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by

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Positive Solutions of Linear Elliptic Equations with Critical Growth in the Neumann Boundary Condition

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Abstract. We study the existence of positive solutions of a linear elliptic equation with critical Sobolev exponent in a nonlinear Neumann boundary condition. We prove a result which is similar to a classical result of Brezis and Nirenberg who considered a corresponding problem with nonlinearity in the equation. Our proof of the fact that the dimension three is critical uses a new Pohožaev-type identity.
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1 Introduction

We investigate the following nonlinear Neumann boundary value problem

$$\begin{aligned} \Delta u + \lambda u &= 0, \quad u > 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ -\frac{\partial u}{\partial x_n} &= u^{\frac{n}{n-2}} && \text{on } \Gamma_N, \end{aligned} \tag{P$_\lambda$}$$

where $n > 2$, $\Omega = B_1^+(0)$ is the half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$, $\Gamma_D = \{x \in \mathbb{R}^n : |x| = 1, x_n \geq 0\}$ and $\Gamma_N = \{x \in \mathbb{R}^n : |x| < 1, x_n = 0\}$. Solutions are understood in the weak $W^{1,2}$-sense. Due to the exponent $\frac{n}{n-2}$ in the boundary condition, the problem is of critical Sobolev-type since the embedding from $W^{1,2}(\Omega)$ into $L^{2^*}(\Gamma_N)$ with $2^* = \frac{2n-2}{n-2}$ is continuous but no longer compact. Thus the variational methods for existence of solutions are not directly applicable. However, due to the presence of the linear term λu, compactness is partially recovered. This was first noticed in a celebrated paper by Brezis and Nirenberg [2] for a corresponding Dirichlet boundary value problem with critical nonlinearity in the equation. For the nonlinear Neumann boundary value problem (P_λ) we have the following result.

Theorem 1. *Let λ_1 be the first eigenvalue of the problem*

$$\Delta\phi + \lambda\phi = 0 \quad in \quad B_1^+(0),$$
$$\frac{\partial\phi}{\partial\nu} = 0 \quad on \quad \Gamma_N, \qquad \phi = 0 \quad on \quad \Gamma_D.$$

(i) For $n \geq 4$ a solution u_λ of (P_λ) exists if and only if $\lambda \in (0, \lambda_1)$.

(ii) For $n = 3$ there is no solution of (P_λ) for $\lambda \geq \lambda_1 = \pi^2$ while for $\pi^2/4 < \lambda < \pi^2$ solutions exist. There is a value $\lambda_* \in (0, \pi^2/4)$ such that no solution exists for $-\infty < \lambda < \lambda_*$. A lower bound for λ_* is 0.772.

The above result is analogous to the Brezis-Nirenberg result [2] for the boundary value problem $\Delta u + \lambda u + u^{\frac{n+2}{n-2}} = 0$ with Dirichlet boundary conditions. There, existence in the full interval $(0, \lambda_1)$ was established for space-dimensions $n \geq 4$. But $n = 3$ was exceptional in the sense, that nonexistence for small positive values of λ occurs on star-shaped domains. Hence $n = 3$ is a *critical* dimension (in the sense of Pucci and Serrin, see [7]) both for the Brezis-Nirenberg problem and for (P_λ).

In the case of a three-dimensional ball Brezis and Nirenberg established $\pi^2/4$ as the *exact* threshold for nonexistence/existence, where the nonexistence was established via a Pohožaev-type identity. This was possible, since by the radial symmetry result of Gidas, Ni, Nirenberg [5] all solutions are radially symmetric and the problem effectively reduces to one dimension, which enabled the derivation of an optimal Pohožaev identity. In our case we make use of the half-ball geometry. The solutions possess the cylindrical symmetry which reduces the problem effectively to two dimensions. Our non-existence result is also proved via a Pohožaev-type identity, cf. Section 4. However, in contrast to the Brezis-Nirenberg situation, our problem is genuinely two-dimensional, which makes the use of the Pohožaev identity harder, and new ideas needed to be developed. It is an open problem to determine the sharp nonexistence/existence threshold.

Equations with critical growth in the Neumann boundary data have been previously investigated by Adimurthi and Yadava [1]. Their results were stated for boundary conditions of the type $\partial_\nu u + \frac{n-2}{n}\beta(x)u = u^{\frac{n}{n-2}}$ on Γ_N. In particular they obtained Part *(i)* of Theorem 1 for space-dimensions $n \geq 5$ but not for the interesting low dimensions $n = 3, 4$. Adimurthi and Yadava also studied another version of (P_λ), namely

$$\begin{aligned} \Delta u &= 0, \quad u > 0 \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \Gamma_D, \\ -\frac{\partial u}{\partial x_n} &= u^{\frac{n}{n-2}} + \mu u \quad \text{on } \Gamma_N, \end{aligned} \tag{Q_μ}$$

where the linear perturbation has been moved into the nonlinear Neumann boundary condition. If we denote by μ_1 the first Steklov eigenvalue of the corresponding linear problem, then in this case existence of positive solutions holds precisely for $0 < \mu < \mu_1$ in *any* space dimension $n \geq 3$. In particular the three dimensional case is no longer critical. Although (P_λ) and (Q_μ) are linear perturbations of the same nonlinear problem, their behavior in dimension $n = 3$ is very different. We will give an explanation for this surprising feature at the end of Section 2.

Adimurthi and Yadava worked on smooth domains where the boundary was split between the Dirichlet boundary Γ_D and the Neumann boundary Γ_N. Their solutions are typically weak $W^{1,2}$-solutions. In contrast, we make use of the half-ball geometry which has two advantages: (i) the solutions are $C^{2,\alpha}(\overline{\Omega})$, and (ii) due to this regularity we have the Pohožaev identity to investigate the nonexistence regions in more detail.

The paper is organized as follows. In Section 2 we prove existence-results for (P_λ) on domains, which generalize the half-balls $B_1^+(0)$. In Section 3 we establish regularity and symmetry results for the solutions of (P_λ) on half-balls. Nonexistence in dimension $n = 3$ for half-balls is proved via a Pohožaev-type identity in Section 4. Finally, in Section 5 we study the λ-behavior of various norms of u_λ.

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2 Existence of positive solutions

In this section we use a variational approach to obtain positive weak solutions of (P_λ). The method goes back to the well known and fundamental work of Brezis, Nirenberg [2]. We will consider more general domains than half-balls. We therefore introduce the following notation: Let $\omega \subset \mathbb{R}^n$ be a bounded domain, whose boundary is decomposed into $\gamma_D \neq \emptyset$ and $\gamma_N \neq \emptyset$. We assume that γ_N is part of a hyperplane (γ_N is then called *flat*), and without loss of generality we suppose that $\gamma_N \subset \partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n = 0\}$ and that $0 \in \text{int}\,(\gamma_N)$, i.e., that the origin lies in the relative interior of γ_N.

The space $W^{1,2}(\omega)$ is defined as the norm closure of the C^∞-functions vanishing on γ_D. As the standard norm we use $\|\nabla u\|_{2,\omega} := \left(\int_\omega |\nabla u|^2\,dx\right)^{1/2}$. From now on we will consider weak $W^{1,2}(\omega)$-solutions of (P_λ) on the set ω. Moreover, we will use the norms $\|u\|_{q,\gamma_N} = (\int_{\gamma_N} |u|^q\,d\sigma)^{1/q}$ and $\|u\|_{2,\omega} = (\int_\omega u^2\,dx)^{1/2}$. We denote by S the best Sobolev constant of the embedding $W^{1,2}(\omega)$ into $L^{2^*}(\gamma_N)$, i.e.,

$$S = \inf_{\psi \in W^{1,2}(\Omega)} \frac{\|\nabla u\|^2_{2,\omega}}{\|u\|^2_{2^*,\gamma_N}}. \tag{1}$$

The constant S is independent of ω and γ_N, and it is not attained on any bounded domain ω. Instead, any positive solution of the half-space problem $\Delta u = 0$ in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ with $\partial u/\partial\nu = u^{\frac{n}{n-2}}$ on $\partial\mathbb{R}^n_+$ is a minimizer. It is given by $u(x) = \alpha^{\frac{n-2}{2}}|x - x_0|^{2-n}$ with x_0 in the lower half-space and $\alpha = (n-2)|x_{0,n}|$. Evaluation of S gives

$$S = (n-2)\sigma_{n-1}^{1/(n-1)} \left(\int_0^\infty r^{n-2}(r^2+1)^{1-n} dr\right)^{1/(n-1)},$$

where σ_{n-1} is the surface-area of the $(n-1)$-dimensional unit sphere.

By λ_1 and ϕ_1 we denote the first eigenvalue and eigenfunction of the problem

$$\Delta\phi + \lambda\phi = 0 \quad \text{in} \quad \omega,$$
$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \gamma_N, \quad \phi = 0 \quad \text{on} \quad \gamma_D.$$

The following theorem is an existence theorem for domains ω where γ_N is flat. It includes in particular the existence parts of *(i)* and *(ii)* in Theorem 1.

Theorem 2. *Let ω be a bounded domain where $\gamma_N \subset \partial\mathbb{R}^n_+$ is flat. Let λ_1 be the first eigenvalue for the linear eigenvalue problem corresponding to* (P_λ) *and let*

$$\lambda^* := \inf_{\substack{u \in W^{1,2}(\omega) \\ u \not\equiv 0}} J(u) \quad \text{with} \quad J(u) := \frac{\|\nabla u\|^2_{2,\omega} - S\|u\|^2_{2^*,\gamma_N}}{\|u\|^2_{2,\omega}}.$$

Then $\lambda^* \in [0, \lambda_1)$ and (P_λ) *has a solution for all $\lambda \in (\lambda^*, \lambda_1)$. Furthermore:*

(i) For $n \geq 4$ we have $\lambda^* = 0$, i.e. a solution u_λ of (P_λ) *exists for all $\lambda \in (0, \lambda_1)$.*

(ii) For $n = 3$ an upper bound for λ^* is given by

$$\lambda^* \leq \tilde{\lambda} := \inf_{x_0 \in \gamma_N} \inf_{\phi \in \mathcal{M}_{x_0}} \int_\omega \frac{|\nabla\phi(x)|^2}{|x - x_0|^2}\,dx \Big/ \int_\omega \frac{\phi(x)^2}{|x - x_0|^2}\,dx$$

where

$$\mathcal{M}_{x_0} = \{\phi \in C^\infty(\mathbb{R}^n) : \phi \equiv 0 \text{ on } \gamma_D,\ \phi_{x_1}(x_0) = \phi_{x_2}(x_0) = 0\}.$$

Hence a solution of (P_λ) *exists for $\lambda^* < \lambda < \lambda_1$.*

Corollary 3. *If $n = 3$ and $\Omega = B_1^+(0)$ then a solution u_λ exists for $\lambda \in (\pi^2/4, \pi^2)$.*

Proof. It is sufficient to choose $x_0 = 0$, $\phi(x) = \cos(\pi|x|/2)$ and then $\tilde{\lambda} \leq \pi^2/4$. □

For the proof of Theorem 2 we consider the following minimization problem

$$S_\lambda^P := \inf_{W^{1,2}(\omega)} \frac{\|\nabla u\|_{2,\omega}^2 - \lambda \|u\|_{2,\omega}^2}{\|u\|_{2^*,\gamma_N}^2}.$$

The definition of λ^* implies that

$$\lambda^* = \sup\Big\{\lambda > 0 \ : \ \|\nabla u\|_{2,\omega}^2 \geq S\|u\|_{2^*,\gamma_N}^2 + \lambda\|u\|_{2,\omega}^2 \quad \text{for all } u \in W^{1,2}(\omega)\Big\}.$$

It easily follows that $S_\lambda^P < S$ for $\lambda > \lambda^*$, and $S_\lambda^P = S$ for $0 \leq \lambda \leq \lambda^*$. Hence, $\lambda \in (\lambda^*, \lambda_1)$ is the case when $0 < S_\lambda^P < S$. A well known consequence of P.L. Lions' concentration-compactness alternative [8] is, that in this case a minimizer exists, and it is a solution of the corresponding Euler-Lagrange equation (P_λ).

Our task is to obtain further information on λ^* depending on the space dimension. We do this by considering the functional J on suitably chosen test-functions. We define $U_\delta(x) = \delta^{\frac{n-2}{2}}|(x, x_n + \delta)|^{2-n}$. Clearly U_δ is a function for which S in (1) is attained. The following constants depend only on n and not on δ:

$$L_1 := \|\nabla U_\delta\|_{2,\mathbb{R}_+^n}^2 = \|U_\delta\|_{2^*,\partial\mathbb{R}_+^n}^{2^*}, \quad L_2 := (L_1)^{2/2^*}.$$

The best Sobolev constant then satisfies $S = L_1/L_2$. Our test function will be

$$v_\delta := U_\delta \phi$$

where ϕ is a suitable cut-off function which will be chosen such that $v_\delta \in W^{1,2}(\omega)$. This choice implies the following lemma that will be proved at the end of this section.

Lemma 4. *For any fixed smooth function $\phi : \bar{\omega} \to \mathbb{R}$ with $\phi \equiv 0$ on γ_D, $\phi(0) = 1$, $D_{x'}^\alpha \phi(0) = 0$ for $1 \leq |\alpha| \leq n-2$ we have the following estimates as $\delta \to 0$:*

$$\|v_\delta\|_{2^*,\gamma_N}^2 = L_2 + O(\delta^{n-1}|\log\delta|) \tag{2}$$

$$\|\nabla v_\delta\|_{2,\omega}^2 = L_1 + \delta^{n-2}\int_\omega \frac{|\nabla\phi|^2}{|(x', x_n + \delta)|^{2n-4}}\,dx + O(\delta^{n-1}|\log\delta|). \tag{3}$$

If we insert v_δ into the functional J we obtain as a consequence of Lemma 4 that

$$\lambda^* \leq J(v_\delta) = \int_\omega \frac{|\nabla\phi|^2}{|(x', x_n + \delta)|^{2n-4}}\,dx \Big/ \int_\omega \frac{\phi^2}{|(x', x_n + \delta)|^{2n-4}}\,dx + o(1) \tag{4}$$

as $\delta \to 0$. Next we follow an observation of Janelli [7]. As $\delta \to 0$ clearly

$$|(x', x_n + \delta)|^{2-n} \to |x|^{2-n} \qquad \text{a.e. in } \omega.$$

In order to take the limit $\delta \to 0$ in (4), we must decide whether the fundamental solution of the Laplacian $F(x, 0) = |x|^{2-n}$ is in L^2_{loc} or not.

First case: $F(x, 0) \notin L^2_{\text{loc}}$. This is the case for $n \geq 4$. We refine our choice of ϕ by requiring that $\phi \equiv 1$ in a neighborhood of $x = 0$. Therefore

$$\int_\omega |\nabla \phi|^2 |(x', x_n + \delta)|^{4-2n}\, dx < \infty \quad \text{as} \quad \delta \to 0,$$

while

$$\int_\omega \phi^2 |(x', x_n + \delta)|^{4-2n}\, dx \to \infty \quad \text{as} \quad \delta \to 0.$$

We conclude from the fact that $\lambda^* \geq 0$ and from (4) that $\lambda^* = 0$. Therefore we have proved part *(i)* of Theorem 1.

Second case: $F(x, 0) \in L^2_{\text{loc}}$. This is the case for $n = 3$. This time we can take the limit in (4) and obtain

$$0 \leq \lambda^* \leq \int_\omega \frac{|\nabla \phi|^2}{|x|^{2n-4}}\, dx \bigg/ \int_\omega \frac{\phi^2}{|x|^{2n-4}}\, dx.$$

Next we decide on the optimal choice of the cut-off function ϕ. If we realize that the choice of the origin was arbitrary in so far that it only had to be on γ_N, we obtain the final characterization of $\tilde{\lambda}$ as an upper bound for λ^*. This establishes Part *(ii)* of Theorem 2.

Proof of Lemma 4. (2): Let $D_\rho \subset \partial \mathbb{R}^n_+$ denote the $(n-1)$-dimensional ball of radius ρ centered at 0. We consider $v_\delta^{2^*}(x', 0) = \delta^{n-1} |(x', \delta)|^{2(1-n)} \phi^{2^*}(x', 0)$.

$$\begin{aligned}
&\|v_\delta\|^{2^*}_{2^*, \gamma_N} \\
&= \delta^{n-1} \int_{D_\rho} \frac{\phi^{2^*}(x', 0)}{|(x', \delta)|^{2(n-1)}}\, dx' + O(\delta^{n-1}) \\
&= \delta^{n-1} \Big(\int_{D_\rho} \frac{dx'}{|(x', \delta)|^{2(n-1)}} + \int_{D_\rho} \frac{\phi^{2^*}(x', 0) - 1}{|(x', \delta)|^{2(n-1)}}\, dx' \Big) + O(\delta^{n-1}) \\
&= \int_{D_{\rho/\delta}} \frac{dy'}{|(y', 1)|^{2(n-1)}} + \int_{D_{\rho/\delta}} \frac{\phi^{2^*}(\delta y', 0) - 1}{|(y', 1)|^{2(n-1)}}\, dy' + O(\delta^{n-1}) \\
&= L_1 - \int_{\partial \mathbb{R}^n_+ \setminus D_{\rho/\delta}} \frac{dy'}{|(y', 1)|^{2(n-1)}} + \int_{D_{\rho/\delta}} \frac{\phi^{2^*}(\delta y', 0) - 1}{|(y', 1)|^{2(n-1)}}\, dy' + O(\delta^{n-1}).
\end{aligned}$$

The first integral behaves like $\int_{\rho/\delta}^\infty r^{-n}\, dr$ and is therefore $O(\delta^{n-1})$. For the second integral we may use $D^\alpha_{x'} \phi(0) = 0$ for $1 \leq |\alpha| \leq n - 2$ to estimate it by $\delta^{n-1}\int_0^{\rho/\delta} r^{2n-3}(1+r^2)^{1-n}\,dr$, which equals $O(\delta^{n-1}) + \int_1^{\rho/\delta}\delta^{n-1}r^{-1}\,dr = O(\delta^{n-1}|\log\delta|)$. This implies (2).

(3): Clearly

$$|\nabla v_\delta|^2 = \frac{\delta^{n-2}}{|(x',x_n+\delta)|^{2n-4}}|\nabla\phi|^2 + \frac{\delta^{n-2}}{|(x',x_n+\delta)|^{2n-2}}\phi^2 + \frac{1}{2}\nabla(\phi^2)\cdot\nabla(U_\delta^2)$$

and hence by the divergence theorem and by the identity $\Delta(U_\delta^2) = 2|\nabla U_\delta|^2 = 2\delta^{n-2}|(x',x_n+\delta)|^{2-2n}$ we see that

$$\begin{aligned}\int_\omega |\nabla v_\delta|^2\,dx &= \int_\omega \frac{\delta^{n-2}}{|(x',x_n+\delta)|^{2n-4}}|\nabla\phi|^2\,dx + \frac{1}{2}\int_{\gamma_N}\phi^2\partial_\nu(U_\delta^2)\,dx' \\ &= \int_\omega \frac{\delta^{n-2}}{|(x',x_n+\delta)|^{2n-4}}|\nabla\phi|^2\,dx + \delta^{n-1}\int_{\gamma_N}\frac{\phi^2}{|(x',\delta)|^{2n-2}}\,dx'.\end{aligned}$$

Now the last integral is the same as in (2) apart from the fact that ϕ^{2^*} is replaced by ϕ^2, which makes no difference in the calculation. Therefore we find

$$\int_\omega |\nabla v_\delta|^2\,dx = L_1 + \int_\omega \frac{\delta^{n-2}}{|(x',x_n+\delta)|^{2n-4}}|\nabla\phi|^2\,dx + O(\delta^{n-1}|\log\delta|),$$

which implies (3). This finishes the proof of Lemma 4. □

We finish this section by comparing the two problems (P_λ) and (Q_μ). Adimurthi and Yadava obtained solutions of (Q_μ) by finding minimizers of

$$S_\mu^Q := \inf_{W^{1,2}(\Omega)} \frac{\|\nabla u\|_{2,\Omega}^2 - \mu\|u\|_{2,\Gamma_N}^2}{\|u\|_{2^*,\Gamma_N}^2}$$

below the critical energy level S. As before the existence-range of μ is given by (μ^*, μ_1) with

$$\mu^* := \inf_{\substack{u\in W^{1,2}(\Omega)\\ u\not\equiv 0}} I(u), \qquad I(u) := \frac{\|\nabla u\|_{2,\Omega}^2 - S\|u\|_{2^*,\Gamma_N}^2}{\|u\|_{2,\Gamma_N}^2}.$$

Repeating the ansatz $v_\delta = U_\delta\phi$ leads with exactly the same calculations as before to the estimate

$$0 \le \mu^* \le I(v_\delta) = \int_\Omega \frac{|\nabla\phi|^2\,dx}{|(x',x_n+\delta)|^{2n-4}} \Big/ \int_{\Gamma_N}\frac{\mu\phi^2\,dx'}{|(x',\delta)|^{2n-4}} + o(1)$$

as $\delta\to 0$. This time we see that the fundamental solution of the Laplacian $F(x,0) = |x|^{2-n}$ is *never* in $L^2_{\mathrm{loc}}(\Gamma_N)$ for any $n\ge 3$. Therefore by choosing $\phi\equiv 1$ in a neighborhood of 0 and letting $\delta\to 0$ we see that $\mu^* = 0$ for any dimension $n\ge 3$. To sum up, we see that the problem (Q_μ) is different from (P_λ) in the sense that the linear perturbation taking place on an $(n-1)$-dimensional hypersurface instead of an n-dimensional domain typically reduces the critical dimension by 1, which in the case of (Q_μ) leads to the fact that there is *no* critical dimension any more.

3 Properties of weak solutions

Lemma 5. *Every weak $W^{1,2}$-solution u_λ of* (P_λ) *is in* $L^\infty(\omega)$ *and in* $L^\infty(\gamma_N)$*.*

The proof is done by the Moser iteration scheme. It is a direct transcription of the standard proof, see for instance Struwe [13], Appendix B, where volume integrals of $u^{\frac{2n}{n-2}}$ are replaced by surface integrals of $u^{\frac{2n-2}{n-2}}$.

In the case where $\Omega = B_1^+(0)$ we can now derive $C^{2,\alpha}$-regularity for weak solutions. This will be important for showing symmetry in Lemma 7 and non-existence in Theorem 10.

Lemma 6. *If* $\Omega = B_1^+(0)$ *then every weak $W^{1,2}$-solution u_λ of* (P_λ) *is in* $C^{2,\alpha}(\overline{B_1^+(0)})$*.*

Proof. Let $I\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ denote the inversion $I(x) = |x|^{-2}x$, let h_x denote the Newton kernel

$$h_x(y) = \frac{1}{(n-2)\sigma_{n-1}} |x-y|^{2-n}, \qquad \sigma_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

For $x \in \mathbb{R}^n \setminus \{0\}$ let h_x^* denote the Kelvin transform of h_x,

$$h_x^* = |x|^{2-n} h_{I(x)}.$$

Now define $\overline{x} = (x_1, x_2, \ldots, x_{n-1}, -x_n)$ for $x = (x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n$ and

$$\Gamma(x,y) = -h_x(y) + h_x^*(y) - h_{\overline{x}}(y) + h_{\overline{x}}^*(y) \quad \text{for} \quad x \neq y \neq \overline{x}, \;\; x, y \in \mathbb{R}^n.$$

Then Γ is Green's function for the problem

$$\begin{aligned} -\Delta u &= f \;\; \text{in } \Omega, \\ u &= 0 \;\; \text{on } \Gamma_D, \\ -\frac{\partial u}{\partial x_n} &= \phi \;\; \text{on } \Gamma_N. \end{aligned} \tag{L}$$

Our aim is to show that if $f \in L^\infty(\Omega)$, $\phi \in L^\infty(\Gamma_N)$ and u is a weak $W^{1,2}$-solution of (L) then u can be represented as follows:

$$u(x) = -\int_\Omega f(y)\Gamma(x,y)\,dy - \int_{\Gamma_N} \phi(y')\Gamma(x,y',0)\,dy' \qquad \text{for a.e. } x \in \Omega.$$

Due to uniqueness of weak $W^{1,2}$-solutions of (L), it is sufficient to show that

$$v(x) := -\int_\Omega f(y)\Gamma(x,y)\,dy, \qquad x \in \mathbb{R}^n,$$

(restricted to Ω) is a weak $W^{1,2}$-solution of the problem

$$\begin{aligned} -\Delta v &= f \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \Gamma_D, \\ -\frac{\partial v}{\partial x_n} &= 0 \quad \text{on } \Gamma_N, \end{aligned}$$

and

$$w(x) := -\int_{\Gamma_N} \phi(y')\Gamma(x, y', 0)\, dy', \qquad x \in \mathbb{R}^n,$$

(restricted to Ω) is a weak $W^{1,2}$-solution of the problem

$$\begin{aligned} -\Delta w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \Gamma_D, \\ -\frac{\partial w}{\partial x_n} &= \phi \quad \text{on } \Gamma_N. \end{aligned}$$

I. (Case of v) Let F be obtained from f by extending it first symmetrically to $B_1(0)$, i.e. $F(x) = f(\overline{x})$ for $\overline{x} \in \Omega$, and then antisymmetrically (with a suitable multiplier) to $\mathbb{R}^n \setminus B_1(0)$, that is

$$F(x) = -|x|^{-n-2} F(I(x)) \quad \text{for } x \in \mathbb{R}^n \setminus B_1(0).$$

Then $F \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $v = F * h_0$ and we have that $v \in C^{1,\beta}(\mathbb{R}^n)$ for any $\beta \in (0,1)$, $v(\overline{x}) = v(x)$ (cf. [3]). Hence, the boundary conditions for v are satisfied in the classical sense. Further, as $F \in L^1(\mathbb{R}^n)$, v is a distributional solution of $-\Delta v = F$ in $\mathbb{R}^n$ (see e.g. Theorem (2.15) in [6]).

To show explicitly that for any $\psi \in C_0^\infty(\mathbb{R}^n \setminus \Gamma_D)$ we have

$$\int_\Omega \nabla v \cdot \nabla \psi \, dx = \int_\Omega f \psi \, dx$$

or even

$$\int_{\mathbb{R}^n_+} \nabla v \cdot \nabla \psi \, dx = \int_{\mathbb{R}^n_+} F \psi \, dx \tag{5}$$

we proceed as follows. Let $R \in (0, \infty)$ be fixed and let $F_0(x) = F(x)$ for $|x| \leq R$ and $F_0(x) = 0$ for $|x| > R$. Further, set $F_\infty(x) = F(x) - F_0(x)$, $v_0 = F_0 * h_0$ and $v_\infty = F_\infty * h_0$. Again, v_∞ is a distributional solution of $-\Delta v_\infty = F_\infty$ and $v_\infty \in C^{1,\beta}(\mathbb{R}^n)$. Clearly, v_∞ is harmonic in $B_R(0)$, $v_\infty(\overline{x}) = v_\infty(x)$, hence, to show that (5) holds for all $\psi \in C_0^\infty(B_R \setminus \Gamma_D)$, it is sufficient to prove

$$\int_{\mathbb{R}^n_+} \nabla v_0 \cdot \nabla \psi \, dx = \int_{\mathbb{R}^n_+} F_0 \psi \, dx \quad \text{for all } \psi \in C_0^\infty(B_R \setminus \Gamma_D). \tag{6}$$

Now standard regularization techniques apply since F_0 is compactly supported. For example, we can take the regularized Newton kernels

$$h^\epsilon(x) = \frac{1}{(n-2)\omega_{n-1}}(|x|^2+\epsilon^2)^{\frac{2-n}{2}},$$

and set $v_0^\epsilon = F_0 * h^\epsilon$. Clearly, the functions v_0^ϵ, $\frac{\partial v_0^\epsilon}{\partial x_j}$ and $\Delta v_0^\epsilon = F_0 * \Delta h^\epsilon$ belong to $C^\infty(\mathbb{R}^n)$. It is easily checked that v_0^ϵ and $\frac{\partial v_0^\epsilon}{\partial x_j}$ converge uniformly on compact sets to v_0 and $\frac{\partial v_0}{\partial x_j}$, respectively. Again, $v_0^\epsilon(\overline{x}) = v_0^\epsilon(x)$ and, in particular, $\frac{\partial v_0^\epsilon}{\partial x_n} = 0$ if $x_n = 0$. To conclude the proof of (6) take $\psi \in C_0^\infty(B_R \setminus \Gamma_D)$ and pass to the limit (as $\epsilon \to 0$) in

$$\int_{\mathbb{R}^n_+} \nabla v_0^\epsilon \cdot \nabla \psi \, dx = \int_{\mathbb{R}^n_+} F_0 * (-\Delta h^\epsilon)\psi \, dx.$$

To do this, observe that $F_0 * (-\Delta h^\epsilon) \to F_0$ in $L^p_{\text{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$ since

$$\Psi^\epsilon := -\Delta h^\epsilon = \frac{n}{\omega_{n-1}}\epsilon^2(|x|^2+\epsilon^2)^{-\frac{n+2}{2}}$$

satisfies

$$\Psi^\epsilon = \epsilon^{-n}\Psi^1\left(\frac{x}{\epsilon}\right) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \Psi^\epsilon \, dx = 1.$$

II. (Case of w) If

$$\Phi(y') := -|y'|^{-n}\phi(I(y',0)) \quad \text{for } y' \in \partial\mathbb{R}^n_+ \setminus \Gamma_N, \quad \Phi(y') = \phi(y') \quad \text{for } y' \in \Gamma_N,$$

then $\Phi \in L^p(\partial\mathbb{R}^n_+)$ for any $1 \le p \le \infty$ and

$$w(x) = 2\int_{\partial\mathbb{R}^n_+} \Phi(y')\Gamma(x,y',0)\, dy', \qquad x \in \mathbb{R}^n.$$

It is known (cf. [3]) that w is bounded and Hölder continuous in $\mathbb{R}^n$, harmonic in $\mathbb{R}^n \setminus \partial\mathbb{R}^n_+$, $w = 0$ for $|x| = 1$ and $\frac{\partial w}{\partial x_n}$ is bounded in $\mathbb{R}^n_+$.

As before, we write a point $x \in \mathbb{R}^n_+$ in the form $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}_+$. Let P_{x_n} denote the Poisson kernel for $\mathbb{R}^n_+$,

$$P_{x_n}(x') = \frac{2}{\omega_{n-1}} \frac{x_n}{(|x'|^2+x_n^2)^{n/2}}.$$

Then

$$-\frac{\partial w}{\partial x_n}(x) = \int_{\partial\mathbb{R}^n_+} \Phi(y')P_{x_n}(x'-y')\, dy', \qquad x \in \mathbb{R}^n_+, \tag{7}$$

$$-\frac{\partial w}{\partial x_j}(x) = \int_{\partial\mathbb{R}^n_+} R_j\Phi(y')P_{x_n}(x'-y')\, dy', \ j = 1,\dots,n-1, \ x \in \mathbb{R}^n_+, \tag{8}$$

here $R_j\Phi$ is the Riesz transform of Φ (in $\partial\mathbb{R}^n_+$),

$$R_j\Phi(y') := \lim_{t\downarrow 0} \frac{2}{\omega_{n-1}} \int_{\partial\mathbb{R}^n_+\setminus B_t(0)} \Phi(y'-x')\frac{x_j}{|x'|^n}\,dx',$$

the convergence is in the L^p–sense and also pointwise almost everywhere.

We will need the known fact from harmonic analysis that

$$R_j \ : \ L^p(\partial\mathbb{R}^n_+) \to L^p(\partial\mathbb{R}^n_+), \qquad 1 < p < \infty,$$

is a bounded linear operator (see [11, Theorems 3, 4, Chapter II], for example). Another known fact we will use is:

$$\|\Phi * P_{x_n} - \Phi\|_{p,\partial\mathbb{R}^n_+} \to 0 \qquad \text{as} \qquad x_n \to 0, \qquad 1 \le p < \infty, \tag{9}$$

see [12], for instance.

Using Hölder's inequality in (7) and (8) we obtain

$$\sup_{x_n>0} \left\| \frac{\partial w}{\partial x_j} \right\|_{2,\partial\mathbb{R}^n_+} \le \text{const}\,\|\Phi\|_{2,\partial\mathbb{R}^n_+} < \infty, \qquad j = 1,\dots,n,$$

and it follows easily that $w \in W^{1,2}(\Omega)$.

To complete the proof it is sufficient to show that for any $\psi \in C_0^\infty$ we have

$$\int_{\mathbb{R}^n_+} \nabla w\cdot\nabla\psi\,dx = \int_{\partial\mathbb{R}^n_+} \Phi\psi\,dx'.$$

Fix such a function ψ and denote $\mathbb{R}^n_t = \{x \in \mathbb{R}^n_+ \ : \ x_n > t\}$, $t > 0$. As w is harmonic in $\mathbb{R}^n_+$, for $t > 0$ it holds

$$\int_{\mathbb{R}^n_t} \nabla w\cdot\nabla\psi\,dx = -\int_{\partial\mathbb{R}^n_+} \frac{\partial w}{\partial x_n}(x',t)\psi(x',t)\,dx'.$$

Now (7) and (9) yield

$$\begin{aligned}\int_{\mathbb{R}^n_+} \nabla w\cdot\nabla\psi\,dx &= \lim_{t\downarrow 0}\int_{\mathbb{R}^n_t} \nabla w\cdot\nabla\psi\,dx \\ &= -\lim_{t\downarrow 0}\int_{\partial\mathbb{R}^n_+} \frac{\partial w}{\partial x_n}(x',t)\psi(x',t)\,dx' = \int_{\partial\mathbb{R}^n_+} \Phi\psi\,dx'.\end{aligned}$$

□

Lemma 7. *If $\Omega = B_1^+(0)$ then every solution u_λ of (P_λ) has cylindrical symmetry, i.e. $u_\lambda(x', x_n)$ only depends on $|x'|$ and x_n. We write $u_\lambda(|x'|, x_n)$. Furthermore $u_\lambda(|x'|, x_n)$ is strictly decreasing as a function of $|x'|$ and x_n.*

This symmetry result appeared as Proposition 5.1 and 5.2 in Chipot et al. [3].

4 Nonexistence of positive solutions

Testing (P_λ) with ϕ_1 immediately shows that positive solutions can only exist for $\lambda < \lambda_1$. If ω is a domain with $\gamma_N \subset \partial\mathbb{R}^n_+$ flat, then ω is called star-shaped with respect to 0 if $x \cdot \nu \geq 0$ on γ_D. The following identity is a generalization of the famous identity found by S.I. Pohožaev [9]:

Lemma 8. *Let ω be a bounded domain where $\gamma_N \subset \partial\mathbb{R}^n_+$ is flat. If u is a classical solution of $\Delta u + f(u) = 0$ in ω with $u = 0$ on γ_D and $\frac{\partial u}{\partial \nu} = g(u)$ on γ_N then the following identity holds*

$$\begin{aligned}\frac{1}{2}\int_{\gamma_D} |\nabla u|^2 h(x)\cdot\nu\,d\sigma - \int_{\gamma_N} \operatorname{div}'(h'(x',0))G(u)\,dx' \qquad (10)\\ = \int_\omega F(u)\operatorname{div} h + \nabla u Dh(x)\nabla u - \frac{1}{2}|\nabla u|^2 \operatorname{div} h\,dx,\end{aligned}$$

where $G(s) = \int_0^s g(t)\,dt$, $F(s) = \int_0^s f(t)\,dt$ and $h : \overline{\omega} \to \mathbb{R}^n$ is a C^1-vector-field which has to satisfy $h(x)\cdot\nu(x) = 0$ on γ_N.

Proof. Let u be a solution. We begin with the following differential identity

$$\begin{aligned}&\operatorname{div}\Big((h\cdot\nabla u)\nabla u - \frac{1}{2}|\nabla u|^2 h\Big) \qquad (11)\\ &= \Delta u(h\cdot\nabla u) + \nabla u Dh\nabla u - \frac{1}{2}|\nabla u|^2\operatorname{div} h.\end{aligned}$$

Integrating identity (11) over ω and observing that $h(x)\cdot\nu(x) = 0$ on γ_N gives

$$\begin{aligned}&\int_{\gamma_D}\frac{1}{2}|\nabla u|^2 h\cdot\nu\,d\sigma + \int_{\gamma_N} h\cdot\nabla u\frac{\partial u}{\partial\nu}\,dx'\\ &= \int_\omega \Delta u\, h\cdot\nabla u + \nabla u Dh\nabla u - \frac{1}{2}|\nabla u|^2\operatorname{div} h\,dx \qquad (12)\\ &= \int_\omega -\operatorname{div}(F(u)h) + F(u)\operatorname{div} h + \nabla u Dh\nabla u - \frac{1}{2}|\nabla u|^2\operatorname{div} h\,dx.\end{aligned}$$

Due to the zero boundary conditions on γ_D and due to the assumption that $h(x)\cdot\nu(x) = 0$ on γ_N the integral over the first divergence-term vanishes. Next we use the flatness of γ_N. Notice that due to $h(x)\cdot\nu(x) = 0$ on γ_N we have $h\cdot\nabla u = h'\cdot\nabla' u$ on γ_N and therefore

$$\begin{aligned}&\operatorname{div}'\Big(h'(x',0)G(u)\Big) - \operatorname{div}'(h'(x,0))\,G(u)\\ &= h'(x',0)\cdot\nabla' G(u) = h(x',0)\cdot\nabla u(x',0)\,g(u).\end{aligned}$$

Integrating the last identity over the flat part γ_N we obtain

$$\int_{\gamma_N} h\cdot\nabla u\,g(u)\,dx' = -\int_{\gamma_N}\operatorname{div}'(h'(x',0))\,G(u)\,dx'.$$

Inserting back into (12) we have proved the lemma. □

Corollary 9. *Let ω be a bounded star-shaped domain where γ_N is flat. Then* (P_λ) *has no classical solution for $\lambda \le 0$.*

Proof. We use the previous lemma with $h(x) = x$. The integral identity (10) then reads

$$\begin{aligned} &\frac{1}{2}\int_{\gamma_D} |\nabla u|^2 x\cdot\nu\, d\sigma - \int_{\gamma_N} (n-1)G(u)\, dx' \\ &= \int_\omega nF(u) + \frac{2-n}{2}|\nabla u|^2\, dx \\ &= \int_\omega nF(u) + \frac{2-n}{2} f(u)u\, dx + \int_{\gamma_N} \frac{2-n}{2} ug(u)\, dx'. \end{aligned} \tag{13}$$

For (P_λ) we apply this identity to $f(s) = \lambda s$, $g(s) = s^{\frac{n}{n-2}}$. Due to the star-shapedness, the boundary integral over γ_D is non-negative. The sum of the integrands in the boundary integrals over γ_N vanish and the integrand of the volume integral is λu^2. For non-positive values of λ this implies that only trivial solutions exist. □

Remark. *We note in passing that Corollary 9 remains valid for weak $W^{1,2}$-solutions. One way to prove this result with the help of maximum principles is as follows, cf. Reichel, Zou [10]: If u is a $W^{1,2}$-solution let v_ρ be the Kelvin-transform of u at the sphere of radius ρ centered at the star-center of ω. Due to the structure of the equations and provided $\lambda \le 0$, the function v_ρ is a supersolution. With the help of a continuous version of the maximum principle one establishes that $u \le v_\rho$ for all values of $\rho \in (0,1)$. Taking the limit $\rho \to 0$ implies $u \equiv 0$. The proof makes use of the fact that $u \in L^{\frac{2n-2}{n-2}}(\gamma_N)$, and therefore it only just works for the critical case and not for supercritical cases.*

The next application of the Pohožaev-type integral identity (10) uses explicitly the special domain $\Omega = B_1^+(0)$.

Theorem 10. *For $n = 3$ and $\Omega = B_1^+(0)$ problem* (P_λ) *has no solution for $-\infty < \lambda \le 1/8$.*

Remark. *The value $1/8$ does not appear to be best possible. A larger, but computationally more involved bound is 0.772. It is an open problem to determine the precise range of λ for nonexistence/existence. A possible conjecture based on the existence interval from Corollary 3 is that nonexistence holds for $\lambda \le \pi^2/4$.*

Proof. By Corollary 9 we only need to investigate positive values of λ. Due to Lemma 7, all solutions u have cylindrical symmetry and are therefore of the form $u(r,z)$, where $r = \sqrt{x^2+y^2}$. For the vector-field $h(x)\colon \Omega \to \mathbb{R}^n$ we make an ansatz which also respects the cylindrical symmetry. If (r,t,z) denote the cylindrical coordinates $x = r\cos t$, $y = r\sin t$ with $t \in [0, 2\pi)$, then we set

$$h(r,t,z) = (k(r,z)\cos t, k(r,z)\sin t, l(r,z)),$$

with two real-valued functions $k(r,z), l(r,z)$. If we use that the gradient of $u(r,z)$ can be written as $\nabla u(r,z) = (u_r \cos t, u_r \sin t, u_z)$ and if we express $Dh(x,y,z)$ in cylindrical coordinates (a relatively long calculation which we do not perform here), we obtain

$$\nabla u Dh \nabla u = u_r^2 k_r + u_z u_r k_z + u_r u_z l_r + u_z^2 l_z. \tag{14}$$

For a successful application of Pohožaev's identity (10) we need to have $\nabla u Dh \nabla u = H(x,y,z)|\nabla u|^2$ with a function H which does not depend on the solution u. By (14) this is only possible if

$$k_z = -l_r \text{ and } k_r = l_z$$

which is equivalent to the requirement that $p(w) = k(r,z) + il(r,z)$ is a holomorphic function of $w = r + iz$. Under these circumstances we have indeed that $\nabla u Dh \nabla u = k_r|\nabla u|^2$. If we furthermore notice that div $h = k_r + k/r + l_z = 2k_r + k/r$ then the Pohožaev identity (10) reduces to

$$\begin{aligned} &\frac{1}{2}\int_{\Gamma_D} |\nabla u|^2 h(x)\cdot\nu \, d\sigma - \int_{\Gamma_N} (k_r(r,0) + k(r,0)/r) G(u)\, d(x,y) \\ &= \int_\Omega (2k_r + k/r)F(u) + (k_r - \frac{1}{2}(2k_r + k/r))|\nabla u|^2 \, d(x,y,z) \\ &= \int_\Omega (2k_r + k/r)F(u) - k/(2r)|\nabla u|^2 \, d(x,y,z). \end{aligned} \tag{15}$$

Next we test the differential equation with $m(r,z)u(r,z)$, where the C^2-function $m(r,z)$ will be chosen later. After observing that div $(\nabla u m u) = |\nabla u|^2 m + \frac{1}{2}\nabla(u^2)\nabla m - f(u)um$ and by using the divergence theorem we obtain

$$\int_\Omega |\nabla u|^2 m - \frac{u^2}{2}\Delta m - f(u)um \, d(x,y,z) = \int_{\Gamma_N} g(u)um - \frac{u^2}{2}\frac{\partial m}{\partial \nu}\, d(x,y).$$

Now we set $m(r,z) = k(r,z)/(2r)$ (which requires $k(r,z)/(2r)$ to be a C^2-function) and substitute $-\int_\Omega |\nabla u|^2 m \, d(x,y,z)$ from the previous integral identity into (15). The calculation of Δm uses the fact that $k_{rr} + k_{zz} = 0$ and gives

$$\Delta m = -\frac{1}{2r^2}\left(k_r - \frac{k}{r}\right).$$

Altogether we have

$$\begin{aligned} &\frac{1}{2}\int_{\Gamma_D} |\nabla u|^2 (k(r,z)r + l(r,z)z)\, d\sigma \\ &\qquad + \int_{\Gamma_N} -\left(k_r(r,0) + \frac{k(r,0)}{r}\right) G(u) + \frac{k(r,0)}{2r} g(u)u + \frac{1}{2}u^2 \frac{k_z(r,0)}{2r}\, d(x,y) \\ &= \int_\Omega \left(2k_r + \frac{k}{r}\right) F(u) - \frac{k}{2r} f(u)u + \frac{u^2}{4r^2}\left(k_r - \frac{k}{r}\right) d(x,y,z), \end{aligned}$$

and if we insert $f(u) = \lambda u$, $g(u) = u^3$ then we finally obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma_D} |\nabla u|^2 (k(r,z)r + l(r,z)z)\, d\sigma \\
&\qquad + \int_{\Gamma_N} \left(-k_r(r,0) + \frac{k(r,0)}{r}\right)\frac{u^4}{4} + \frac{u^2}{4}\,\frac{k_z(r,0)}{r}\, d(x,y) \\
&\qquad = \int_\Omega \left(4\lambda k_r + \frac{k_r}{r^2} - \frac{k}{r^3}\right)\frac{u^2}{4}\, d(x,y,z).
\end{aligned}$$

In order to obtain non-existence of positive solutions the following set of conditions needs to be satisfied

(i)	k_r	$\le$	$\frac{k}{r}$	on Γ_N,	(iv)	$k(r,z)r + l(r,z)z$	$\ge$	0	on Γ_D,
(ii)	k_z	$\ge$	0	on Γ_N,	(v)	$k_r(4\lambda + r^{-2})$	$\le$	kr^{-3}	in Ω,
(iii)	$l(r,0)$	$=$	0	on Γ_N,	(vi)	$\frac{1}{r}k(r,z) \in C^2(\overline{\Omega})$,			

where (iii) stems from the condition $h(x) \cdot \nu(x) = 0$ on Γ_N in Lemma 8. In the next step we need to decide on the choice of the holomorphic function $p(w) = k(r,z) + il(r,z)$ with $w = r + iz$. If we choose $p(w) = w$, then we have only created the vector-field $h(x,y,z) = (x,y,z)$ and we obtain non-existence only for $\lambda \le 0$. A more successful choice is $p(w) = w - \alpha w^3$, where α is determined a-posteriori. This leads to $k(r,z) = r - \alpha r^3 + 3\alpha r z^2$ and $l(r,z) = z - 3\alpha r^2 z + \alpha z^3$ and we immediately see that (ii), (iii) and (vi) are fulfilled. Condition (v) is equivalent to

$$(1 + 3\alpha(z^2 - r^2))\lambda \le \alpha/2 \quad \text{ for } 0 < r < 1 \text{ and } 0 < z < \sqrt{1 - r^2},$$

which shows that α must be positive (take $z = r = 1/\sqrt{2}$). Since $z^2 \le 1 - r^2$ a necessary and sufficient condition for (v) is $(1 + 3\alpha(1 - 2r^2))\lambda \le \alpha/2$. The left hand side takes its maximum at $r = 0$ and hence we need to require that

$$\lambda \le \frac{\alpha}{2 + 6\alpha} \text{ for } \alpha > 0. \tag{16}$$

Due to the positivity of α we find that (i) is also fulfilled and it remains to investigate (iv) on Γ_D, were $z = \sqrt{1 - r^2}$. Thus, (iv) takes the form $1 + \alpha - 2\alpha r^2 \ge 0$ for $0 \le r \le 1$. By monotonicity in r this only needs to be checked at $r = 1$, which leads to the restriction $0 < \alpha \le 1$. Hence $\alpha = 1$ in (16) gives us the largest non-existence interval $\lambda \in (0, 1/8)$. □

Remark. *The larger non-existence interval $\lambda \in (0, 0.772)$ was obtained by choosing the holomorphic function $p(w) = \sin(\alpha w)$, where α is determined a-posteriori as $\pi - 0.097$.*

5 Properties of u_λ for $\lambda \to 0$

Proposition 11. *Let ω be a bounded star-shaped domain where $\gamma_N \subset \partial\mathbb{R}^n_+$ is flat. If $n \ge 4$ and u_λ is any sequence of solutions of (P_λ) with $\lambda \to 0$ then necessarily $\|u\|_{\infty,\omega} \to \infty$ as $\lambda \to 0$.*

Proof. If $\|u_\lambda\|_{\infty,\omega}$ stays bounded we may extract a subsequence u_λ which converges in $W^{1,2}(\omega)$ to a weak solution u of (P_0). By the Pohožaev-type non-existence result and in particular the remark following Corollary 9 it follows that $u \equiv 0$. However, near $u \equiv 0$ the linearization of (P_λ) has a bounded inverse. Hence the uniqueness part of the implicit function theorem shows that it is impossible for nontrivial solutions u_λ to approach the trivial solution in the $W^{1,2}(\omega)$-norm as $\lambda \to 0$. Therefore $\|u_\lambda\|_{\infty,\omega}$ must become unbounded as $\lambda \to 0$. □

Proposition 12. *Let* $\Omega = B_1^+(0)$.

(i) If u_λ *is any sequence of solutions of* (P_λ) *then, for the entire sequence,* $\|u_\lambda\|_{q,\Gamma_N}$ *is uniformly bounded in* λ *for all* $q \le \frac{n}{n-2}$.

(ii) Let $\epsilon > 0$*. On* $\Gamma_{N,\epsilon} = \{x' \in \mathbb{R}^{n-1} : \epsilon < |x'| \le 1\}$ *and on* $\Omega_\epsilon = \{x \in \mathbb{R}^n : \epsilon < |x| \le 1\}$ *we have that* $\|u_\lambda\|_{\infty,\Gamma_{N,\epsilon}}$ *and* $\|u_\lambda\|_{\infty,\Omega_\epsilon}$ *are uniformly bounded in* λ.

Proof. (i) Let λ_s, ϕ_s be the first Steklov eigenvalue and positive eigenfunction of

$$\Delta\phi = 0 \text{ in } \Omega, \qquad \frac{\partial\phi}{\partial\nu} = \lambda_s\phi \text{ on } \Gamma_N, \quad \phi = 0 \text{ on } \Gamma_D.$$

By testing (P_λ) with ϕ_s we obtain

$$\begin{aligned}\int_{\Gamma_N} \phi_s u_\lambda^{\frac{n}{n-2}}\,dx' &= \int_\Omega (\nabla u_\lambda \nabla\phi_s - \lambda u_\lambda \phi_s)\,dx \\ &\le \int_\Omega \nabla u_\lambda \nabla \phi_s\,dx = \lambda_s \int_{\Gamma_N} u_\lambda \phi_s\,dx'.\end{aligned}$$

After using Hölder's inequality this implies that $\int_{\Gamma_N} \phi_s u_\lambda^{\frac{n}{n-2}}\,dx' \le K$ uniformly in λ. Since $u_\lambda(x',0)$ is monotone decreasing as a function of $|x'|$ by Lemma 7, we obtain the full result $\int_{\Gamma_N} u_\lambda^{\frac{n}{n-2}}\,dx' \le K$ uniformly in λ.
(ii) By *(i)* we find the following estimate

$$K \ge \int_{\Gamma_N} \phi_s u_\lambda^{\frac{n}{n-2}}\,dx' \ge \int_{|x'|\le\epsilon} \phi_s u_\lambda^{\frac{n}{n-2}}\,dx' \ge K_\epsilon \|u_\lambda\|_{\infty,\Gamma_{N,\epsilon}}^{\frac{n}{n-2}},$$

where in the last step we have used the monotonicity result $u_\lambda(x',0) \ge u_\lambda(\xi',0)$ for $|x'| \le \epsilon \le |\xi'| \le 1$. This establishes the claim for $\|u_\lambda\|_{\infty,\Gamma_{N,\epsilon}}$. The estimate for $\|u_\lambda\|_{\infty,\Omega_\epsilon}$ follows in the set $\tilde{\Omega}_\epsilon = \{(x',x_n) \in \Omega : x' \in \Gamma_{N,\epsilon}\}$ by monotonicity in the x_n-direction from the estimate on $\Gamma_{N,\epsilon}$. The estimate in the remaining parts of Ω_ϵ follows by Harnack's inequality and in a neighborhood of the pole $(0,\ldots,0,1)$ again by monotonicity in the x_n-direction. □

Remark. *Part* (i) *of the Proposition holds for any bounded domain* ω *with* γ_N *flat and* $\gamma_D \in C^2$*. To show this, the simple monotonicity argument in our proof for* $B_1^+(0)$ *has to be replaced by a more refined monotonicity argument based on the Kelvin transform as in de Figueiredo et al.* [4], *Section* 1.2.

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