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by

Mohammad Reza Pakzad

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Max-Planck-Institute for Mathematics in the Sciences
Inselstr. 22 - 26, 04103 Leipzig, Germany
pakzad@mis.mpg.de

Abstract

We prove that every $W^{2,2}$ isometric immersion from a convex regular domain of $\mathbb{R}^2$ into $\mathbb{R}^3$ can be approximated in $W^{2,2}$-norm by smooth isometric immersions from the same domain into $\mathbb{R}^3$.

1 Introduction

1.1 Motivation

In this paper we study isometric immersions with square integrable curvatures. The motivation for this is two-fold. First the existence, rigidity and flexibility of isometric immersions of an m-dimensional manifold into $\mathbb{R}^n$ is a long standing problem in differential geometry. The second motivation arises from the study of curvature functionals in elasticity.

In his pioneering paper [28] John Nash proved that every n-dimensional C^3-smooth Riemannian manifold can be confined in an Euclidean space of dimension $\frac{1}{2}n(n+1)(3n+11)$, i.e. it can be embedded isometrically in any small portion of this space (See [10], [11] for further developments). On the other hand there are rigidity results. A classical result in the differential geometry of surfaces is that an isometric embedding of the 2-dimensional disk into a ball of radius less than 1/2 in $\mathbb{R}^3$ cannot be C^2-smooth; for generalizations for isometric immersions $M^m \to \mathbb{R}^n$, $n < 2m$, see [37]. One important insight was that the regularity of the immersion plays a crucial role. Hilbert showed that C^2 immersions of S^2 into $\mathbb{R}^3$ have to be rigid motions. However, Nash [27] constructed C^1 isometric immersions of S^n into $\mathbb{R}^m$, $m \geq n+2$, whose image is contained in an arbitrarily small ball. This result was extended to $m = n+1$ by Kuiper [17].

Here we study isometric immersions in the class $W^{2,2}$ which, roughly speaking, lies between C^1 and C^2. We prove that isometrically immersed surfaces in $\mathbb{R}^3$ with $W^{2,2}$ regularity are developable. Therefore we can deduce that the unit disc is not $W^{2,2}$-confinable in a ball of radius less than 1/2.

The surfaces with L^2 integrable second fundamental form have been studied from several points of view. T.Toro [35] proved the existence of bilipschitz parameterizations for the graphs of $W^{2,2}$ functions on $\mathbb{R}^2$. S.Müller and V.Šverák [26], studying surfaces with finite total curvature, improved Toro's result by showing the existence of conformal parameterizations with continuous metric for these graphs. L.Simon [33] considered another interesting curvature functional, the Willmore functional. In the context of nonlinear elasticity, the energy functional for the Föppl-von Karman equations ([19], [22]) is largely studied in plate theories for thin elastic bodies (See e.g. [1], [5], [21] and [36]).

Here we focus on nonlinear bending theory of Kirchhoff [16] which consists in minimizing the bending energy among immersions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ subject to the isometry constraint

$$(\nabla u)^T \nabla u = Id. \tag{1.1}$$

The bending energy is given by

$$E_b(u) = \int_\Omega |II(u)|^2, \tag{1.2}$$

where $II(u)$ is the second fundamental form of u.

The finite energy solutions are exactly the $W^{2,2}$ isometric immersions of Ω into $\mathbb{R}^3$ studied in this paper. The smooth solutions are of particular interest. In [31] O.Pantz mentioned that a density result for smooth maps in the space of Sobolev isometric immersions would be sufficient in order to prove that Kirchhoff's nonlinear plate theory is the Γ-limit of a constrained nonlinear elasticity theory. Γ-convergence of the standard 3-dimensional nonlinear elasticity was proven later on by G.Friesecke, R.D.James and S.Müller [8], [9] (See also [20], [2], [30]). As the main result of our paper we prove that smooth maps are strongly dense in the Sobolev space of $W^{2,2}$ isometric immersions from a convex domain of $\mathbb{R}^2$ into $\mathbb{R}^3$.

There are several other interesting related aspects to consider. Given a closed manifold N, we may pose the more general problem of the density of smooth maps in a given Sobolev space with the constraint $\nabla u \in N$. The case of $W^{2,2}$ isometric immersions of m-dimensional domains into $\mathbb{R}^n$ for $n \geq m$ is of particular interest here. Another important issue involves the approximate minimizers of (1.2) which do satisfy (1.1) except on a set of small measure, where the stretching energy concentrates, subjected to various boundary constraints. We refer to [21], [2], [5] and [36] for the detailed discussion of this domain. Many aspects remain however unclear, as for example the distribution of the concentration set in the domain.

1.2 Survey of results

Let Ω be a bounded regular convex domain in $\mathbb{R}^2$ with Lipschitz boundary. We define the space of $W^{2,2}$ local isometries from Ω in the 3 dimensional Euclidean space

$$I^{2,2}(\Omega, \mathbb{R}^3) := \{u \in W^{2,2}(\Omega, \mathbb{R}^3);\ \nabla u \in O(2,3)\ a.e.\}$$

where

$$O(2,3) := \left\{M \in M^{3\times 2};\ M^T M = Id\right\}.$$

This space inherits the strong and weak topology of $W^{2,2}(\Omega, \mathbb{R}^3)$ and is closed under the weak convergence.

Our main result is the following:

Theorem I *Smooth maps are strongly dense in $I^{2,2}(\Omega, \mathbb{R}^3)$. Equivalently, for any map $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ with a.e. singular symmetric gradient there exists a sequence f_m of smooth maps satisfying the same condition and converging to f in $W^{1,2}(\Omega, \mathbb{R}^2)$.*

Remark 1.1 *We do not require our approximating sequences to be smooth at the boundary.*

To put this result in perspective, we recall that the problem of density of smooth maps in Sobolev spaces between manifolds $W^{1,p}(M, N)$ where the constraint is on the value of the map itself has been a quite active domain of research in recent years. The respective topologies of M and N and Integer part of p determine whether smooth maps are dense in this space [3], [4], [12], [29] and [13]. However the nature of the problem we consider in this paper seems to be different and is to our knowledge the first density result for Sobolev spaces of maps with constraints on the value of the gradient.

In order to prove Theorem I we need to show that $W^{2,2}$ isometrically immersed surfaces in $\mathbb{R}^3$ are developable. More exactly we have

Theorem II *Let $u \in I^{2,2}(\Omega, \mathbb{R}^3)$. Then for every point $\mathrm{x} \in \Omega$, there exists either a neighborhood U of x, or a segment passing through it and joining $\partial\Omega$ at its both ends, on which u is affine.*

Remark 1.2 *Note that if $\mathrm{x} \in \Omega$ is of the first type then the $\partial U \setminus \partial\Omega$ is a piecewise affine curve for the maximal open set U having the above property.*

Corollary 1.1 *(See [37]) There is no $W^{2,2}$ embedding of the 2-dimensional disc into a three dimensional Euclidean ball of radius $r < 1/2$.*

The heuristic idea behind Theorem II is the following. Since the immersion u has L^2 integrable second derivatives, the Gaussian curvature of the immersed surface should be an L^1 integrable function on its domain. In contrast in the case of a cone which is the typical example of a nondevelopable singular surface, the Gaussian curvature is the Dirac mass concentrated at the vertex.

The main ingredient of the proof of Theorem II is the following result on the degenerate Monge-Ampère equations:

Proposition 1.1 *Let $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ be a map with almost everywhere symmetric singular (i.e. of zero determinant) gradient. Then for every point $\mathrm{x} \in \Omega$, there exists either a neighborhood U of x, or a segment passing through it and joining $\partial\Omega$ at its both ends, on which f is constant.*

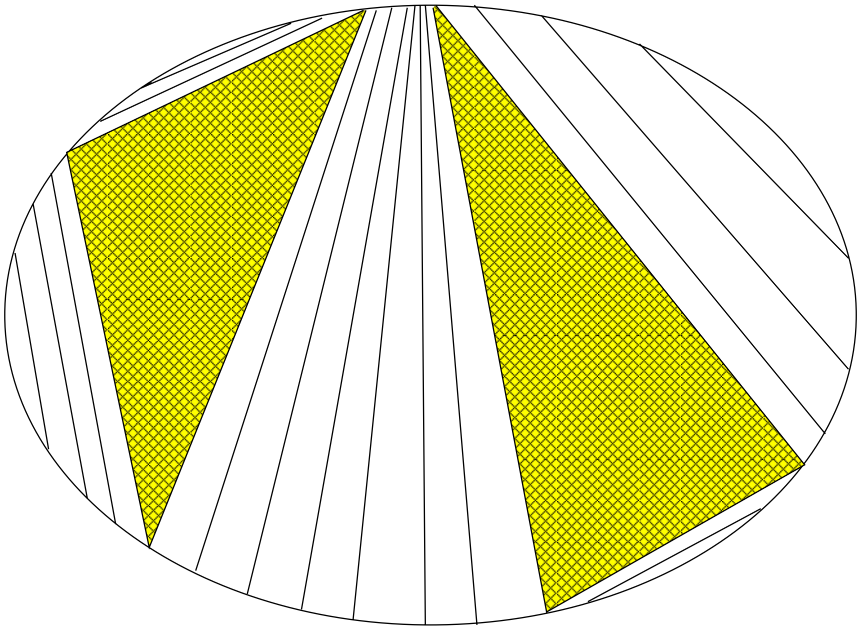

Figure 1: The inverse images of f

Corollary 1.2 *In the setting of Proposition 1.1, on the regions where f is not constant the normal vector field orthogonal to the inverse images is locally Lipschitz and thus integrable.*

Remark 1.3 *Since ∇f is symmetric, we can write $f = \nabla\phi$ for some $\phi \in W^{2,2}(\Omega, \mathbb{R})$ and we have*

$$\det(\nabla^2\phi) = 0$$

which is the degenerate Monge-Ampère equation. Note that the graph of ϕ is a flat surface in $\mathbb{R}^3$.

Remark 1.4 *An interesting counter example which shows that the conditions are optimal is the radial map $\frac{x}{\|x\|} \in W^{1,2-\varepsilon}(\mathbb{B}^2, \mathbb{R}^2)$ for $\varepsilon > 0$.*

B.Kirchheim has proved Proposition 1.1 under an extra assumption of the Lipschitz continuity of f satisfying (See [15], Proposition 2.29). The proof we present here for $W^{1,2}$ case is essentially the same.

The paper is organized as follows. In Section 2 we give a prove of Proposition 1.1 based on which we prove the developability theorem for $W^{2,2}$ isometric immersions. Then, in the last section we prove our main density result. The main idea of the proof of Theorem I is to use the developability of the image. We reparameterize the map $u \in I^{2,2}(\Omega, \mathbb{R}^3)$ along its *leading curves*; the curves in the domain and on the image surface which are orthogonal to the directions in which one of the principle curvatures vanish. Then by writing the equations of geodesic and normal curvature of a leading curve in the image related to its Darboux frame we manage to reduce the problem to the approximation of these functions by smooth functions with respect to suitable norms. This gives rise to an interesting correspondence between maps in $I^{2,2}(\mathbb{B}^2, \mathbb{R}^3)$ which have no affine part, and two functions defined on an interval in $\mathbb{R}$ which play the role of the curvature of a leading curve inside the domain and the geodesic and normal curvature of its image. We point out that using this correspondence we can indeed construct nonsmooth examples of maps in $I^{2,2}(\Omega, \mathbb{R}^3)$.

2 $W^{2,2}$ isometrically immersed surfaces are developable

2.1 Proof of Proposition 1.1

For the convenience of the readers the proof is divided into several lemmas.

Lemma 2.1 *Let $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that ∇f is symmetric and singular. Then f is continuous in Ω. Moreover, for any $A \subset \Omega$ we have $f(\overline{A}) = f(\partial A)$.*

Proof: For $\delta > 0$ define $f_\delta(x, y) := f(x, y) + \delta(-y, x)$. Then f_δ converges uniformly to f when $\delta \to 0$ and $det(\nabla f_\delta) = \delta^2$. So f_δ is a map in $W^{1,2}(\Omega, \mathbb{R}^2)$ with positive Jacobian and integrable dilatation, hence open and continuous [14]. By passing to the limit, we obtain the continuity for f.

Since f_δ is open we get $\partial f_\delta(B) \subset f_\delta(\partial B)$ for any $B \subset \Omega$. As a result, f_δ is spherically pseudomonotone and by [23] we can apply the coarea formula to obtain $|f_\delta(A)| = \delta^2|\Omega| \to 0$ as $\delta \to 0$. Now let us suppose that $y \in f(\overline{A}) \setminus f(\partial A) \subset f(A^\circ) \neq \emptyset$. There exists small ε for which $B(y, \varepsilon) \cap \partial f_\delta(A^\circ) \subset B(y, \varepsilon) \cap f_\delta(\partial A) = \emptyset$ for $0 < \delta < C\varepsilon$ otherwise, y would have belonged to $f(\partial A)$. Also $B(y, \varepsilon)$ should intersect $f_\delta(A^\circ)$ for small δ because of the uniform convergence of f_δ towards f. As a consequence, $B(y, \varepsilon) \subset f_\delta(A^\circ)$ for every δ which is a contradiction with the fact that $f_\delta(A)$ converges to zero in measure. ∎

Lemma 2.2 *In order to prove Proposition 1.1 it is sufficient to assume that f in continuous on the closure of Ω and that $f|_{\partial\Omega} \in W^{1,2}(\partial\Omega, \mathbb{R}^2)$.*

Proof: Assume that the proposition is true in this case. Put

$$\Omega_\delta := \{\mathrm{x} \in \Omega;\ \mathrm{dist}(\mathrm{x}, \partial\Omega) > \delta\}.$$

By Fubini theorem and the regularity of the boundary we observe that for almost all small enough values of δ, $f|_{\partial\Omega_\delta} \in W^{1,2}(\partial\Omega_\delta, \mathbb{R}^2)$. So by our assumption the proposition is true for $f_\delta := f|_{\Omega_\delta}$. Let $\mathrm{x} \in \Omega$ be a point around which f in not constant. For suitable δ_1 small enough, there is a segment joining $\partial\Omega_{\delta_1}$ at both sides on which f is constant. Let $[\mathrm{y}_1, \mathrm{y}_2]$ be the biggest segment in this direction on which f is constant. We prove that y_1 lies on

$\partial\Omega$. Suppose for a contradiction that $y_1 \in \Omega_{\delta_2}$ for some $\delta_2 < \delta_1$. The map f_{δ_2} cannot be constant in a neighborhood of y_1, since otherwise we could have prolonged the maximal segment $[y_1, y_2]$ in the direction of the vector $y_1 - y_2$. So there is another segment $[z_1, z_2]$, passing through y_1 and joining $\partial\Omega_{\delta_2}$ at its both ends which cannot be parallel to $[y_1, y_2]$ and on which f, by continuity, takes the same value. If $y_2 \in \Omega_{\delta_2}$ we can repeat the same argument and find a segment $[w_1, w_2]$ non-parallel to $[y_1, y_2]$ which joins $\partial\Omega_{\delta_2}$ at both ends. In this case, we assume that w_1 is at the same side of $[y_1, y_2]$ as z_1. If $y_2 \notin \Omega_{\delta_2}$, we let w_1 be the intersection of $[x, y_2]$ and $\partial\Omega_{\delta_2}$. Consider the region Δ formed between $\partial\Omega_{\delta_2}$ and the segments $[w_1, y_2]$, $[y_2, y_1]$ and $[y_2, z_1]$. Note that no segment joining $\partial\Omega_{\delta_2}$ cannot leave a point in Δ without intersecting $[y_2, y_1] \cup [y_2, z_1] \cup [w_1, y_2]$. As a consequence, f_{δ_2} is constant on Δ. Repeating the argument on the other side of $[y_2, y_1]$, we obtain that f is constant in a neighborhood of x which contradicts our earlier assumptions. ■

From now on we assume that f in continuous on $\overline{\Omega}$ and that $f|_{\partial\Omega} \in W^{1,2}(\partial\Omega, \mathbb{R}^2)$.

Lemma 2.3 *The set $T := f(\partial\Omega)$ is compact, arc-wise connected and 1-rectifiable. Moreover, if f is not constant, for $\mathcal{H}^1$-almost every $y \in f(\Omega)$, $f^{-1}(y)$ is a 1-rectifiable set with positive bounded measure. Also, for every $y \in T$, any connected component of $f^{-1}(y)$ is a closed set touching the boundary.*

Proof: The map f restricted to $\partial\Omega$ is in $W^{1,2}(\partial\Omega, \mathbb{R}^2)$. By [25] the change of variable formula is true for $f|_{\partial\Omega}$ and therefore $f(\Omega)$ is of bounded $\mathcal{H}^1$ measure. The first claim follows using ([6], lemma 3.12 and corollary 3.15). The second claim is proved considering the fact that the map f is spherically pseudomonotone since $f(B) = f(\partial B)$ for every ball $B \subset \Omega$. Therefore using the same arguments as in [23] and regarding ([7], theorem 3.2.22) we can use the coarea formula for $f : \Omega \to T$:

$$\int_T \mathcal{H}^1(f^{-1}(y))\, dy = \int_\Omega |J_1 f| = \int_\Omega |\nabla f| < \infty.$$

This proves that $f^{-1}(y)$ is a 1-rectifiable set with finite $\mathcal{H}^1$ measure for almost every $y \in T$.

Let C be any connected component of $f^{-1}(y)$. Note that $f^{-1}(y)$ is closed in $\overline{\Omega}$ since f is continuous. Therefore C is closed too. We argue by contradiction and assume $C \cap \partial\Omega = \emptyset$. In this case, we claim we can write $f^{-1}(y)$ as a union of two disjoint and non-empty closed and open subsets: $C_1 \subset \Omega$ and C_2. This will lead to a contradiction: Let A_δ be any δ-neighborhood of C_1 in Ω which has no intersection with $C_2 \cup \partial\Omega$. This is possible since both C_1 and $C_2 \cup \partial\Omega$ are compact and hence have positive distance from each other. As a consequence,

$$\partial A_\delta \cap (C_1 \cup C_2) = \partial A_\delta \cap f^{-1}(y) = \emptyset.$$

This contradicts Lemma 2.1 because $y \in f(A_\delta) \subset f(\partial A_\delta)$.

So to finish the proof we shall prove the above claim. This is a consequence of the equivalence of components and quasicomponents for compact spaces (See [18], §42, II.2). By definition, a quasicomponent of a topological space is the intersection of closed and open subsets of the space. So $C = \bigcap_j U_j$ where each U_j is open and closed in $f^{-1}(y)$. Hence

$$f^{-1}(y) \cap \partial\Omega \subset f^{-1}(y) \setminus C = \bigcup_j \left(f^{-1}(y) \setminus U_j\right).$$

This gives an open covering of the compact set $f^{-1}(y) \cap \partial\Omega$. Thus for some $N > 0$

$$f^{-1}(y) \cap \partial\Omega \subset C_2 := \bigcup_{j=1}^{N} \left(f^{-1}(y) \setminus U_j\right).$$

Therefore $C_1 := f^{-1}(y) \setminus C_2$ is a subset of Ω containing C. Both C_1 and C_2 are open and closed in $f^{-1}(y)$ because the U_j are open and closed, which proves our claim. ∎

Lemma 2.4 *For almost all $y \in T$ any connected component of $f^{-1}(y)$ is a segment which joins $\partial\Omega$ at one of its ends.*

Proof: Let $T_1 \subset T$ be the set of points y in T for which $f^{-1}(y)$ is a 1-rectifiable set with finite $\mathcal{H}^1$ measure and that T has a tangent at y. Let $A_0 \subset \Omega$ be the set of all points $\mathrm{x} \in \Omega$ for which $\nabla f(\mathrm{x})$ is either zero or is not a singular symmetric matrix and let D_1 be the set of all points $\mathrm{x} \in \Omega$ for which $\nabla f(\mathrm{x})$ is the total differential of f at x, i.e.

$$f(\mathrm{y}) - f(\mathrm{x}) - \nabla f(\mathrm{x})(\mathrm{y} - \mathrm{x}) = o(|\mathrm{y} - \mathrm{x}|).$$

By ([23], Theorem 3.3) $D_0 := \Omega \setminus D_1$ is of measure zero. Thus, by coarea formula, we obtain that for some $T_0 \subset T_1$ of null $\mathcal{H}^1$-measure

$$\mathcal{H}^1(f^{-1}(y) \cap (A_0 \cup D_0)) = 0\,;\ \forall y \in T_2 := T_1 \setminus T_0.$$

Let $y \in T_2$ and let C be any connected component of $f^{-1}(y)$. By Lemma 2.3 there is a point $\mathrm{x}_0 \in C \cap \partial\Omega$. By [6], lemma 3.12 and corollary 3.15, C is arc-wise connected and 1-rectifiable. Note that by rectifiability, C is locally the graph of some Lipschitz map. Therefore for any point $\mathrm{x} \in C \cap \Omega$ there exists a Lipschitz curve $\phi : [0, 1] \to C$ such that $\phi(0) = \mathrm{x}_0$, $\phi(1) = \mathrm{x}$ and $\phi'(t) \neq 0$ for almost all $t \in [0, 1]$. Since $y \notin T_0$, we can write ∇f along the image of ϕ as

$$\nabla f(\phi(t)) = \lambda(t) a(t) \otimes a(t) \quad \text{a.e.}\, t \in [0, 1],$$

where $a(t) \in S^1$. Since $\nabla f(\phi(t))(\mathbf{v})$ for $\mathbf{v} \in \mathbb{R}^2$ is tangent to T at y, we deduce that for almost every $t \in [0, 1]$, $a(t)$ is parallel to T_y, the tangent to T at y. Meanwhile $g := f \circ \phi$ is constant on $[0, 1]$ and f has a total differential in almost every point of $g([0, 1])$ since $y \in T_2$. Therefore as in ([24], Theorem 4.2) the chain rule applies to g and we obtain

$$\nabla f(\phi(t))(\phi'(t)) = 0.$$

Thus $\phi'(t)$ is orthogonal to $a(t)$ for almost every t (Note that $\lambda(t) \neq 0$ a.e.). As a consequence $\phi'(t)$ is orthogonal almost every where to the tangent T_y, hence ϕ lies on a straight line. As a conclusion C is a segment in Ω one of whose ends is x_0. ∎

Lemma 2.5 *For almost every $y \in T$ any connected component of $f^{-1}(y)$ is a segment joining $\partial\Omega$ at its both ends.*

Proof: For $k = 1, 2$ let $\pi_k : T \to \mathbb{R}$ be the projection of T on the the kth axis and put $f_k : \pi_k \circ f$. Since $\mathcal{H}^1(T) < \infty$, $\mathcal{H}^1(T \cap L) > 0$ only for countably many straight lines $L \subset \mathbb{R}^2$. Thus by rotating the coordinate system in $\mathbb{R}^2$ we can assume that

$$\mathcal{H}^1(\pi_k^{-1}(y_k)) = 0 \ \forall y_k \in \pi_k(T) \ k = 1, 2. \tag{2.1}$$

We claim that for almost all value $y \in T_2$, f is constant on any connected component of $f_k^{-1}(y_k)$ and the constant value is in T_2. Let

$$T_3 := T_2 \setminus \bigcup_{k=1,2} \pi_k^{-1}(\pi_k(T \setminus T_2)).$$

By (2.1), $\mathcal{H}^1(T_2 \backslash T_3) = 0$. For $y \in T_3$ and C_k any connected component of $f_k^{-1}(y_k)$, $f(C_K) \subset \pi_k^{-1}(y_k) \subset T_2$ is connected and of measure zero, hence a singleton. As a consequence for $y \in T_3$, the connected components of $f_k^{-1}(y_k)$ are segments joining $\partial\Omega$ at one side. Again by using the coarea formula for f_1 and f_2 we can be assured that $\mathcal{H}^1(f_k^{-1}(y_k)) < \infty$ for almost all $y \in T_3$. Let T_4 be the set of point in T_3 which satisfy this last condition.

To prove Lemma 2.5 it is sufficient to prove that the $\mathcal{H}^1$-measure of the set of points in T_4 for which the conclusion of the lemma fails is zero. Let $y \in T_4$ be such a point and let C be any connected component of $f^{-1}(y)$. By Lemma 2.4, C is a segment joining $\partial\Omega$ at one of its ends x_0. So x_0', the other end of C, belongs to Ω.

In the first step we prove that for some $\delta > 0$

$$f_k^{-1}(y_k) \cap B(\mathrm{x}_0', \delta) = C \cap B(\mathrm{x}_0', \delta), \ k = 1, 2. \tag{2.2}$$

Otherwise there is a sequence of points $\mathrm{x}_i \in f_k^{-1}(y_k) \setminus C$ converging to x_0'. Note that by passing to a subsequence we can assume that for $i \neq j$, x_i and x_j belong to different connected components of $f_k^{-1}(y_k)$. Let C_i be the connected component passing through x_i. Since C_i is a segment joining the boundary of Ω, we obtain

$$|C_i| \geq \mathrm{dist}(\mathrm{x_i}, \partial\Omega) \geq \mathrm{dist}(\mathrm{x}_0', \partial\Omega) - \delta > \rho > 0, \ \ \forall i \in \mathbb{N}$$

This contradicts $\mathcal{H}^1(f_k^{-1}(y_k)) < \infty$. Hence (2.2) is true and as a consequence x_0' is a local extremum for f_1 and f_2. Indeed since $B(\mathrm{x}_0', \delta) \setminus C$ is connected, $I_k = f_k(B(\mathrm{x}_0', \delta) \setminus C)$ is a connected subset of $\mathbb{R}$. By (2.2), $y_k \notin I_k$ thus I_k lies entirely on one side of y_k which proves our claim.

Let B_0 be the set of local extrema of f_1 and f_2 in Ω. In order to finish the proof of the lemma, it is sufficient to prove that $\mathcal{H}^1(f(B_0)) = 0$. Assume that this is not true and suppose, for instance, $\mathcal{H}^1(f_1(B_0)) > 0$. Let E be the set of all local extrema of f_1 in Ω. Then $E = \bigcup E_m$, where E_m consists of all the points $\mathrm{x} \in \Omega$ such that x is a local extremum in the ball of radius $1/m$ around itself. If $f_1(E)$ is uncountable, there exists $m \in \mathbb{N}$ for which $f_1(E_m)$ is uncountable too. Hence there exists E', an uncountable subset of E_m, such that $f_1(\mathrm{x}) \neq f_1(\mathrm{x}')$ whenever $\mathrm{x}, \mathrm{x}' \in E'$ are different. Since E' is infinite, it has an accumulation point in $\overline{\Omega}$ and we can find $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3 \in E'$ such that $0 < |\mathrm{x}_i - \mathrm{x}_j| < 1/2m$. But each x_i must have been a local extremum in a ball of radius $1/m$ around itself which leads to a contradiction. ∎

We finish the proof of Proposition 1.1. Let x $\in \Omega$ be an arbitrary point and assume that f is not constant in its neighborhood. By continuity of f, $\mathcal{H}^1(f(B(\mathrm{x},\delta)) > 0$ for any small enough $\delta > 0$. So by Lemma 2.5 we can find segments joining $\partial\Omega$ arbitrarily near to x on which f would have pairwise distinct constant values, i.e. no two such segments can intersect each other in Ω, particularly in a neighborhood of x. By choosing a sequence of these segments converging to x in distance we obtain a segment passing through x and joining $\partial\Omega$ on its both ends on which f must be constant by continuity. ∎

2.2 Proof of Theorem II

Consider the second fundamental form of $u \in I^{2,2}(\Omega, \mathbb{R}^3)$:

$$II(u) := \begin{bmatrix} u_{,xx} \cdot \mathbf{n} & u_{,xy} \cdot \mathbf{n} \\ u_{,yx} \cdot \mathbf{n} & u_{,yy} \cdot \mathbf{n} \end{bmatrix} \in L^2(\Omega, M^{2\times 2}).$$

where $\mathbf{n} := u_x \wedge u_y$ is the unit normal field to the image $u(\Omega)$.

Lemma 2.6 *The following equations are satisfied in the sense of distributions*

$$\frac{\partial II_{11}}{\partial y} = \frac{\partial II_{12}}{\partial x}, \quad \frac{\partial II_{21}}{\partial y} = \frac{\partial II_{22}}{\partial x}, \quad detII = 0. \tag{2.3}$$

Proof: By classical mollifying technics we can approximate u in the $W^{2,2}$ norm by a sequence of smooth maps $u_m \in W^{2,2}(\Omega, \mathbb{R}^3)$. Moreover we can assume that the u_m are immersions satisfying $1/2 < |\nabla u_m|_\infty \leq 1$ [32]. Let $g_{ij}(u_m)$ be the coefficients of the metric matrix $(\nabla u_m)^T \nabla u_m$. As a consequence $g_{ij}(u_m) \in W^{1,2} \cap L^\infty$ converge to δ_{ij} in the $W^{1,2}$ norm and $\Gamma^h_{ij}(u_m)$, the Christoffel symbols of u_m, converge in L^2 to 0. Also $l_{ij}(u_m) := (u_m)_{,ij}.\mathbf{n}_m$ converge to II_{ij} in L^2. According to the Codazzi-Mainardi equations ([34], p.79) the following equations are satisfied for u_m

$$\begin{cases} \dfrac{\partial l_{12}}{\partial x} - \dfrac{\partial l_{11}}{\partial y} + \displaystyle\sum_{h=1}^{2} \Gamma^h_{12} l_{h1} - \sum_{h=1}^{2} \Gamma^h_{11} l_{h2} = 0. \\[2ex] \dfrac{\partial l_{22}}{\partial x} - \dfrac{\partial l_{21}}{\partial y} + \displaystyle\sum_{h=1}^{2} \Gamma^h_{22} l_{h1} - \sum_{h=1}^{2} \Gamma^h_{21} l_{h2} = 0. \end{cases}$$

By passing to the limit we obtain the first two identities in (2.3) in the sense of distributions. We note that since u is an isometry, $u_{,ij}$ is orthogonal to $u_{,x}$ and $u_{,y}$ almost everywhere. Therefore

$$detII = u_{,xx} \cdot u_{,yy} - u_{,xy} \cdot u_{,xy} \,.$$

Using the following identity for the smooth u_m

$$-2\frac{\partial^2 u_m}{\partial x^2} \cdot \frac{\partial^2 u_m}{\partial y^2} + 2\frac{\partial^2 u_m}{\partial x \partial y} \cdot \frac{\partial^2 u_m}{\partial x \partial y} = \frac{\partial^2 g_{xx}}{\partial x^2}(u_m) + \frac{\partial^2 g_{yy}}{\partial y^2}(u_m) - 2\frac{\partial^2 g_{xy}}{\partial x \partial y}(u_m)$$

and passing to the limit in the sense of distributions the third identity is verified. ∎

As a consequence, there exists $f_u \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $\nabla f_u = II$ is symmetric and singular. As in lemma 2.2 we may assume that $f_u|_{\partial\Omega} \in W^{1,2}(\partial\Omega, \mathbb{R}^2)$ and is continuous in $\overline{\Omega}$. Applying Proposition 1.1, it is sufficient to prove that u is affine on the connected components of almost every inverse image of f_u. Note that for almost every $z \in T_{f_u}$, $\mathcal{H}^1(f_u^{-1}(z)) < \infty$ and

$$\int_{f_u^{-1}(z)} |II|\, d\mathcal{H}^1 = \int_{f_u^{-1}(z)} |\nabla f_u|\, d\mathcal{H}^1 < \infty \tag{2.4}$$

since by Lemma 2.3 the coarea formula is applicable to f_u and we have

$$\int_{T_{f_u}} dz \int_{f_u^{-1}(z)} |\nabla f_u|\, d\mathcal{H}^1 = \int_\Omega |\nabla f_u|^2 < \infty.$$

Let C be a connected component of such $f_u^{-1}(z)$, which is a segment in the direction of $\vec{v} \in \mathbb{R}^2$ and touching $\partial\Omega$ on its both ends. By (2.4), $II \in L^1(C, \mathbb{R}^{2\times 2})$. Since f_u is constant on C we obtain $II\,\vec{v} = 0$. The conclusion easily follows. ■

Remark 2.1 *In order to prove Theorem II we could have also applied Proposition 1.1 to the vector field* $\mathbf{n} : \Omega \to S^2$ *which satisfies* $J_2\mathbf{n} = 0$ *for* $u \in I^{2,2}(\Omega, \mathbb{R}^3)$.

3 Approximating isometric immersions

In this section we show how to exploit the developability of $u(\Omega)$ to approximate u by smooth maps in $I^{2,2}(\Omega, \mathbb{R}^3)$.

3.1 Preliminaries

Let us introduce some notions:

Definition 3.1 *We say that a differentiable curve* $\gamma : [0, l] \to \Omega$, *parameterized by arclength and joining two distinct points, is a leading curve if it is orthogonal to the inverse images of* f_u *in the regions where* f_u *is not constant.*

By corollary 1.2 the leading curves exist and have bounded curvature κ defined by

$$\gamma'' = \kappa \mathbf{N}$$

where $\mathbf{N} := (-\gamma_2', \gamma_1')$ is the unit normal vector of the curve γ. So by the Sobolev injection Theorem the leading curves are $C^{1,\alpha}$ for $\alpha < 1$. Also the leading front of γ at the point $t \in [0, l]$ is defined to be

$$F_\gamma(t) := \{\gamma(t) + s\mathbf{N}(t);\ s \in \mathbb{R}\}.$$

Definition 3.2 *We say that a curve γ covers the domain $S \subset \Omega$ if*

$$S \subset \{\gamma(t) + s\mathbf{N}(t)\,;\, s \in \mathbb{R},\, t \in [0, l]\,\}\,.$$

By $\Omega(\gamma)$ we refer to the biggest set covered by γ in Ω.

Lemma 3.1 *Every covered domain is convex.*

Proof: It is a direct consequence of the convexity of Ω and the shape of covered domains. ■

To prove the theorem we first divide Ω into domains on which u is affine or which are covered by leading curves, such that each two domains encounter on a segment joining $\partial\Omega$ at its both ends (Remark 1.2). As a result more than two such domains can join together only at a corner on the boundary and a covered domain can at most have two segments as its boundary inside Ω. Then what remains is to approximate u on each covered domain successively in a way that the boundary values match smoothly.

To persuade the readers that we can perfectly match the boundary values we have no other way than to show how we construct our approximation. So let us first prove the theorem for a covered domain $\Omega(\gamma)$.

Define the leading curve corresponding to γ in $u(\Omega)$ to be $\tilde{\gamma} := u \circ \gamma$ and consider the Darboux frame $(\mathbf{t}, \mathbf{v}, \mathbf{n})$ of this curve in the surface:

$$\begin{cases} \mathbf{t} := \tilde{\gamma}' \\ \mathbf{v} := \nabla u(\mathbf{N}) \\ \mathbf{n} := \mathbf{t} \times \mathbf{v} \end{cases}$$

Since u is an isometric affine map along the $\mathbf{N}$ direction we obtain

$$u(\gamma(t) + s\mathbf{N}(t)) = \tilde{\gamma}(t) + s\mathbf{v}(t) \tag{3.1}$$

for all t, s for which the identity makes sense. By deriving with respect to t we realize

$$\mathbf{v}' = -\kappa\mathbf{t} \tag{3.2}$$

is the necessary and sufficient condition for that $\nabla u \in O(2,3)$ almost everywhere. Let us consider the derivative of the Darboux frame of $\tilde{\gamma}$ as in ([34], p.277):

$$\begin{cases} \mathbf{t}' = & & \kappa_g\mathbf{v} & +\kappa_n\mathbf{n} \\ \mathbf{v}' = & -\kappa_g\mathbf{t} & & +\tau_g\mathbf{n} \\ \mathbf{n}' = & -\kappa_n\mathbf{t} & -\tau_g\mathbf{v} & \end{cases} \tag{3.3}$$

Hence, by (3.2),

$$\begin{cases} \kappa_g = \kappa \\ \\ \tau_g = 0 \end{cases} \tag{3.4}$$

is the necessary and sufficient condition that (3.1) defines an isometry on $\Omega(\gamma)$.

Before proceeding let us also study the map f_u as defined in section 2. The map f_u is constant on the front lines, therefore :

$$f_u(\gamma(t) + s\mathbf{N}(t)) = f_u \circ \gamma(t) = \Gamma(t)$$

for some curve $\Gamma \in W^{1,2}([0, l], \mathbb{R}^2)$. Straightforward calculations yield

$$\Gamma'(t) = \kappa_n(t)\gamma'(t).$$

The natural way to analyze the situation would be to write u in terms of the new coordinate system defined by γ and the orthogonal segments. So let us put

$$\Phi_\gamma : \Omega(l, s_-^\gamma, s_+^\gamma) \to \Omega(\gamma), \quad \Phi_\gamma(t, s) := \gamma(t) + s\mathbf{N}(t) \tag{3.5}$$

where

$$\begin{aligned} \Omega(l, s_-^\gamma, s_+^\gamma) \; &:= \{(t, s) \in [0, l] \times \mathbb{R};\ \gamma(t) + s\mathbf{N} \in \Omega(\gamma)\} \\ &= \{(t, s) \in [0, l] \times \mathbb{R};\ s_-^\gamma(t) < s < s_+^\gamma(t)\} \end{aligned}$$

where $s_-^\gamma < 0 < s_+^\gamma$ are continuous functions on $[0, l]$ ($|s_\pm^\gamma(t)| > 0$ since $\gamma(t) \in \Omega$).

Let us also define

$$I_\gamma := \left\{t \in [0, l]; F_\gamma(t) \cap F_\gamma(t') \cap \overline{\Omega} = \emptyset \ \forall t' \in [0, l]\right\}$$

Because of continuous dependence of $F_\gamma(t)$ on t, I_γ is an open subset of $[0, l]$. Observe that if $t \in I_\gamma$, then $1 - s\kappa(t) > 0$ for $s_-^\gamma(t) \le s \le s_+^\gamma(t)$. Put

$$I_0 := [0, l] \setminus I_\gamma.$$

Assume that $\mathrm{x} \in F_\gamma(t) \cap F_\gamma(t') \cap \overline{\Omega}$ for $t < t'$. A simple geometric observation shows that $[t, t'] \subset I_0$, because the barrier formed by $F_\gamma(t)$ and $F_\gamma(t')$ over γ blocks the way to the leading fronts of γ between t and t' to get out of $\overline{\Omega}$ without touching these two lines. Doing straightforward calculations we obtain

$$u_x \circ \Phi(t, s) = \gamma_1'(t)\mathbf{t}(t) - \gamma_2'(t)\mathbf{v}(t), \quad u_y \circ \Phi(t, s) = \gamma_2'(t)\mathbf{t}(t) + \gamma_1'(t)\mathbf{v}(t) \tag{3.6}$$

and

$$\begin{cases} \displaystyle\int_{\Omega(\gamma)} |u|^2 d\mathrm{x} &= \displaystyle\int_{\Omega(l, s_-^\gamma, s_+^\gamma)} \|\tilde{\gamma}(t) + s\mathbf{v}(t)\|^2 (1 - s\kappa(t)) dt ds \\ \displaystyle\int_{\Omega(\gamma)} |\nabla u|^2 d\mathrm{x} &= 2|\Omega(\gamma)| \\ \displaystyle\int_{\Omega(\gamma)} |\nabla^2 u|^2 d\mathrm{x} &= \displaystyle\int_{\Omega(l, s_-^\gamma, s_+^\gamma)} \zeta(t, s) dt ds \end{cases} \tag{3.7}$$

where

$$\zeta(t,s) := \begin{cases} \dfrac{\kappa_n^2(t)}{1-s\kappa(t)} & \text{if} \quad t \in I_\gamma \\ \\ 0 & \text{otherwise} \end{cases}$$

The geometric interpretation of the last equality is that if for $t_1 < t_2$,

$$\mathrm{x} \in F_\gamma(t_1) \cap F_\gamma(t_2) \cap \overline{\Omega}(\gamma) \neq \emptyset,$$

then u should be an affine map on the domain covered by $\gamma|_{[t_1,t_2]}$ otherwise the $W^{2,2}$ norm would explode around the vertex x. Hence, $\kappa_n = 0$ in $[t_1, t_2]$ too. Applying again the same change of variable computations to f_u we have the following equality:

$$\|\nabla f_u\|_{L^2}^2 = \int_{\Omega(l,s_-^\gamma,s_+^\gamma)} \zeta(t,s)dtds.$$

To approximate u on $\Omega(\gamma)$ we proceed in this way: We search for smooth curves γ_m and $\tilde{\gamma}_m$ approximating γ and $\tilde{\gamma}$ in suitable norms and satisfying equations (3.3) and (3.4). Based on the above observations, the map u_m defined by

$$u_m(\gamma_m(t) + s\mathbf{N_m}(t)) := \tilde{\gamma}_m(t) + s\mathbf{v_m}(t) \tag{3.8}$$

would be a smooth isometry which we would hope to be our approximating sequence. However, a major obstacle to this enterprise is that $\Omega(\gamma_m)$ may not coincide with $\Omega(\gamma)$ while we want our approximating sequence to be defined on the latter. But since γ_m converges to γ, its covered domain will cover a major part of $\Omega(\gamma)$ and we would be able to extend our maps to this domain.

3.2 Approximation process for $u|_{\Omega(\gamma)}$

In the first step we modify γ properly to avoid some technical difficulties.

Lemma 3.2 *We can assume, if necessary by modifying γ, that*

$$F_\gamma(t) \cap F_\gamma(t') \cap \Omega = \emptyset$$

for all $t \neq t'$.

Proof: It is sufficient to modify the leading fronts on the connected components of I_0. This is possible since u is affine on the region covered by $\gamma|_{I_0}$. If $[t_1, t_2]$ is such an interval, we can consider the new leading fronts of γ inside $\bigcup_{t\in[t_1,t_2]} F_\gamma(t)$ to be the lines passing through $F_\gamma(t) \cap F_\gamma(t) \in \mathbb{R}^2 \setminus \Omega$. Then, to obtain a new leading curve, we integrate the unit normal vector field to our new set of leading fronts. ■

Let us now define two new functions on $[0, l]$:

$$S_+^\gamma(t) := \inf \{s > 0;\, \gamma(t) + s\mathbf{N}(t) \notin F_\gamma(t') \text{ if } F_\gamma(t') \neq F_\gamma(t)\} \geq s_+^\gamma\,,$$

and

$$S_-^\gamma(t) := \sup \{s < 0;\, \gamma(t) + s\mathbf{N}(t) \notin F_\gamma(t') \text{ if } F_\gamma(t') \neq F_\gamma(t)\} \leq s_-^\gamma\,.$$

Since γ and $\mathbf{N}$ are continuous and $\Omega(\gamma)$ is convex with regular upper and lower boundaries, $S_\pm^\gamma$ are continuous. We have

$$\frac{1}{S_-^\gamma(t)} \leq \kappa(t) \leq \frac{1}{S_+^\gamma(t)}.$$

Put

$$\varphi_\pm^\gamma := S_\pm^\gamma - s_\pm^\gamma$$

and

$$\varphi_m(t) := \min \{\varphi_+^{\gamma_m}(t), -\varphi_-^{\gamma_m}(t)\}\,.$$

Notice that

$$\mu\left(\varphi^{-1}(0) \setminus \kappa_n^{-1}(0)\right) = 0, \tag{3.9}$$

otherwise the L^2 norm of the second derivatives of u in (3.7) would not be finite.

Proposition 3.1 *There exists a sequence of isometires $u_m \in W^{2,2}(\Omega(\gamma), \mathbb{R}^3)$ converging strongly to u for which $\varphi_m(t) > \rho_m > 0$ for a suitable leading curve γ_m.*

Remark 3.1 *We construct this sequence such that $\gamma_m(0) = \gamma(0)$ and $\gamma'(0) = \gamma_m'(0)$ for the corresponding leading curves. In other words $F_{\gamma_m}(0) = F_\gamma(0)$.*

Proof: Consider $D_m : \mathbb{R}^2 \to \mathbb{R}^2$ to be the dilation in the plane centered at $\mathrm{x}_0 = \gamma(0)$ and defined by

$$D_m(\mathrm{x}) := \frac{m}{m-1}(\mathrm{x} - \mathrm{x}_0) + \mathrm{x}_0.$$

Also consider the dilation in $\mathbb{R}^3$ centered at $\mathrm{y}_0 = u(\mathrm{x}_0)$ and defined by

$$\tilde{D}_m(\mathrm{y}) := \frac{m}{m-1}(\mathrm{y} - \mathrm{y}_0) + \mathrm{y}.$$

Put $\Omega_m := D_m(\Omega(\gamma))$. We define the sequence $\tilde{u}_m : \Omega_m \to \mathbb{R}^3$ as follows

$$\tilde{u}_m := \tilde{D}_m \circ u \circ D_m^{-1}$$

Note that since $\Omega(\gamma)$ lies entirely in one side of $F_\gamma(0)$, $\Omega(\gamma) \subset \Omega_m$, so $\tilde{u}_m$ is well defined over $\Omega(\gamma)$. The sequence $\tilde{u}_m$ converges strongly to u in $W^{2,2}(\Omega(\gamma), \mathbb{R}^3)$ and it is an isometric immersion. We should modify it slightly to satisfy the condition $\varphi_m > 0$. Note that the curve

$$\gamma_m(t) := D_m \circ \gamma(\frac{m-1}{m}t)$$

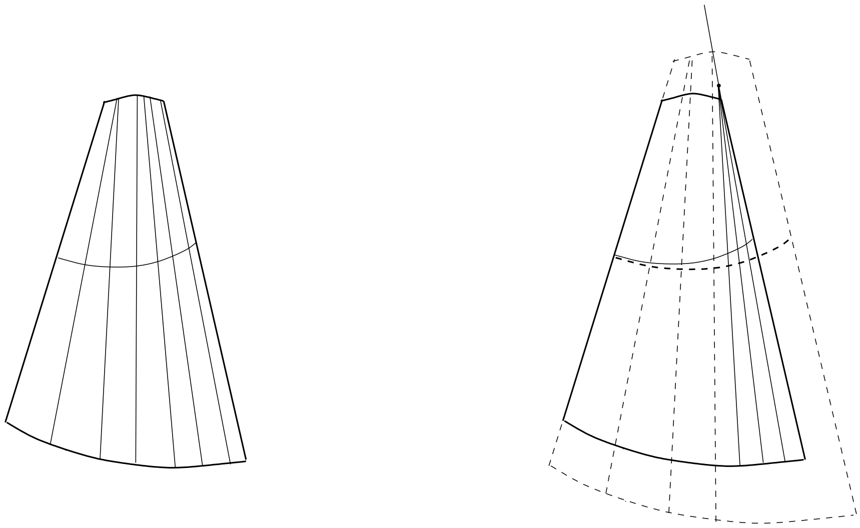

Figure 2: Dilation of $\Omega(\gamma)$

defined on $[0, \frac{m}{m-1}l]$ is a leading curve for $\tilde{u}_m$. Put

$$l_m^* := \sup\left\{t;\ \gamma_m(t) \in \Omega(\gamma) \text{ and } F_{\gamma_m}(t) \cap F_\gamma(l) \cap \overline{\Omega}(\gamma) = \emptyset\right\}$$

and define $u_m : \Omega(\gamma) \to \mathbb{R}^3$ by

$$u_m(\mathrm{x}) := \begin{cases} \tilde{u}_m(\mathrm{x}) \\ \text{if} \quad \mathrm{x} \in F_{\gamma_m}(t) \quad \text{for some } t \leq l_m^* - \dfrac{1}{m} \\ \\ \nabla\tilde{u}_m\left(\gamma_m(l_m^* - \dfrac{1}{m})\right)\left(\mathrm{x} - \gamma_m(l_m^* - \dfrac{1}{m})\right) + \tilde{u}_m\left(\gamma_m(l_m^* - \dfrac{1}{m})\right) \\ \text{otherwise} \end{cases}$$

The maps u_m are well defined since by (3.6) $\nabla\tilde{u}_m$ is constant on $F_{\gamma_m}(l_m^* - \frac{1}{m})$. Note that u_m is an extension over $\Omega(\gamma)$ of the restriction of $\tilde{u}_m$ to $\bigcup_{t \leq l_m^* - \frac{1}{m}} F_{\gamma_m}(t) \cap \Omega(\gamma)$ by an affine map. The sequence u_m is in $I^{2,2}(\Omega(\gamma), \mathbb{R}^3)$ and converges strongly to u since $l_m^* \to l$ for $m \to \infty$.

The first observation is that $\varphi_m(t) > 0$ for $t \in [0, l_m^* - \frac{1}{m}]$. Note that the vertex of the dilation x_0 belongs to Ω. Thus because of the convexity of $\Omega(\gamma)$ no portion of $\partial\Omega$ can be mapped to itself under D_m. Therefore, since we have done an expansion, the focal points of the curve γ_m have positive distance from $\partial\Omega$ for these values of t as it is illustrated in Figure 3.2. Remember that according to Lemma 3.2 the focal points of γ are out of Ω, at worst on its boundary. If necessary we modify γ_m as in Lemma 3.2 and rearrange the leading fronts for $t > l_m^* - \frac{1}{m}$ in such a way that $\varphi_m(t) > 0$ for all t. ■

Proposition 3.2 *Assume that $\varphi(t) > \rho > 0$ on $[0,l]$. Then we can construct a sequence of smooth maps in $I^{2,2}(\Omega(\gamma), \mathbb{R}^3)$ converging strongly to u.*

Proof: Let $u : \Omega(\gamma) \to \mathbb{R}^3$ be such that $\varphi(t) > \rho > 0$. We choose uniformly bounded continuous functions $\frac{1}{S_-^\gamma} \leq \kappa_m \leq \frac{1}{S_+^\gamma}$ which converge to κ almost everywhere in $[0,l]$. We define γ_m to be the curve with curvature κ_m and with initial conditions $\gamma_m(0) = \gamma(0)$ and $\gamma_m'(0) = \gamma'(0)$. This curve is uniquely defined and smooth. A simple observation using the Poincaré inequality for intervals shows that $\gamma_m \to \gamma$ in $W^{2,p}$ for every $p < \infty$. Put

$$\begin{cases} l_m^* := l & \text{if} \quad \Omega(\gamma) \subset \Omega(\gamma_m) \\ \\ l_m^* := \sup\left\{t;\ F_{\gamma_m}(t) \cap F_\gamma(l) \cap \overline{\Omega}(\gamma) = \emptyset\right\} & \text{otherwise.} \end{cases}$$

Note that $l_m^* \to l$ as $m \to \infty$. Also, since $\mathbf{N}_m$ converges uniformly to $\mathbf{N}$ and $\Omega(\gamma)$ is convex, $S_\pm^{\gamma_m}$ and $s_\pm^{\gamma_m}$ converge uniformly to $S_\pm^\gamma$ and $s_\pm^\gamma$ respectively.

To define the curves $\tilde{\gamma}_m$, first we choose a suitable converging sequence of smooth functions $g_m \in L^2([0,l])$ such that

$$g_m \longrightarrow \kappa_n \quad \text{a.e. in} \quad [0,l].$$

Recall that by (3.7) $\kappa_n \in L^2([0,1])$ so this is possible. Let ψ be any smooth positive function which is 0 on $[-1,\infty)$ and 1 out of $(-2,\infty)$. We put

$$\kappa_{n;m}(t) := \psi\left(m(t - l_m^*)\right)\sqrt{\frac{\varphi_m}{\varphi}(t)}g_m(t), \quad t \in [0,l]$$

and we solve the following linear system for the initial values $\mathbf{t}_m(0) = \mathbf{t}(0)$, $\mathbf{v}_m(0) = \mathbf{v}(0)$ and $\mathbf{n}_m(0) = \mathbf{n}(0)$:

$$\begin{pmatrix} \mathbf{t}_m' \\ \mathbf{v}_m' \\ \mathbf{n}_m' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_m & \kappa_{n;m} \\ -\kappa_m & 0 & 0 \\ -\kappa_{n;m} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}_m \\ \mathbf{v}_m \\ \mathbf{n}_m \end{pmatrix} \tag{3.10}$$

The solution is a unique moving orthonormal frame in $\mathbb{R}^3$ because the matrix is skew-symmetric. The curve $\tilde{\gamma}_m : [0,l] \to \mathbb{R}^3$, whose Darboux frame is $(\mathbf{t}_m, \mathbf{v}_m, \mathbf{n}_m)$, is defined by the following equation

$$\tilde{\gamma}_m' = \mathbf{t}_m, \quad \tilde{\gamma}_m(0) = \tilde{\gamma}(0)$$

and is smooth because of the smoothness of $\kappa_{n;m}$ and κ.

We can now proceed to define our approximating sequence u_m on $\Omega(\gamma)$. For $(t,s) \in \Omega(l, s_-^{\gamma_m}, s_+^{\gamma_m})$ we put

$$u_m(\Phi_m(t,s)) := \tilde{\gamma}_m(t) + s\mathbf{v}_m(t)$$

where $\Phi_m := \Phi_{\gamma_m}$ is as in (3.5).

Proposition 3.3 *u_m is a well defined smooth isometry over $\Omega(\gamma) \cap \Omega(\gamma_m)$ and can be extended by an affine isometry over $\Omega(\gamma)$.*

Proof:

(i) u_m is well defined: If for $t_1 < t_2 < l_m^*$, $\mathrm{x} = \Phi_m(t_1, s_1) = \Phi_m(t_2, s_2)$, we have $t_1, t_2 \in I_{m,0}$ and therefore, by our construction, $\kappa_{n;m} = 0$ on $[t_1, t_2]$ which means that $\tilde{\gamma}_m$ is a flat curve in $[t_1, t_2]$, i.e. an Euclidean transformation of γ_m, and therefore

$$\tilde{\gamma}_m(t_1) + s_1 \mathbf{v}_m(t_1) = \tilde{\gamma}_m(t_2) + s_2 \mathbf{v}_m(t_2).$$

Note that in this case u_m is an affine isometry in a neighborhood of x. So far we have proved that u_m is well defined over the region covered by $\gamma_m|_{[0,l_m^*]}$. Note that $\kappa_{n;m} = 0$ also for $t > l_m^* - \frac{1}{m}$, hence u_m is affine on the region covered by $\gamma_m|_{[l_m^* - \frac{1}{m}, l]}$ and as a consequence we can extend it by the same map over $\Omega(\gamma)$.

(ii) u_m is an isometry: It is true since the equations of (3.4) are satisfied for γ_m and $\tilde{\gamma}_m$.

(iii) u_m is smooth: Observe that

$$det(\nabla \Phi_m)(t, s) = 1 - s\kappa_m(t).$$

If $\mathrm{x} = \Phi_m(t, s)$ is the crossing point of two leading fronts of γ_m, as above, u_m would be affine around x and so smooth. Otherwise, $det(\nabla \Phi_m)(t, s) \neq 0$ and therefore Φ_m has a smooth inverse around the point, which completes the proof. ■

To finish the proof of Proposition 3.2 we prove the following claim:

Proposition 3.4 *The sequence $u_m : \Omega(\gamma) \to \mathbb{R}^3$ converges strongly in $W^{2,2}(\Omega(\gamma), \mathbb{R}^3)$ to u.*

Proof: The first observation, using (3.7), is that the sequence u_m is uniformly bounded in $W^{1,2}(\Omega, \mathbb{R}^3)$. We estimate the L^2-norm of $|\nabla^2 u_m|$. We have for $(t, s) \in \Omega(l, s_+^{\gamma_m}, s_-^{\gamma_m})$

$$\frac{\kappa_{n;m}^2(t)}{1 - s\kappa_m(t)} \leq \begin{cases} 2g_m^2(t) & \text{if} \quad \frac{1}{2S_-^\gamma(t)} \leq \kappa_m(t) \leq \frac{1}{2S_+^\gamma(t)} \\ \\ \dfrac{g_m^2(t)}{|\kappa_m|\varphi} \leq \min\{S_+^\gamma(t), -S_-^\gamma(t)\} \dfrac{g_m^2(t)}{\rho} & \text{otherwise} \end{cases} \tag{3.11}$$

Now note that $\kappa_{n;m} \to \kappa_n$ a.e. in $[0, l]$ since

$$\varphi_m(t) \to \varphi(t) > \rho > 0$$

and

$$\psi\left(m(t - l_m^*)\right) \to 1 \quad (\text{since } l_m^* \to l).$$

By the convergence of g_m in L^2 and (3.11), the sequence $\kappa_{n;m}^2(t)/(1 - s\kappa_m(t))$ is bounded by a sequence of maps converging in $L^2([0, l])$. Using the Lebesgue dominant convergence Theorem and (3.7), and considering that $\Omega(l, s_+^{\gamma_m}, s_-^{\gamma_m})$ is converging to $\Omega(l, s_+^\gamma, s_-^\gamma)$ in measure, we obtain

$$\int_{\Omega(\gamma)} |\nabla^2 u_m|^2 d\mathrm{x} \longrightarrow \int_{\Omega(\gamma)} |\nabla^2 u|^2 d\mathrm{x}. \tag{3.12}$$

As a consequence, there is a subsequence of u_m converging weakly in $W^{2,2}(\Omega(\gamma), \mathbb{R}^3)$ to some map $\tilde{u} : \Omega(\gamma) \to \mathbb{R}^3$. By Sobolev embedding Theorem, the convergence is strong in $W^{1,p}$ for some $p > 2$, which means that u_m should converge to $\tilde{u}$ uniformly. Hence, $u_m(\Phi_m(t,s))$ converges to $\tilde{u}(\Phi(t,s))$ almost everywhere. Meanwhile

$$u_m(\Phi_m(t,s)) = \tilde{\gamma}_m(t) + s\mathbf{v}_m(t) \longrightarrow \tilde{\gamma}(t) + s\mathbf{v}(t) = u(\Phi(t,s)).$$

Therefore $\tilde{u} = u$ and by (3.12) the convergence of u_m to u in $W^{2,2}(\Omega(\gamma))$ is strong. ■

Combining Propositions 3.1 and 3.2 we get a smooth approximation sequence for any map $u \in W^{2,2}(\Omega(\gamma), \mathbb{R}^3)$.

3.3 Proof of Theorem I

We say a connected maximal sub-domain on which u is affine is a *body* if its boundary contains more than two segments inside Ω. Respectively an *arm* is a maximal subdomain covered by some leading curve γ. By Theorem II, Ω is partitioned into bodies and arms.

Lemma 3.3 *To prove Theorem I we can assume the number of bodies to be finite.*

Proof: In cases where the above condition is not satisfied, the number of the vertices of bodies on $\partial\Omega$ would be infinite. This yields that for all the bodies except for a finite number, we can extend the map by affine extension to the boundary without changing much the $W^{2,2}$ norm of u. Hence the maps with finite number of bodies are strongly dense in $I^{2,2}(\Omega, \mathbb{R}^3)$ and our assumption is justified. ■

Lemma 3.4 *Assume that the number of bodies is finite. Then to prove Theorem I we can assume that the complement of the union of the bodies is covered by a finite number of arms.*

Proof : We can write the complement of bodies in Ω as a finite union of its connected components $\bigcup_{j=1}^N \Delta_j$. If the region $\Delta = \overline{\Delta_j}$ is between two bodies A and B, we can consider it as a union of segments joining $\partial\Omega$ at their both sides in whose directions u is affine. We denote these segments by the term *leading segments*. The middle points of the leading segments form a Lipschitz curve parameterized by arclength $\gamma_0 : [0,T] \to \Omega$ which joins the segment $A \cap \Delta$ to the segment $B \cap \Delta$. Note that γ_0 may not be a leading curve. We consider the unit vector field $\mathbf{v}$ in Δ which is orthogonal to these covering segments everywhere and is pointing towards B. Let $\mathrm{x}_0 = \gamma_0(0)$. For given $\mathrm{x}_{i-1} \in \Omega$, we define the curve $\gamma_i : [0, T_i] \to \overline{\Omega}$ to be the maximal solution of

$$\gamma_i'(t) = \mathbf{v}(\gamma_i(t)); \quad \gamma_i(0) = \mathrm{x}_{i-1}$$

and we define x_i to be the middle point of a leading segment passing through $\gamma_i(T_i)$.

We claim that a finite number of γ_i is sufficient to cover all Δ. Otherwise, for every i, $\gamma_i(T_i)$ does not belong to $B \cap \Delta$. Note that we can prolong γ_i inside Δ as far as it does

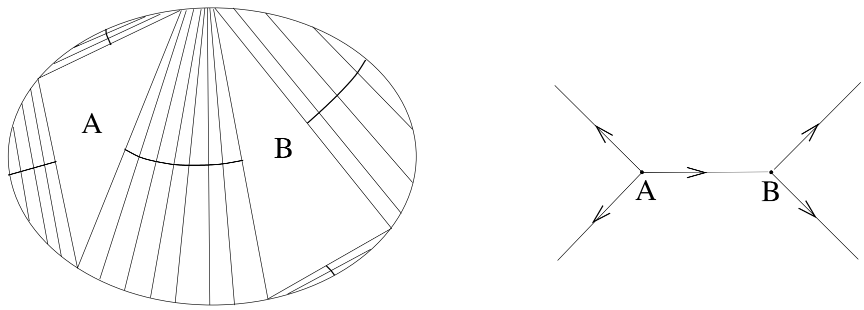

Figure 3: Bodies A and B in Ω and the graph obtained by their retraction

not touch $\partial\Delta$. As a consequence T_{i+1} would be at least equal to the distance of x_i and $\partial\Omega$ which is uniformly greater than some positive constant $\rho > 0$.

However, following the same calculations as in the previous section we have

$$|\Omega(\gamma_i)| = \int_{\Omega(\gamma_i)} d\mathrm{x} = \int_0^{T_i} \int_{s_-^{\gamma_i}(t)}^{s_+^{\gamma_i}(t)} (1 - s\kappa_i(t))\, dtds \geq \frac{\rho l}{2}$$

where l is the minimum length of the leading segments. The claim is proved since the area of Δ is not infinite.

Indeed it is sufficient to prove the approximation Theorem I for when all Δ_i are as above. Otherwise, if the region $\Delta = \Delta_j$ has common boundary with at most one body, we approximate the map u by a sequence of maps whose corresponding Δ_i's satisfy always the above condition: Consider again the curve $\gamma_0 : [0, T] \to \Delta$ as above. For $t \in [0, T]$ we denote the leading segment passing through $\gamma_0(t)$ by $E(t)$. For $\varepsilon > 0$, we modify u by affine extension over the region $\bigcup_{t<\varepsilon} E(t) \cup \bigcup_{t>T-\varepsilon} E(t)$. The modified maps converge in $W^{2,2}$ norm to u and by what preceded each of them divides Ω into a finite number of bodies and arms. ■

Now, since Ω is convex it is simply-connected. We claim that two bodies are connected only through one chain of bodies and arms: It suffices to consider the graph obtained by retracting bodies to vertices and arms to edges. This graph is simply-connected because it is a deformation retract of Ω. Therefore every two vertices are connected through only one chain of edges, which proves the claim (Figure 3). This helps us to construct our approximation maps on whole Ω without any problem: We begin by a central body A and define our approximation sequence on its arms as in previous section. Note that by our construction the approximation sequence is affine near the free hands of our arms. If necessary we continue our approximation on the chain of arms getting out of one side of A until we reach to another body B. So in each step we fix the boundary value of the map in one side of the arm and the value at the other side is given by an affine map. This assures us that the new maps are smooth at these joints. When we reach to body B, all we need is to modify the map on B by an affine transformation in $\mathbb{R}^3$ such that it equals the affine map on the free hand of the last arm in the chain and that we can perfectly match the two values on the connecting edge. Then we continue our construction using B as a new point of depart. Note that we will never come back to the body A or any of its arms because of what we said in the beginning of the paragraph, hence the construction is consistent. Since the operation is finished in finitely many steps, we can repeat the arguments of the last section on the whole Ω and prove the theorem. ■

The proof for the second part of the theorem follows the same lines.

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