(x_1, x_2, 0) + x_3 H_3(x_1) = G_1(x_1, 0, 0) + x_2 H_2(x_1) + x_3 H_3(x_1).$$

Setting $A(x_1) := R^T R_{,1}$ we have found that

$$G_1(x) = G_1(x_1, 0, 0) + A(x_1) \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}. \tag{3.10}$$

In order to identify the remaining columns of G, let us define

$$\alpha^{(h_j)}(x) := \frac{R^{(h_j)}(x_1)^T \frac{1}{h_j} y^{(h_j)} - x_2 e_2 - x_3 e_3}{h_j}. \tag{3.11}$$

It is easy to check that

$$\alpha^{(h_j)}_{,k} = G^{(h_j)}_k \quad \text{for } k = 2, 3, \tag{3.12}$$

where $G^{(h_j)}_k$ denotes the k-th column of $G^{(h_j)}$. If we set now $\alpha^{(h_j)}_0(x_1) := \int_S \alpha^{(h_j)}(x)\, dx_2 dx_3$, by Poincaré inequality we have that

$$\int_S |\alpha^{(h_j)}(x) - \alpha^{(h_j)}_0(x_1)|^2 dx_2 dx_3 \le C \int_S (|\alpha^{(h_j)}_{,2}(x)|^2 + |\alpha^{(h_j)}_{,3}(x)|^2)\, dx_2 dx_3$$

for a.e. $x_1 \in (0, L)$. Integrating with respect to x_1, we deduce

$$\|\alpha^{(h_j)} - \alpha^{(h_j)}_0\|^2_{L^2(\Omega)} \le C(\|\alpha^{(h_j)}_{,2}\|^2_{L^2(\Omega)} + \|\alpha^{(h_j)}_{,3}\|^2_{L^2(\Omega)}).$$

Since the right-hand side is bounded, we can conclude that $\alpha^{(h_j)} - \alpha^{(h_j)}_0$ weakly converges to some α in $L^2(\Omega)$. From (3.12) it follows that

$$\alpha_{,k} = G_k \quad \text{for } k = 2, 3 \tag{3.13}$$

and therefore $\alpha_{,k} \in L^2(\Omega; \mathbb{R}^3)$ for $k = 2, 3$. Combining (3.10) and (3.13), and setting

$$\tilde{\alpha}(x_2, x_3) := \alpha(x_2, x_3) - x_2 \int_S \alpha_{,2}\, dx_2 dx_3 - x_3 \int_S \alpha_{,3}\, dx_2 dx_3, \tag{3.14}$$

we can write

$$G = \left(G_1(x_1,0,0) \;\Big|\; \int_S \alpha_{,2} \;\Big|\; \int_S \alpha_{,3} \right) + \left(A \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \Big| \tilde{\alpha}_{,2} \Big| \tilde{\alpha}_{,3} \right). \tag{3.15}$$

By expanding the quadratic form Q_3 and by using the fact that the first matrix on the right-hand side of (3.15) is independent of x_2, x_3 and that $\int_S \tilde{\alpha}_{,k}\, dx_2 dx_3 = 0$ for $k = 2, 3$, we find in combination with (2.1) that

$$\begin{aligned} \int_S Q_3(G(x))\, dx_2 dx_3 = & \int_S Q_3 \left(G_1(x_1,0,0) \;\Big|\; \int_S \alpha_{,2} \;\Big|\; \int_S \alpha_{,3} \right) dx_2 dx_3 \\ & + \int_S Q_3 \left(A \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \Big| \tilde{\alpha}_{,2} \Big| \tilde{\alpha}_{,3} \right) dx_2 dx_3. \end{aligned} \tag{3.16}$$

Dropping the first term on the right-hand side, which is nonnegative, and using the definition of Q_2, we have

$$\int_\Omega Q_3(G(x))\, dx \geq \int_0^L Q_2(A(x_1))\, dx_1 = \int_0^L Q_2(R^T R_{,1})\, dx_1,$$

where in the last equality we have simply applied the definition of the matrix A. This finishes the proof of the liminf estimate.

(ii) To prove the limsup estimate, let $(y, d_2, d_3) \in \mathcal{A}$. Assume in addition $y \in C^2([0,L];\mathbb{R}^3)$, $d_2, d_3 \in C^1([0,L];\mathbb{R}^3)$. For every $h > 0$ let us consider the function

$$y^{(h)}(x) := y(x_1) + h x_2 d_2(x_1) + h x_3 d_3(x_1) + h^2 \beta(x) \tag{3.17}$$

with $\beta \in C^1(\overline{\Omega};\mathbb{R}^3)$. Then

$$\nabla_h y^{(h)} = R + h(x_2 d_{2,1} + x_3 d_{3,1} \,|\, \beta_{,2} \,|\, \beta_{,3}) + h^2(\beta_{,1} \,|\, 0 \,|\, 0).$$

If we set

$$\begin{aligned} B^{(h)} & := \frac{R^T \nabla_h y^{(h)} - Id}{h} \\ & = R^T(x_2 d_{2,1} + x_3 d_{3,1} \,|\, \beta_{,2} \,|\, \beta_{,3}) + h R^T(\beta_{,1} \,|\, 0 \,|\, 0), \end{aligned}$$

then for h sufficiently small (in such a way that, for a.e. $x \in \Omega$, the matrix $Id + hB^{(h)}(x)$ belongs to the neighbourhood of Id where W is of class C^2) we have by Taylor expansion

$$\frac{1}{h^2} W(Id + hB^{(h)}) \to \frac{1}{2} Q_3(R^T(x_2 d_{2,1} + x_3 d_{3,1} \,|\, \beta_{,2} \,|\, \beta_{,3})) \quad \text{a.e.},$$

and

$$\frac{1}{h^2} W(Id + hB^{(h)}) \leq C|B^{(h)}|^2 \leq C(|d_{2,1}|^2 + |d_{3,1}|^2 + |\nabla \beta|^2) \in L^1(\Omega).$$

By the dominated convergence theorem

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h^2}\int_\Omega W(\nabla_h y^{(h)})\,dx &= \lim_{h\to 0}\frac{1}{h^2}\int_\Omega W(Id+hB^{(h)})\,dx\\ &= \frac{1}{2}\int_\Omega Q_3(R^T(x_2 d_{2,1}+x_3 d_{3,1}\,|\,\beta_{,2}\,|\,\beta_{,3}))\,dx. \quad (3.18)\end{aligned}$$

Consider now the general case: let $(y,d_2,d_3)\in\mathcal{A}$ and let $\alpha(x_1,\cdot)\in V$ be the solution of the minimum problem defining $Q_2(R^T R_{,1})$. To conclude it remains to exhibit a sequence converging to (y,d_2,d_3) and whose energy converges to the right-hand side of (3.18) with $R^T\beta$ replaced by α. Since α and $\alpha_{,k}$ (for $k=2,3$) belong to $L^2(\Omega;\mathbb{R}^3)$, we can construct by convolution a sequence $(\alpha^{(j)})\subset C^1(\overline{\Omega};\mathbb{R}^3)$ such that $\alpha^{(j)}\to\alpha$, $\alpha^{(j)}_{,k}\to\alpha_{,k}$ (for $k=2,3$) in $L^2(\Omega)$. Moreover, we can find $(\tilde{R}^{(j)})\subset C^1([0,L];\mathbb{M}^{3\times 3})$ such that $\tilde{R}^{(j)}\to R$ in $W^{1,2}(0,L)$; by Sobolev embedding theorem this implies that $\tilde{R}^{(j)}\to R$ uniformly on $[0,L]$. In order to obtain an approximating sequence of orthogonal matrices, we take $R^{(j)}:=\Pi\,\tilde{R}^{(j)}$, where $\Pi:\mathbb{M}^{3\times 3}\to\mathbb{M}^{3\times 3}$ is a smooth function defining a projection from a neighbourhood of $\mathrm{SO}(3)$ onto $\mathrm{SO}(3)$, and we set

$$y^{(j)}(x_1):=\int_0^{x_1}R^{(j)}(s)e_1\,ds,\qquad d_k^{(j)}(x_1):=R^{(j)}(x_1)e_k \text{ for } k=2,3.$$

Then $(y^{(j)},d_2^{(j)},d_3^{(j)})\in\mathcal{A}$, $y^{(j)}\in C^2([0,L];\mathbb{R}^3)$, $d_2^{(j)},d_3^{(j)}\in C^1([0,L];\mathbb{R}^3)$, and $(y^{(j)}_{,1}\,|\,d_2^{(j)}\,|\,d_3^{(j)})=R^{(j)}$ is converging to R strongly in $W^{1,2}(0,L)$ and uniformly on $[0,L]$. Finally, we can assume, up to subsequences, that

$$\begin{aligned}&\frac{1}{2}\int_\Omega Q_3(x_2(R^{(j)})^T d^{(j)}_{2,1}+x_3(R^{(j)})^T d^{(j)}_{3,1}\,|\,\alpha^{(j)}_{,2}\,|\,\alpha^{(j)}_{,3})\,dx\\ &\quad\le \frac{1}{2}\int_\Omega Q_3(x_2R^Td_{2,1}+x_3R^Td_{3,1}\,|\,\alpha_{,2}\,|\,\alpha_{,3})\,dx+\frac{1}{j}\\ &\quad= I(y,d_2,d_3)+\frac{1}{j};\end{aligned}$$

here we have used the fact that the functional on the left-hand side is continuous with respect to the kind of convergence we have shown for $(R^{(j)})$ and $(\alpha^{(j)})$.

Now, given any positive (h_m) converging to 0, by (3.18) we can find a subsequence (that we denote by (h_j) with an abuse of notation) such that the sequence (3.17) with $y=y^{(j)}$, $d_k=d_k^{(j)}$, $\beta=R^{(j)}\alpha^{(j)}$ and $h=h_j$ satisfies

$$\frac{1}{h_j^2}I^{(h_j)}(y^{(h_j)})\le I(y,d_2,d_3)+\frac{2}{j},$$

and $y^{(h_j)}\to y$ in $W^{1,2}$, $(\frac{1}{h_j}y^{(h_j)}_{,2},\frac{1}{h_j}y^{(h_j)}_{,3})\to(d_2,d_3)$ in L^2, as required. □

Remark 3.5 (Isotropic case) Assume that the stored energy function W in (1.1) is isotropic, that is, W satisfies the following condition:

iv) $W(F) = W(FR)$ for every $R \in \mathrm{SO}(3)$.

Then, the quadratic form Q_3 is equal to

$$Q_3(G) = 2\mu \left| \frac{G + G^T}{2} \right|^2 + \lambda(\operatorname{trace} G)^2$$

for some constants $\lambda, \mu \in \mathbb{R}$. In this case it is easy to find an explicit solution to the system (3.2)-(3.3) and therefore the explicit expression of Q_2.

Indeed the system of equations (3.2)-(3.3) splits in the two following systems:

$$\begin{cases} \Delta\alpha_1 = 0 & \text{in } S, \\ \partial_\nu \alpha_1 = -a_{23}(x_3, -x_2) \cdot \nu & \text{on } \partial S, \end{cases} \tag{3.19}$$

and

$$\begin{cases} \operatorname{div}((2\mu+\lambda)\alpha_{2,2} + \lambda\alpha_{3,3}, \mu\alpha_{2,3} + \mu\alpha_{3,2}) = -\lambda a_{12} & \text{in } S, \\ \operatorname{div}(\mu\alpha_{2,3} + \mu\alpha_{3,2}, \lambda\alpha_{2,2} + (2\mu+\lambda)\alpha_{3,3}) = -\lambda a_{13} & \text{in } S, \\ ((2\mu+\lambda)\alpha_{2,2} + \lambda\alpha_{3,3}, \mu\alpha_{2,3} + \mu\alpha_{3,2}) \cdot \nu = -\lambda(a_{12}x_2 + a_{13}x_3)\nu_2 & \text{on } \partial S, \\ (\mu\alpha_{2,3} + \mu\alpha_{3,2}, \lambda\alpha_{2,2} + (2\mu+\lambda)\alpha_{3,3}) \cdot \nu = -\lambda(a_{12}x_2 + a_{13}x_3)\nu_3 & \text{on } \partial S. \end{cases} \tag{3.20}$$

If we denote by φ the torsion function, i.e., a function solving the Neumann problem

$$\begin{cases} \Delta\varphi = 0 & \text{in } S, \\ \partial_\nu \varphi = -(x_3, -x_2) \cdot \nu & \text{on } \partial S, \end{cases}$$

then it is straightforward to show that the solution to (3.19)-(3.20) belonging to the space V is provided by $\alpha_1(x_2, x_3) = a_{23}\, \varphi(x_2, x_3)$ and

$$\begin{aligned} \alpha_2(x_2, x_3) &= -\frac{1}{4}\frac{\lambda}{\lambda+\mu}(a_{12}x_2^2 - a_{12}x_3^2 + 2a_{13}x_2x_3), \\ \alpha_3(x_2, x_3) &= -\frac{1}{4}\frac{\lambda}{\lambda+\mu}(-a_{13}x_2^2 + a_{13}x_3^2 + 2a_{12}x_2x_3). \end{aligned}$$

Now, computing the value of the functional at the minimum point we have found, we obtain

$$Q_2(A) = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \left(a_{12}^2 \int_S x_2^2 \, dx_2 dx_3 + a_{13}^2 \int_S x_3^2 \, dx_2 dx_3 \right) + \mu\tau a_{23}^2,$$

where the constant τ is the so-called torsional rigidity, defined as

$$\tau(S) := \int_S (x_2^2 + x_3^2 - x_2\varphi_{,3} + x_3\varphi_{,2})\, dx_2 dx_3.$$

If, in addition, S has circular cross section, i.e. $S = \{(x_2, x_3) : \; x_2^2 + x_3^2 = \frac{1}{\pi}\}$, then $\varphi = 0$ and $\int_S x_2^2 \, dx_2 dx_3 = \frac{1}{4\pi}$, so that

$$Q_2(A) = \frac{1}{2\pi} \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}(a_{12}^2 + a_{13}^2) + \frac{\mu}{2\pi} a_{23}^2.$$

Remark 3.6 (Rods with circular cross section) Assume S is a circle of radius $1/\sqrt{\pi}$ centred at the origin. In this case the quadratic form Q_2 can be computed by a pointwise minimization as follows:

$$Q_2(A) = \frac{1}{4\pi} \min_{u,v,w \in \mathbb{R}^3} Q_3 \begin{pmatrix} a_{12} & u_1 & v_1 \\ 0 & u_2 & v_2 \\ -a_{23} & u_3 & v_3 \end{pmatrix} + Q_3 \begin{pmatrix} a_{13} & v_1 & w_1 \\ a_{23} & v_2 & w_2 \\ 0 & v_3 & w_3 \end{pmatrix}. \tag{3.21}$$

Fix $A \in \mathbb{M}^{3\times 3}_{skew}$ and let α be a function in V. For notation convenience we set

$$H^\alpha(x_2, x_3) := \begin{pmatrix} a_{12}x_2 + a_{13}x_3 & \alpha_{1,2} & \alpha_{1,3} \\ a_{23}x_3 & \alpha_{2,2} & \alpha_{2,3} \\ -a_{23}x_2 & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix}.$$

Let us define the following vectors in $\mathbb{R}^3$:

$$B := 4\pi \int_S x_2 \alpha_{,2} \, dx_2 dx_3, \qquad C := 4\pi \int_S x_2 \alpha_{,3} \, dx_2 dx_3,$$
$$D := 4\pi \int_S x_3 \alpha_{,2} \, dx_2 dx_3, \qquad E := 4\pi \int_S x_3 \alpha_{,3} \, dx_2 dx_3.$$

A crucial remark is that, since S is a circle, the two vectors C and D are in fact equal; indeed, by Green's formula we have

$$\int_S (-x_3 \alpha_{,2} + x_2 \alpha_{,3}) \, dx_2 dx_3 = \int_{\partial S} \alpha(-x_3, x_2) \cdot \nu \, d\sigma = 0,$$

where in the last equality we have used the fact that the normal vector ν to ∂S at a point (x_2, x_3) is given by $\sqrt{\pi}(x_2, x_3)$. We now consider the function

$$\beta(x) := \frac{1}{2} B x_2^2 + C x_2 x_3 + \frac{1}{2} E x_3^2,$$

and we want to prove that

$$\int_S Q_3(H^\alpha) \, dx_2 dx_3 \geq \int_S Q_3(H^\beta) \, dx_2 dx_3. \tag{3.22}$$

If we write the quadratic form Q_3 as

$$Q_3(G) = \sum_{i,j,k,l=1}^{3} \tilde{q}_{ijkl} G_{ij} G_{kl} \quad \text{for every } G \in \mathbb{M}^{3\times 3},$$

then we can expand $Q_3(H^\alpha)$ as follows:

$$Q_3(H^\alpha) = Q_3(H^\beta) + Q_3(H^\alpha - H^\beta) + 2 \sum_{i,j,k,l=1}^{3} \tilde{q}_{ijkl} H_{ij}^\beta (H_{kl}^\alpha - H_{kl}^\beta). \tag{3.23}$$

We claim that for every i, j and k, l

$$\int_S \tilde{q}_{ijkl} H_{ij}^\beta (H_{kl}^\alpha - H_{kl}^\beta)\, dx_2 dx_3 = 0. \tag{3.24}$$

Indeed, since H_{ij}^β is a linear combination of x_2 and x_3 for every i, j, it is enough to show that

$$\int_S x_m (H_{kl}^\alpha - H_{kl}^\beta)\, dx_2 dx_3 = 0 \tag{3.25}$$

for $m = 2, 3$ and for every k, l. For $l = 1$ the assertion is trivial. For $l = 2$ we have

$$\begin{aligned}
&\int_S x_m (H_{k2}^\alpha - H_{k2}^\beta)\, dx_2 dx_3 \\
&\quad = \int_S x_m \alpha_{k,2}\, dx_2 dx_3 - \int_S x_m B_k x_2\, dx_2 dx_3 - \int_S x_m C_k x_3\, dx_2 dx_3 \\
&\quad = \int_S x_m \alpha_{k,2} dx_2 dx_3 - \frac{1}{4\pi} B_k \delta_{m2} - \frac{1}{4\pi} C_k \delta_{m3} \;=\; 0
\end{aligned}$$

since $\int_S x_m x_l dx_2 dx_3 = \frac{1}{4\pi}\delta_{ml}$. Similarly,

$$\begin{aligned}
&\int_S x_m (H_{k3}^\alpha - H_{k3}^\beta)\, dx_2 dx_3 \\
&\quad = \int_S x_m \alpha_{k,3}\, dx_2 dx_3 - \int_S x_m C_k x_2\, dx_2 dx_3 - \int_S x_m E_k x_3\, dx_2 dx_3 \\
&\quad = \int_S x_m \alpha_{k,3} dx_2 dx_3 - \frac{1}{4\pi} C_k \delta_{m2} - \frac{1}{4\pi} E_k \delta_{m3} \;=\; 0.
\end{aligned}$$

Thus, the claim (3.25) is proved.

From (3.23) and (3.24) it follows that

$$\begin{aligned}
\int_S Q_3(H^\alpha)\, dx_2 dx_3 &= \int_S Q_3(H^\beta)\, dx_2 dx_3 + \int_S Q_3(H^\alpha - H^\beta)\, dx_2 dx_3 \\
&\geq \int_S Q_3(H^\beta)\, dx_2 dx_3,
\end{aligned}$$

since Q_3 is nonnegative. So, (3.22) is shown. This proves that, when S is a circle, it is enough to compute the minimum in (3.1) on the class of polynomials of degree 2 in x_2, x_3.

Now, if α is any polynomial of degree 2 in x_2, x_3, i.e., $\alpha(x_2, x_3) = \frac{1}{2}u\, x_2^2 + v\, x_2 x_3 + \frac{1}{2} w\, x_3^2$ with $u, v, w \in \mathbb{R}^3$, then

$$H^\alpha(x_2, x_3) = \begin{pmatrix} a_{12}x_2 + a_{13}x_3 & u_1 x_2 + v_1 x_3 & v_1 x_2 + w_1 x_3 \\ a_{23}x_3 & u_2 x_2 + v_2 x_3 & v_2 x_2 + w_2 x_3 \\ -a_{23}x_2 & u_3 x_2 + v_3 x_3 & v_3 x_2 + w_3 x_3 \end{pmatrix}.$$

Expanding again Q_3 and using the fact that $\int_S x_2 x_3\, dx_2 dx_3 = 0$ by (2.1), we obtain

$$\begin{aligned} &\int_S Q_3(H^\alpha)\, dx_2 dx_3 \\ &= \int_S x_2^2\, dx_2 dx_3\ Q_3\begin{pmatrix} a_{12} & u_1 & v_1 \\ 0 & u_2 & v_2 \\ -a_{23} & u_3 & v_3 \end{pmatrix} + \int_S x_3^2\, dx_2 dx_3\ Q_3\begin{pmatrix} a_{13} & v_1 & w_1 \\ a_{23} & v_2 & w_2 \\ 0 & v_3 & w_3 \end{pmatrix}, \end{aligned}$$

and this yields (3.21) since $\int_S x_2^2\, dx_2 dx_3 = \int_S x_3^2\, dx_2 dx_3 = (4\pi)^{-1}$.

4 Refined Γ-convergence and director theories

In this section we reformulate the theorem of the previous section as a Γ-convergence result for a functional depending on more variables. We need to introduce some new definitions. Given a sequence $(y^{(h)}) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ we set

$$\begin{aligned} y_0^{(h)}(x_1) &:= \int_S y^{(h)}(x)\, dx_2 dx_3, \qquad F^{(h)}(x_1) := \int_S \nabla_h y^{(h)}(x)\, dx_2 dx_3, \\ \beta^{(h)}(x) &:= \frac{y^{(h)}(x) - y_0^{(h)}(x_1)}{h^2} - \frac{1}{h} F^{(h)}(x_1) \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}, \qquad (4.1) \\ S^{(h)}(x_1) &:= \frac{[F^{(h)}(x_1)^T F^{(h)}(x_1)]^{1/2} - Id}{h}. \end{aligned}$$

Theorem 4.1 *Let $(y^{(h)})$ be a sequence in $W^{1,2}(\Omega; \mathbb{R}^3)$ such that*

$$\limsup_{h \to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)})\, dx < +\infty. \qquad (4.2)$$

Then there exists a subsequence (not relabelled) such that the following properties are satisfied:

(1) $y^{(h)} \to y$ *in* $W^{1,2}(\Omega)$, $y \in W^{2,2}(\Omega; \mathbb{R}^3)$, $y_{,2} = y_{,3} = 0$;

(2) $F^{(h)} \to R$ *in* $L^2(0, L)$, $R \in W^{1,2}(\Omega; \mathbb{M}^{3\times 3})$, $Re_1 = y_{,1}$, $R \in \mathrm{SO}(3)$ *a.e.;*

(3) $\beta^{(h)} \rightharpoonup \beta$ *in* $L^2(\Omega)$, $\int_S \beta\, dx_2 dx_3 = \int_S \beta_{,k}\, dx_2 dx_3 = 0$ *for* $k = 2, 3$;

(4) $S^{(h)} \rightharpoonup \overline{G}$ *in* $L^p(0, L)$ *for every* $p < 2$, $\overline{G} \in \mathbb{M}^{3\times 3}_{sym}$.

Moreover,

$$\liminf_{h\to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)})\, dx$$
$$\geq \frac{1}{2}\int_0^L Q_3(\overline{G})\, dx_1 + \frac{1}{2}\int_\Omega Q_3 \left(R^T R_{,1} \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \Big| R^T \beta_{,2} \Big| R^T \beta_{,3} \right) dx$$
$$=: F(\overline{G}, R, \beta). \tag{4.3}$$

Remark 4.2 As before the matrix $R^T R_{,1}$ describes bending and torsion effects, averaged over the cross section. Concerning the new additional variables $\overline{G}$ and β, the quantity $\overline{G}_{11}$ is related to the scaled stretch, while $\overline{G}_{21}, \overline{G}_{31}$ to the scaled shear; the remaining entries G_{ij}, $j \geq 2$, and the function β take into account the scaled cross sectional deformations.

Proof of Theorem 4.1. – Let $(y^{(h)})$ be a sequence satisfying (4.2). From Theorem 2.1 it follows that $\nabla_h y^{(h)} \to R$ in $L^2(\Omega)$, where $R \in W^{1,2}(\Omega; \mathbb{M}^{3\times 3})$, $R \in SO(3)$ a.e., and R does not depend on x_2, x_3. So, properties (1) and (2) are proved.

Let $R^{(h)}, G^{(h)}, \alpha^{(h)}, \alpha_0^{(h)}$ be as in the proof of Theorem 3.1-(i). By the definition of $F^{(h)}$ we have that

$$(R^{(h)})^T F^{(h)} = Id + h \int_S G^{(h)} dx_2 dx_3 = Id + h\tilde{G}^{(h)}, \tag{4.4}$$

where we have set $\tilde{G}^{(h)}(x_1) := \int_S G^{(h)}(x) dx_2 dx_3$; using (4.4) and (3.11) we can write

$$(R^{(h)})^T \beta^{(h)} = \alpha^{(h)} - \alpha_0^{(h)} - \tilde{G}^{(h)} \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.5}$$

In particular, we have $(R^{(h)})^T \beta^{(h)}_{,k} = \alpha^{(h)}_{,k} - \tilde{G}^{(h)}_k$ for $k = 2, 3$. Since $\alpha^{(h)} - \alpha_0^{(h)} \rightharpoonup \alpha$ in $L^2(\Omega)$, $G^{(h)} \rightharpoonup G$ in $L^2(\Omega)$, and $R^{(h)} \to R$ in $L^2(0, L)$, the equality (4.5) implies that

$$\beta^{(h)} \rightharpoonup \beta := R\alpha - R\tilde{G} \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{in } L^2(\Omega),$$

where $\tilde{G}(x_1) := \int_S G(x)\, dx_2 dx_3$. Moreover, $\beta^{(h)}_{,k} \rightharpoonup \beta_{,k}$ in $L^2(\Omega)$ for $k = 2, 3$, since $\alpha^{(h)}_{,k} \rightharpoonup \alpha_{,k}$ in $L^2(\Omega)$ for $k = 2, 3$. It is easy to check that $\int_S \beta^{(h)} dx_2 dx_3 = \int_S \beta^{(h)}_{,k} dx_2 dx_3 = 0$ (for $k = 2, 3$) for every h; therefore, the same properties hold for β. Assertion (3) is proved. For further references we notice that $\beta = \tilde{\alpha}$ by (3.13), where $\tilde{\alpha}$ is the function defined in (3.14).

In order to show (4), let $\Phi : \mathbb{M}^{3\times 3} \to \mathbb{M}^{3\times 3}$ be the function defined by $\Phi(F) := (F^T F)^{1/2} - Id$. By (4.4) we have $S^{(h)} = \frac{1}{h}\Phi(Id + h\tilde{G}^{(h)})$. Notice also that Φ is C^1 in a neighbourhood of Id and globally Lipschitz continuous. Then, given any test function $\varphi \in L^{p'}((0,L);\mathbb{M}^{3\times 3})$ with $p'>2$, we have

$$\begin{aligned}
\int_0^L S^{(h)}\varphi\,dx_1 &= \int_0^L \frac{1}{h}\Phi(Id+h\tilde{G}^{(h)})\varphi\,dx_1 \\
&= \int_0^L \frac{\Phi(Id+h\tilde{G}^{(h)})-\Phi(Id)}{h}\varphi\,dx_1 \\
&= \int_0^L \left(\frac{\Phi(Id+h\tilde{G}^{(h)})-\Phi(Id)}{h} - \Phi'(Id)\tilde{G}^{(h)}\right)\varphi\,dx_1 \\
&\quad + \int_0^L \Phi'(Id)\tilde{G}^{(h)}\varphi\,dx_1. \qquad (4.6)
\end{aligned}$$

The first integral on the last right-hand side converges to 0; indeed, since Φ' is continuous in a neighbourhood of the identity and globally bounded, for every $\varepsilon>0$ we have that for some $\delta=\delta(\varepsilon)$

$$\begin{aligned}
&\left|\int_0^L \left(\frac{\Phi(Id+h\tilde{G}^{(h)})-\Phi(Id)}{h} - \Phi'(Id)\tilde{G}^{(h)}\right)\varphi\,dx_1\right| \\
&\leq \varepsilon\int_{\{h|\tilde{G}^{(h)}|<\delta\}}\left|\tilde{G}^{(h)}\varphi\right|\,dx_1 + C\int_{\{h|\tilde{G}^{(h)}|\geq\delta\}}\left|\tilde{G}^{(h)}\varphi\right|\,dx_1 \\
&\leq C\|\varphi\|_{L^2}\varepsilon + C'\|\varphi\|_{L^{p'}}\mathcal{L}^1(\{h\left|\tilde{G}^{(h)}\right|\geq\delta\})^{\frac{2}{2-p}}.
\end{aligned}$$

Passing to the limit as $h\to 0$ and using the fact that ε is arbitrary, we obtain the claim. As for the second integral in (4.6), since $\tilde{G}^h \rightharpoonup \tilde{G}$ in $L^2(0,L)$ and $\Phi'(Id)F = \operatorname{sym} F$, we have

$$\lim_{h\to 0}\int_0^L \Phi'(Id)\tilde{G}^h\varphi\,dx_1 = \int_0^L \operatorname{sym}\tilde{G}\,\varphi\,dx_1.$$

Therefore, property (4) is proved with $\overline{G} = \operatorname{sim}\tilde{G}$.

To conclude it is enough to repeat the proof of Theorem 3.1-(i) up to the equality (3.16). From (3.15) it follows that

$$\tilde{G} = \left(G_1(x_1,0,0)\ \middle|\ \int_S \alpha_{,2}\ \middle|\ \int_S \alpha_{,3}\right).$$

Using this fact and the equalities $A = R^T R_{,1}$ and $\tilde{\alpha} = R^T\beta$, (3.16) can be rewritten as

$$\int_\Omega Q_3(G(x))\,dx = \int_0^L Q_3(\tilde{G})\,dx_1 + \int_\Omega Q_3\left(R^T R_{,1}\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}\middle| R^T\beta_{,2}\middle| R^T\beta_{,3}\right)dx.$$

Since $Q_3(F)$ depends only on the symmetric part of F, we can replace $\tilde{G}$ by $\overline{G}$ in the equality above. The thesis follows now from (3.6). $\square$

Theorem 4.3 *Let $y \in W^{2,2}((0,L);\mathbb{R}^3)$, let $R \in W^{1,2}((0,L);\mathbb{M}^{3\times 3})$ be such that $Re_1 = y_{,1}$ and $R \in \mathrm{SO}(3)$ a.e., let $\beta \in L^2(\Omega;\mathbb{R}^3)$ be such that $\int_S \beta\, dx_2 dx_3 = 0$, $\beta_{,k} \in L^2(\Omega;\mathbb{R}^3)$ and $\int_S \beta_{,k}\, dx_2 dx_3 = 0$ for $k = 2,3$, and let $\overline{G} \in L^2((0,L);\mathbb{M}^{3\times 3})$ be symmetric. Then there exists a sequence $(y^{(h)}) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that the properties (1)-(4) of Theorem 4.1 are satisfied, and*

$$\lim_{h\to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)})\, dx = F(\overline{G}, R, \beta),$$

where F is the functional defined in (4.3).

Proof. – Let the functions $y, \beta, R, \overline{G}$ be as in the statement. Assume in addition that $R, \overline{G} \in C^1([0,L];\mathbb{M}^{3\times 3})$ and $\beta \in C^1(\overline{\Omega};\mathbb{R}^3)$. Set

$$\gamma(x_1) = \int_0^{x_1} R(s)\overline{G}_1(s)\, ds, \qquad B_k(x_1) = R(x_1)\overline{G}_k(x_1) \text{ for } k = 2,3.$$

For every $h > 0$ consider the functions

$$y^{(h)}(x) = y(x_1) + h\gamma(x_1) + hR(x_1) \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + h^2\beta(x) + h^2 B_2(x_1)x_2 + h^2 B_3(x_1)x_3.$$

It is easy to see that $(y^{(h)})$ satisfies all the properties (1)-(4). Moreover, using the Taylor expansion of W around the identity and the dominated convergence theorem, we obtain

$$\lim_{h\to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)})\, dx = \frac{1}{2} \int_\Omega Q_3 \left(R^T(x_2 R_{,1}e_2 + x_3 R_{,1}e_3 + \gamma_{,1} \,|\, \beta_{,2} + B_2 \,|\, \beta_{,3} + B_3) \right) dx.$$

If we expand the quadratic form Q_3 and we use the fact that $\int_S \beta_{,k}\, dx_2 dx_3 = 0$ for $k = 2,3$, we have

$$\begin{aligned} &\int_\Omega Q_3 \left(R^T(x_2 R_{,1}e_2 + x_3 R_{,1}e_3 + \gamma_{,1} \,|\, \beta_{,2} + B_2 \,|\, \beta_{,3} + B_3) \right) dx \\ &= \int_0^L Q_3(R^T(\gamma_{,1} \,|\, B_2 \,|\, B_3))\, dx_1 + \int_\Omega Q_3 \left(R^T(x_2 R_{,1}e_2 + x_3 R_{,1}e_3 \,|\, \beta_{,2} \,|\, \beta_{,3}) \right) dx \\ &= \int_0^L Q_3(\overline{G})\, dx_1 + \int_\Omega Q_3 \left(R^T(x_2 R_{,1}e_2 + x_3 R_{,1}e_3 \,|\, \beta_{,2} \,|\, \beta_{,3}) \right) dx. \end{aligned}$$

In the general case it is enough to act by density, that is, to show that for any $y, \beta, R, \overline{G}$ as in the statement, we can construct approximating sequences $(y^{(j)}), (\beta^{(j)}), (R^{(j)}), (\overline{G}^{(j)})$ satisfying the extra assumption of C^1 regularity and such that

$$\lim_{j\to\infty} F(\overline{G}^{(j)}, R^{(j)}, \beta^{(j)}) = F(\overline{G}, R, \beta). \tag{4.7}$$

As in the proof of Theorem 3.1-(ii), we can construct $(y^{(j)}) \subset C^2([0,L];\mathbb{R}^3)$ and $(R^{(j)}) \subset C^1([0,L];\mathbb{M}^{3\times 3})$ such that $R^{(j)}e_1 = y^{(j)}_{,1}$, $R^{(j)} \in \mathrm{SO}(3)$ a.e., and $R^{(j)} \to R$ in $W^{1,2}(0,L)$. By mollification we can find $(G^{(j)}) \subset C^1([0,L];\mathbb{M}^{3\times 3})$ such that $G^{(j)} \to \overline{G}$ in $L^2(0,L)$, and $(\tilde{\beta}^{(j)}) \subset C^1(\overline{\Omega};\mathbb{R}^3)$ such that $\tilde{\beta}^{(j)} \to \beta$, $\tilde{\beta}^{(j)}_{,k} \to \beta_{,k}$ (for $k = 2,3$) in $L^2(\Omega)$. Then we define $\overline{G}^{(j)} := \mathrm{sym}\, G^{(j)}$ and

$$\beta^{(j)} := \tilde{\beta}^{(j)} - \int_S \tilde{\beta}^{(j)} dx_2 dx_3 - x_2 \int_S \tilde{\beta}^{(j)}_{,2}\, dx_2 dx_3 - x_3 \int_S \tilde{\beta}^{(j)}_{,3}\, dx_2 dx_3.$$

Now it is easy to check that (4.7) is satisfied. □

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